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ELEMENTS OF SURVEYING

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ELEMENTS OF SURVEYING

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PREFACE

This book has been prepared for the specific purpose of providing a text to meet the needs of the classroom in the less comprehensive courses in surveying given by engineering schools and colleges usually to other than civil engineering students.

It is essentially an abridgment of "Surveying Theory and Practice," first published in 1928, and it has been written in response to the requests of a considerable number of teachers who have been using the authors' earlier book for their more extensive courses and who desire a briefer treatise of similar arrangement for their shorter courses.

In determining the character and scope of the material here presented, the authors have been guided not only by their own experience in the classroom, but also in a large measure by the helpful advice and suggestions of teachers in various engineering colleges throughout the United States. Brevity has been recognized as desirable, and to this end the more highly specialized branches of surveying have been eliminated from consideration, and detailed discussions have been avoided. But the necessity of a thorough grounding in essentials has been deemed of paramount importance, and words have not been spared in presenting fundamental principles and practices which apply to the elementary operations of surveying and which are largely common to all branches.

Chapters I to IV inclusive may be considered as introductory to the subject. This material is designed to give the student a speaking acquaintance with surveys and their purposes and with the rudimentary operations of field and office. It is material which might be assigned for general reading but which might not receive formal treatment in the classroom. It is especially designed for those courses for which field and office practice is given concurrently with the work of the classroom.

Chapter V presents the subject of errors, an understanding of which is so essential to sound surveying practice. In this chapter special attention is given to accidental errors and the law of probability. Errors are further discussed throughout the following chapters, where the kinds, sources, and distribution of errors are considered in connection with the various operations of surveying.

Chapters VI to XVII inclusive deal with the fundamental operations of surveying, including the use and adjustment of surveying instruments, and the methods of measuring and laying off angles, distances, and differences in elevation in the field, and of calculating these and related quantities in the office. A knowledge of the material of these chapters is regarded as prerequisite to all surveying practice, and special attempt has been made to develop thoroughly the principles and to describe clearly and concisely the operations upon which surveying in all its branches is based. Although in many short courses limitations of time make it impossible to consider otherwise than superficially the problems of determining true position and direction by observations on the sun and stars, the authors see danger in merely reciting a few facts without at least making the reasons therefor easily available to the student. For this reason, the subject of field astronomy has been treated at somewhat greater length than is customary in a book of this kind, considerable space being devoted to describing the fundamental concepts of the celestial sphere.

Chapters XVIII to XXIV inclusive are concerned with the practice of surveying as extended to the more common kinds of surveys, including surveys for the establishing of boundaries of rural and urban lands, for locating and constructing railroads, highways, and other routes, and for topographic mapping.

The authors desire especially to make the following acknowledgments:

Prof. S. Einarsson, Department of Astronomy, University of California, has reviewed the manuscript for Chaps. XV and XVI and has made suggestions which have resulted in substantially strengthening the material on astronomy.

The Surveyor General of the General Land Office has kindly furnished data for tables bearing on the United States system of subdividing the public lands described in Chap. XX, and the "Manual of Instructions of the General Land Office" has been freely quoted.

The Director of the U. S. Coast and Geodetic Survey has supplied data for Tables IV and V and has furnished photographs from which have been prepared a considerable number of the illustrations for Chaps. XXIII, XXIV, and XXV.

The Director of the U. S. Geological Survey has furnished photographs for numerous halftones for Chaps. XXII, XXIII, XXIV; and C. H. Birdseye, Chief Topographic Engineer, and W. H. Herron, Geographer in Charge of Central Division, both of the U. S. Geological Survey, have rendered very generous help in the preparation of those portions of the manuscript dealing with topographic mapping.

To John Wiley & Sons, Inc., credit is due for permission to use Table IX, which is from Johnson's "Theory and Practice of Surveying," and Table X, which is from Searles and Ives' "Field Engineering."

Scattered throughout the text are halftones of surveying instruments, the photographs having been furnished by various manufacturers. In this connection it is desired to acknowledge the assistance of W. and L. E. Gurley, Troy, N. Y.; Keuffel and Esser Company, New York; the A. Lietz Company, San Francisco, Calif.; and C. L. Berger and Sons, Boston, Mass.

R. E. D.
F. S. F.
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ELEMENTS OF SURVEYING

CHAPTER I

INTRODUCTION TO SURVEYING

1. Surveying.—Surveying has to do with the determination of the relative position of points on or near the surface of the earth. It is the art of measuring distances between terrestrial objects, of measuring angles between terrestrial lines, of determining the direction of lines, and of establishing points by predetermined angular and linear measurements.

Incidental to the actual measurements or surveying are mathematical calculations. Distances, angles, directions, positions, areas, and volumes are thus determined from data secured during the progress of the survey. Much of the information of the survey is portrayed graphically, involving the construction of maps, profiles, cross-sections, and diagrams.

Thus it is that the process of surveying may be divided into two parts: first, the work of taking measurements in the field, commonly designated as the *field work;* and second, the computing and drawing necessary to the purpose of the survey, usually termed the *office work.*

2. Classification of Surveys.—The earliest surveys known were for the purpose of establishing the boundaries of land, and today land surveys are the important work of a large number of surveyors.

Every project of any magnitude in the field of construction is based to a greater or less degree upon measurements taken during the progress of a survey, and is constructed about lines and points established by the surveyor. Aside from surveys made primarily for fixing the boundaries of landed property, practically all surveys of a private nature, and most of those conducted by government agencies, if not made directly for, are of assistance to the engineer in the conception, design, and execution of his work.

For many years the government has conducted surveys over large areas. The purposes of these surveys have been many. The principal work so far accomplished consists of the fixing of national and state boundaries, charting the coast lines and navigable streams and lakes, the precise location of certain definite points throughout the country, and the topographic mapping of large portions of the interior.

Summing up, it is seen that surveys have been divided into three classes: (1) those the primary purpose of which is to establish the boundaries of landed properties; (2) those forming the basis of a study for or necessary to the construction of public or private works; and (3) those of large magnitude conducted by the government.

3. The Earth.—The earth is an oblate spheroid of revolution, the length of its polar axis being somewhat less than that of its equatorial axis. The lengths of these axes are:

Polar axis	41,710,242 ft.
Equatorial axis	41,852,124 ft.

Comparing the distances, it will be seen that the polar axis is the shorter by 141,882 ft. or about 26.87 miles. Relative to the diameter of the earth this is a very small quantity, less than 0.34 per cent.

Consider that the irregularities of the earth have been removed. The surface of this imaginary spheroid is a curved surface called a *level surface* every element of which is normal to the plumb line. The surface at sea level is termed the mean surface of the earth.

Imagine a plane passing through the center of the earth, as in Fig. 3*a*. Its intersection with the level surface forms a continuous line around the earth. Any portion of such a line is termed a level line, and the circle defined by the intersection of such a plane with the mean level of the earth is termed a great circle of the earth. The distance between two points on the earth, as A and B (Fig. 3*a*), is the length of the arc of the great circle passing through the points, and is always somewhat more than the chord intercepted by this arc. The arc is a level line; the chord is a mathematically straight line.

If a plane is passed through the poles of the earth and any other point on the earth's surface, as A (Fig. 3*b*), the line defined by the intersection of the level surface and plane is called a meridian. Imagine two such planes as passing through two points as A and B (Fig. 3*b*) on the earth, and the section between the two planes removed like the slice of an orange, as in Fig. 3*c*. At the equator the two meridians are parallel; above and below the equator they converge, and the angle of convergence increases as the poles are approached. No two meridians are parallel except at the equator.

Imagine radial lines drawn on the two cut surfaces of the slice. If the earth be regarded as a perfect sphere these lines converge at a point at the center of the earth. Considering the lines on either or both the cut surfaces, no two are parallel. The radial lines may be considered as vertical or plumb lines, hence it appears that all plumb lines converge at the earth's center, and no two are parallel.

Consider three points on the mean surface of the earth. Let us make these three points the vertices of a triangle, as in Fig. 3*d*.

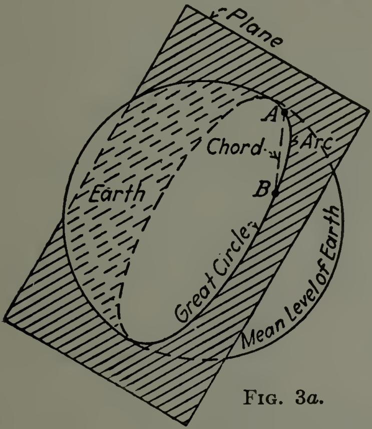

Plane
A
Chord
Arc
Earth
B
Great Circle
Mean Level of Earth

Fig. 3*a*.

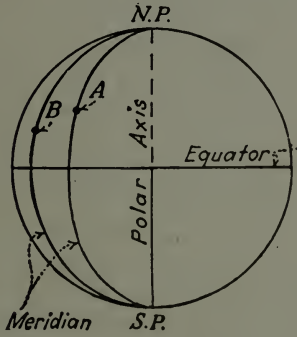

N.P.
A
B
Axis
Equator
Polar
Meridian
S.P.

Fig. 3*b*.

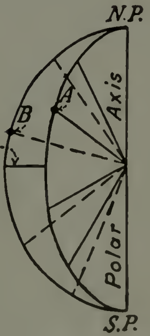

N.P.
B
A
Axis
Polar
S.P.

Fig. 3*c*.

The surface within the triangle *ABC* is a curved surface, and the lines forming its sides are arcs of great circles. The figure is a spherical triangle. In the figure the dotted lines represent the plane triangle whose vertices are points *A*, *B*, and *C*. Lines drawn tangent to the sides of the spherical triangle at its vertices are shown. The angles *a*, *b*, and *c* of the spherical triangle are seen to be greater than the corresponding angles *a′*, *b′*, and *c′* of the plane triangle. Thus it is seen that angles on the surface of the earth are spherical angles.

In everyday life we are not concerned with these facts. We think of a line passing along the surface of the earth directly between two points as being a straight line, we think of plumb-lines as being parallel, we think of a level surface as a flat surface, and we think of angles between lines in such a surface as being plane angles.

As to whether the surveyor must regard the earth as a sphere or may regard it as a disc depends upon the character and magnitude of the survey and the precision required. The latter premise is much the simpler.

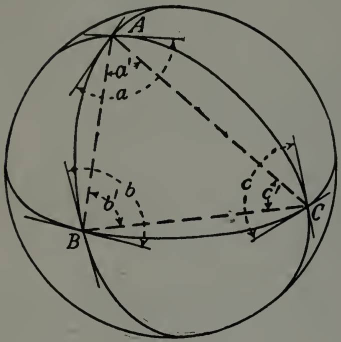


Fig. 3*d*.

4. Plane Surveying.—That type of surveying in which the mean surface of the earth is considered as a plane, or in other words, the spheroidal shape of the earth is neglected, is generally defined as *plane surveying*. From this assumption it follows that a level line or a line lying within a level surface is considered as mathematically straight, that the direction of the plumb-line at any point within the limits of the survey is considered as parallel to the direction of the plumb-line at any other point, and that angles of polygons are considered as plane angles.

Surveys for the location and construction of railroads, highways, canals, and, in general, the surveys necessary for the works of man are of this type, as are also the surveys made for the purpose of establishing boundaries, except state and national.

The operation of determining elevation is usually considered as a division of plane surveying. Elevations are referred to a spheroidal surface, a tangent at any point in the surface being normal to the plumb line at that point. The procedure ordinarily used in determining elevations automatically takes into account the curvature of the earth, and elevations referred to the surface of reference or "datum plane" are secured without extra effort. Imagine a plane, tangent to the surface of mean sea level at a given point. At a horizontal distance of 10 miles from the point of tangency the vertical distance (or elevation) of the plane above the surface represented by mean sea level is 67 ft. and at a distance of 100 miles from the point of tangency the elevation of the plane is 6,670 ft. above mean sea level. Evidently the curvature of the earth's surface is a factor which cannot be neglected in obtaining even very rough values of elevations.

The United States system of subdividing the public lands employs the methods of plane surveying, but takes into account the shape of the earth in the location of certain of the primary lines of division.

This book deals with the methods of plane surveying.

5. Geodetic Surveying.—That type of surveying which takes into account the shape of the earth is defined as geodetic surveying. All surveys employing the principles of geodesy are of high precision and, in general, extend over large areas.

The data of the various geodetic surveys are of great importance in that they furnish precise points of reference to which the multitude of surveys of less precision may be tied.

6. Definitions.—A *level surface* is one parallel with the mean spheroidal surface of the earth. A body of still water provides the best example.

A *horizontal plane* is a plane tangent to a level surface.

A *horizontal line* is a line tangent to a level surface.

A *horizontal angle* is an angle formed by the intersection of two lines in a horizontal plane.

A *vertical line* is a line normal to a level surface. A plumb-line is an example.

A *vertical plane* is a plane of which the vertical line is an element.

A *vertical angle* is an angle between two intersecting lines in a vertical plane. In surveying it is commonly understood that one of these lines is horizontal and a vertical angle to a point is understood to be the angle in a vertical plane between a line to that point and the horizontal plane.

At any point there can be but one horizontal plane and but one vertical line, but there may be an infinite number of horizontal lines and vertical planes.

In surveying, measured angles are either vertical or horizontal. In plane surveying, distances measured along a level line are termed horizontal distances. The distance between two points, as for example, the corners of a lot, is commonly understood to be the horizontal distance from the plumb-line through one point to the plumb-line through the other.

The *elevation* of a point is its vertical distance above (or below) some arbitrarily assumed level surface, called a "*datum*" or a "*datum plane.*" The datum to which most elevations are referred is *mean sea level,* the normal level of the ocean at mean tide.

The vertical distance between two points is termed *the difference in elevation.* It is the distance between an imaginary level surface containing the high point and a similar surface containing the low point.

7. Units of Measurement.—The units of angular measurement are the degree, minute, and second.

In the United States, Great Britain, and Canada the common units of linear measurement are the yard, the foot, and the inch. On most surveys in these countries, distances are measured in feet, tenths of feet, and hundredths of feet, and surveyors' tapes are usually graduated in these units.

Formerly the *rod* and the *Gunter's chain* were units much used in land surveying. The length of the Gunter's chain is 66 ft. A convenient relation exists between the rod, the Gunter's chain, the acre, and the mile.

In the United States the units of area commonly used are the square foot and the acre. Formerly the square rod and the square Gunter's chain were also used.

43,560 sq. ft. = 160 sq. rods = 10 sq. Gunter's chains = 1 acre.

The units of volumetric measurement are the cubic foot and the cubic yard.

8. Kinds and Operations of Surveying.—The nature of the measurements made by the surveyor has been indicated in preceding articles.

In *land surveying* his work consists of:

1. Rerunning old land lines to determine their length and direction.
2. Reestablishing obliterated land lines from recorded lengths and directions and such other information as it is possible to secure.
3. Subdividing lands into parcels of predetermined shape and size.
4. Setting monuments to preserve the location of land lines.
5. Locating the position of such monuments with respect to permanent landmarks.
6. Calculating areas, distances, and angles, or directions.
7. Portraying the data of the survey on a *land map.*
8. Writing descriptions for deeds.

A *topographic survey* is a survey made to secure data from which may be made a map indicating the relief. The result of such a survey is the *topographic map.* The work consists of:

1. Establishing by angular and linear measurements the horizontal location of certain points for the control of the survey within the area to be mapped, termed the *horizontal control.*
2. Determining the elevation of control points by the operation of leveling, termed the *vertical control.*
3. Determining the horizontal location and elevation of a sufficient number of ground points to provide data for the map.
4. Locating such other natural or artificial details as the requirements of the survey demand.

5. Calculating angles, distances, and elevations.
6. Plotting and finishing the topographic map.

Route surveying as the term is here used has reference to those surveys necessary for the location and construction of lines of transportation or communications, such as highways, canals, railroads, transmission lines, and pipe lines. Such surveys are often called line surveys. The preliminary work for such a line usually consists of a topographic survey. The location and construction surveys may further consist of:

1. Locating the center line by stakes at short intervals.
2. Running levels to determine the profile of the ground along the center line.
3. Plotting such profile, and fixing grades.
4. Taking cross-sections.
5. Calculating earth work.
6. Measuring drainage areas.
7. Laying out structures, such as culverts and bridges.
8. Locating right-of-way boundaries.

Laying out lots and the municipal surveys made in connection with the construction of streets, water supply systems, and sewers is frequently termed *city surveying.*

Cadastral surveying, as the term has come to be used in the United States, has particular reference to those extensive surveys of some of our large cities, made for the purpose of fixing with great precision a system of reference monuments and locating property lines and improvements in great detail.

CHAPTER II

FIELD WORK

9. General Remarks.—The nature of field measurements has already been indicated. Field work consists of:

1. Adjusting instruments and caring for field equipment.
2. Determining the position of or establishing stones, stakes, or other more or less permanent monuments for the control of the survey or for other purposes.
3. Fixing the horizontal position of objects or points by horizontal angles and distances.
4. Determining the elevations of objects or points by one or another of the methods of leveling.
5. Making a record of the field work, usually in the form of field notes in a book called the field notebook, but sometimes directly in the form of a map drawn to scale.

10. Some Surveying Terms.—For a better understanding of the following articles brief definitions of a few of the terms of surveying are appropriate.

Chaining is the operation of measuring horizontal or inclined distances with the tape. The persons who make such measurements are called chainmen.

Flagman.—A person whose duty it is to hold the flag or range pole at selected points, as directed by the transitman or other person in charge.

Rodman.—A person whose duty it is to hold the rod and otherwise assist the leveler or topographer.

Backsight.—(1) A sight taken with the level to a point of known elevation. (2) A sight or observation taken with the transit along a line of known direction to a reference point generally in the rear.

Foresight.—(1) A sight taken with the level to a point the elevation of which is to be determined. (2) A sight taken with the transit along a line whose direction is to be determined to a point generally in advance.

Hub.—A point over which the transit is set, usually a heavy stake set nearly flush with the ground, with a tack in the top marking the point.

Line refers to the path or route between points of control along which measurements are taken to determine distance. To *give line* is to direct the placing of a range pole, pin, or other object on line.

Turning Point.—A fixed point or object temporarily used in leveling where the rod is held first for a foresight, then for a backsight.

Benchmark.—A fixed reference point or object, more or less permanent in character, the elevation of which is known.

11. Checking.—No measurement should be regarded as correct until verified and in as far as it is practicable, methods of verification should differ from the methods used in original measurements. All persons are liable to mistakes, but mistakes in field work become discreditable to the maker if he allows any other than himself to discover the discrepancy.

11a. Consistent Accuracy.—The accuracy of measurements should be consistent with the purposes of the survey. Surveyors often fail to comprehend the different degrees of precision necessary for the different kinds of work, or fail to maintain a consistent degree of accuracy throughout any one survey. No definite rules for the relative accuracy of different classes of surveys can be established, for the objects and conditions are too many and too complicated.

Hence, before beginning a survey the surveyor must establish the limit of error for himself, using his own judgment and the experience of others to guide him. The best surveyor is not necessarily one who is extremely accurate, but one whose measurements are of a precision consistent with the purpose of the survey.

12. Relation between Angles and Distances.—If measurements are to be consistent, it is evident that the precision of angles should correspond to the precision of related distances—or in other words, the percentage of error in angle should equal the percentage of error in distance. Thus in Fig. 12, the position of the point B with respect to the line AC is represented by the point B, its true position, and B', its erroneous position resulting from the error e_d in the measured distance AB, and the displacement e_a due to the error e_α in the measured angle.

Evidently a consistent relation between errors in angle and errors in distance would require that the distances e_a and e_d should be equal, or nearly so. Accordingly, it may be stated that a consistent relation between precision of angles and distances will be maintained if the tangent of the allowable error in angle e_α equals the allowable error in the distance (expressed as a ratio).

It is impracticable to maintain an exact equality between these ratios; but in general, surveys should be so conducted that the variation between precision of angle and precision of distance will not be large. Since the accuracy of a survey is often judged by the number of significant places in the recorded distances, it is always best to measure angles with a precision at least equal to the precision of the distances. It is instructive to note that an angular error of 01′ corresponds to a linear error of 0.291 ft. or nearly 3.5 in. per 1,000 ft., a ratio of precision of $\frac{1}{3,440}$. For a length other than 1,000 ft. the linear error is in direct proportion.

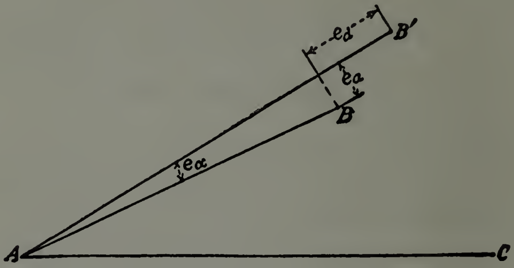


Fig. 12.

Where distances are roughly determined, the angles are frequently measured with more than the required accuracy without increased effort. For example, in rough chaining the ratio of precision might be $\frac{1}{1,000}$, corresponding to an angular error of 03′. But with the ordinary transit, the angles could be determined to the nearest 01′ as quickly as to the nearest 03′.

13. Angles Used in Trigonometric Computations.—Very often field measurements are made the basis of computations involving the trigonometric functions, and it is necessary that the computed results be of a required precision. If the values of these functions were exactly proportional to the size of the angles—or in other words, if any increase in the size of an angle were accompanied by a proportional increase or decrease in the value of a function—the problem of determining the precision of angular measurements would resolve itself into that explained in the preceding article; but since the ratios of the rates of change to the values of the sines of small angles, the cosines of angles near 90°, and the tangents and cotangents of small and large angles are relatively large, it is evident that the precision with which an angle is determined should be made to depend upon the size of the angle and the function to be used in the computations. It is not practicable to measure each angle with exactly the precision necessary to insure sufficiently accurate computed values, but at least the surveyor should have a sufficiently comprehensive knowledge of the purpose of the survey and of the properties of the trigonometric functions to keep the angles within the required precision.

The curves of Figs. 13*a* and 13*b* show the ratios of precision corresponding to various angular errors from 05″ to 01′ for sines, cosines, tangents, and cotangents. For the function under consideration these curves may be utilized to determine the ratio of precision corresponding to a given angular error and angle, or to determine the maximum or minimum angle that for a given angular error will furnish the required ratio of precision, or to determine the precision with which angles of a given size must be measured to maintain a required ratio of precision in computations.

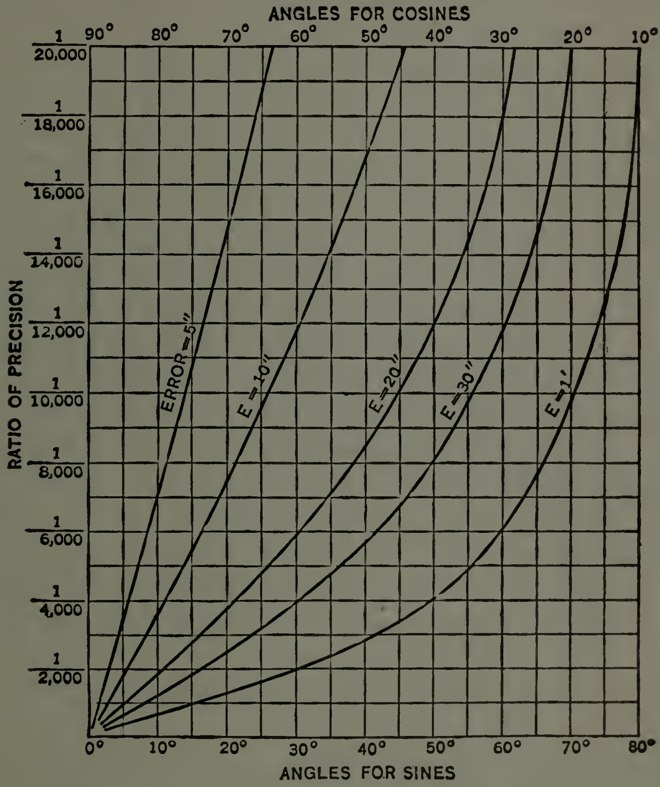


Fig. 13*a*.—Ratio curves for sines and cosines.

14. Signals.—Except for short distances, a good system of hand signals between different members of the party makes a more efficient means of communication than is possible by word of mouth. The number of signals necessary will depend upon the character of the work and the nature of the country. A few of the more common are given below:

"*Right*" or "*Left.*"—The arm is extended in the direction of the desired movement, the right arm being extended for a movement to the right and the left arm for a movement to the left. A long, slow, sweeping motion of the hand indicates a long movement; a short, quick motion indicates a short movement. This signal may be given by the transitman in directing the chainman on line, by the leveler in directing the rodman for a turning point, by the chief of the party to any member, or by one chainman to another chainman.

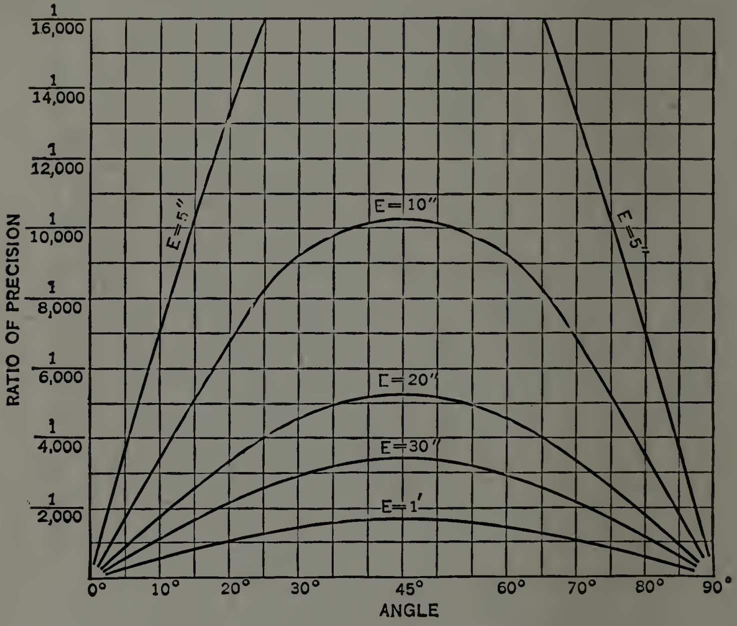


Fig. 13*b*.—Ratio curves for tangents and cotangents.

"*Up*" or "*Down.*"—The arm is extended upward or downward, with wrist straight. When the desired movement is nearly completed, the arm is moved towards the horizontal. The signal is given by the leveler.

"*All Right.*"—Both arms are extended horizontally and the forearms waved vertically. The signal may be given by any member of any party.

"Plumb the Flag" or *"Plumb the Rod."*—The arm is held vertically and moved in the direction that the flag or rod is to be plumbed. It is given by the transitman or leveler.

"Give a Foresight."—The instrumentman holds one arm vertically above his head.

"Establish a Turning Point" or *"Set a Hub."*—The instrumentman holds one arm above his head and waves it in a circle.

"Give Line."—The flagman holds the flag horizontally in both hands above his head and brings it down and turns it to a vertical position. If he desires to set a hub, he waves the flag with one end in the ground, from side to side.

"Turning Point" or *"Bench Mark."*—In profile leveling the rodman holds the rod horizontally above his head and then brings it down on the point.

"Wave the Rod."—The leveler holds one arm above his head and moves it from side to side.

"Pick Up the Instrument."—Both arms are extended outward and downward, then inward and up, as one would do in grasping the legs of the tripod and shouldering the instrument. It is given by the chief of the party or by the head chainman when the transit is to be moved to another point.

15. Field Notes.—The field notes should constitute a permanent record of the survey with data in such form as to be interpreted with ease by anyone having a knowledge of surveying. The notes consist of numerical data, explanatory notes, and sketches. In addition, the record of every survey should include the date, the weather conditions, the names of the surveyor and his assistants, and a title indicating the location of the survey and its nature or purpose.

No matter what the character of the survey, all field notes should be taken in the field at the time the work is being done. They are generally recorded in pencil.

The observer should realize that the notes may be used by persons unfamiliar with the locality who must rely entirely upon what he has recorded. For this reason, not only should the notebook contain all necessary information, but data should be recorded in a form which will admit of only one interpretation, and that the correct one. A good sketch perhaps helps more than anything else to convey a correct impression to others, and for this reason sketches should be used freely. In general, sketches are not drawn to scale, but in most work it is better to construct them approximately to scale. If the note keeper will bear in mind what use is to be made of the notes, it will aid him greatly in deciding what data are necessary and what are not. To make the notes clear, the recorder should put himself in the place of one who is not on the ground at the time the survey is made; and to make the notes complete, he should thoroughly understand for what purpose they are to be used. Before making any survey, the necessary data to be collected should be carefully considered; and when doing the field work, all such data should be obtained, but no more.

While a few convenient forms of notes are in common usage, generally it will be necessary to supplement these, and in many cases it will be necessary for the surveyor to devise his own form of record for field data.

CHAPTER III

COMPUTATIONS

16. General Remarks.—Calculations of one kind or another form a large part of the work of surveying, and the ability to compute with speed and accuracy is an important qualification of the surveyor. Computing is an art not to be acquired without practice, but no amount of experience will make an efficient computer unless he possesses a knowledge of the precision of measurements and the effect of errors in given data upon the precision of values calculated therefrom, and is familiar with the algebraic and graphical processes and mechanical devices by means of which the labor of computing may be reduced to a minimum.

Calculations are made algebraically through the use of the simple arithmetical processes, logarithms, and the trigonometric functions; graphically by accurately scaled drawings; or by mechanical devices such as the slide rule and computing machines. A knowledge of the short methods of arithmetic is of value. The ability to make mental calculations quickly is desirable. Facility in the use of the ordinary slide rule is important.

Before making calculations of any great importance or extent, the computer should carefully plan a clear and orderly arrangement. This is necessary: (1) to save time; (2) to prevent mistakes; (3) to make the calculations legible to another; (4) to afford proper checks; and (5) to facilitate the work of the checker.

17. Checking.—In practice the necessity of checking every computation is recognized. *No confidence is placed in results that have not been checked,* and important results are checked by more than one method.

Most problems can be solved by more than one exact method. Since by using the same method in checking, the same error is likely to occur, the computer should check results by a different method when this is feasible. Approximate checks may be obtained in many cases through such mechanical devices as the slide rule, the planimeter, and the protractor. Graphical methods may often be used to good advantage, as an approximate check.

18. Significant Figures.—The term *significant figures* is used to refer to those digits in a number which have meaning; *i.e.*, those digits whose values are known. Confusion in the matter of computations involving measured quantities arises from the failure of the novice to distinguish between exact numbers and numbers which carry with them the inevitable errors of measured quantities. An example of the former is the number of eggs in a grocer's basket, which may be exactly counted and an example of the latter is the weight of the sugar that remains in his barrel, which cannot be exactly determined. If he weighs the sugar on his scales, he may find a weight of 57¼ lb. But if for any reason it should be necessary to know the weight more precisely, a more accurate scales would be used and the weight might be found to be 57 lb. 5 oz.

Similarly if we measure roughly a given distance with a steel tape we may find it to be 732 ft., but if we measure it more carefully it may be found to be 732.4 ft., or by still greater refinements it may be determined as 732.38 ft. The number of significant figures in these three results is 3, 4, and 5, respectively.

Obviously then, the number of digits that have meaning and that may be used to indicate the length of this line is strictly limited by the precision with which the measurement has been made.

This precision is not always apparent in measurements, as will now be explained.

When direct measurements are taken, the number of significant figures in the result is evident. Thus, suppose the surveyor measures a distance with the steel tape (graduated to hundredths of a foot) and finds it to be 37.42 ft. There is no question but that the number of significant figures in this result is four. Suppose the distance is much greater, however, so that the tape must be stretched several times over rough ground and the total distance is determined as 623.58 ft. It is now very doubtful if the last digit is correct, even though it can be read with certainty on the tape, because the indirect measurement has introduced a number of sources of error (as marking the ends of the tape, keeping the tape level, etc.) which renders the accuracy of the last digit, and possibly the last two digits, uncertain. Hence, it cannot be said, offhand, how many significant figures there are in any measured quantity until the character and magnitude of the errors have been examined.

In a decimal the number of significant figures is not necessarily the number of decimal places, as the following examples will illustrate:

0.0000065 contains two significant figures.
0.00000650 contains three significant figures.
10.00000650 contains ten significant figures.
0.08000650 contains seven significant figures.

19. Trigonometric Tables.—The number of places to be used in trigonometric tables will depend upon the ratio of precision of the particular value of the function involved rather than upon the angular error. The number of significant figures for a particular angle, angular error, and function may be readily determined by inspecting either the Ratio Curves (Figs. 13*a* and 13*b*), or from trigonometric tables. *Assuming that the precision of angle is the governing precision*, for tangents and cotangents three places will be sufficient for angles in error more than 02′; four places, for angles in error less than 02′ but more than 10″; and five places, for angles in error less than 10″ but more than 01″.

For sines and cosines of angles of *average size* it will be seen that four places are sufficient for angles in error not less than 20″; five places for angles in error less than 20″ but more than say 05″; and six places for angles in error less than 05″ but more than ½″. Since angles are likely to be other than average size, the preceding limits should, in general, be doubled, though for sines of very large angles and for cosines of very small angles the necessary number of places should be ascertained by first determining the ratio of precision corresponding to the given angular error. For example, the ratio of the tabular difference for 10″ to sin 80° (or cos 10°) is about $\frac{1}{120,000}$, therefore six places will be necessary for angles as small as 10° when the error is 10″; or from the Ratio Curve for Sines it will be seen that for angles greater than 70° when the angular error is 01′, five places will be required.

20. Graphical and Mechanical Methods.—These methods are of particular value in approximately checking more precise computations. In general, graphical results may be achieved with less labor than by arithmetic and mistakes are less likely to occur. Frequently, combined graphical and mechanical methods may be utilized in conjunction with algebraic processes.

For example, earthwork cross-sections may be plotted to scale (graphical), the area of the cross-sections may be measured with the planimeter (mechanical), and the volume of earthwork may be determined by arithmetic.

Again, the area of a field may be determined by logarithmic computations of its partial areas, and by adding its partial areas on the adding machine. The result may be checked against large mistakes by use of the slide rule.

Similarly, unknown lengths and angles which have been algebraically or mechanically computed may be approximately checked by plotting the known data, measuring the unknown length with a scale, and measuring the unknown angle with a protractor.

The use of the common mechanical devices is later described. It is pertinent here to draw attention to the advantages of some of these.

The most common mechanical aid available to the surveyor is the ordinary 10-in. slide rule. This rule greatly facilitates calculations involving no more than three significant figures and is in every way the equivalent of a three-place table of logarithmic functions. Probably no other calculating device yet invented has as wide a range of usefulness and certainly none other can compare with it in the rapidity with which computations can be made.

For results of more than three significant figures, the calculating machine (of which there are several types adapted to the work of the surveyor) is coming largely to replace other methods of computing in many offices. With the improved types, the operations of multiplication, division, squaring, and taking the square root are instantly proven, so that further checking is not required, and for most calculations involving more than three significant figures the result may be obtained more quickly by the calculating machine than by any other method.

Another device frequently utilized by the surveyor is the polar planimeter. It is of great value in finding the areas of figures plotted to scale. The precision with which results may be obtained depends upon a number of factors but principally depends upon the skill of the operator in traversing the lines of the drawing with the tracing point. In general, results may be determined to three significant figures, a precision in keeping with much of the field data upon which area calculations are based. It is a simple instrument to operate, and it furnishes the most efficient means of determining the areas of figures with irregular or curved boundaries.

21. Use of the Slide Rule.—Special books of instruction for each of the several varieties of slide rules are issued by the manufacturers. Space will not permit a detailed discussion of the use of each of these rules, but some of the more frequent calculations which may be performed on all rules of the Mannheim type will be described.

In Fig. 21 is shown one style of 10-in. slide rule, on the face of which are four graduated scales. The two scales on the body of the rule are lettered *A* and *D* and those on the slide are lettered *B* and *C*. The rectangular glass runner may be moved to any position along the rule, its setting being indicated by a fine line etched on the glass at right angles to the axis of the rule.

The *C* and *D* scales are exactly alike and are graduated with numbers from 1 to 10 within the 10-in. length. The *A* and B scales are similar to the *C* and *D* scales but the corresponding graduations are only one-half as great. All four of the scales are logarithmic, and if the properties and use of logarithms are borne in mind, facility in calculating with the slide rule will be more quickly acquired. On the *C* and *D* scales logarithms are to the scale of 1 = 25 cm. and on the *A* and *B* scales logarithms are to the scale of 1 = 12½ cm. The figures shown are for numbers. The distance from a graduation corresponding to a given number to the 1 at the left of the *C* or *D* scale (left index) represents to the scale of 1 = 25 cm. the mantissa of the logarithm of that number. If the distance from the 2 graduation to the left index were measured with a 25-cm. scale it would be found to be 0.301 of the length of the scale, which is the value of the mantissa of the logarithm of 0.2, 2.0, 2,000, or any number having the same sequence of figures.

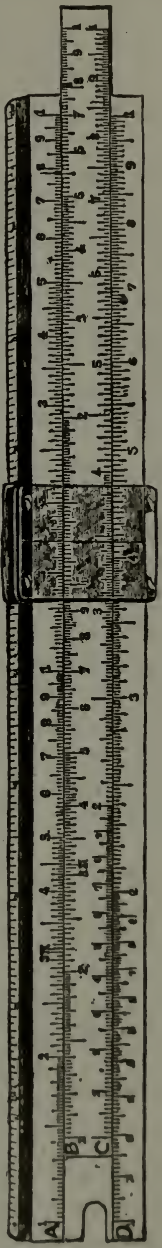


Fig. 21*a*.—Slide rule (Mannheim type).

Almost every rule has a scale divided into 100 equal parts. It is called the scale of equal parts. In the rule shown in the figure it is on the back of the slide and its graduations are numbered from the right end. From this scale the logarithm of a number may be found as follows: Set the runner on the number on the *D* scale. Move the slide until the left index is at the number, as indicated by the runner. Read the mantissa from the scale of equal parts. This amounts to the same thing as scaling the distance from the left-hand index to the number.

21a. In logarithmic computations two numbers are multiplied together by adding their logarithms. This operation may be mechanically performed by using the slide rule as illustrated by the following examples:

Example 1: Multiply 4 by 2.

Set the runner at 2 on the *D* scale, move the slide until the left index is at the runner. (Graphically the mantissa of 2 has now been laid off.) Move the runner to 4 on the *C* scale. (Graphically the mantissa of 4 has been added to that of 2.) On the *D* scale read 8.

Example 2: Multiply 8.2 by 7.3.

Set runner to 8.2 on *D* scale; right index to runner; runner to 7.3 on *C* scale; read answer 59.9 on *D* scale. The position of the decimal point is determined by mental calculation.

Had the initial setting been made with the left index at 8.2 the result would have been off the rule, at a *logarithmic* distance of 1 to the right of the final setting as determined in the example. For a logarithmic change of 1, the mantissa, and hence the sequence of numbers, is the same. Therefore settings may be made with either index, that one being chosen which will bring the final result within the length of the *D* scale.

Example 3: Find the product of 8.2, 7.3, 9.1, and 0.151.

Set runner at 8.2 on *D* scale; right index to runner; runner to 7.3 on *C* scale; right index to runner; runner to 9.1 on *C* scale; left index to runner; runner to 151 on *C* scale; read answer 82.3 on *D* scale.

22. Division of one number by another is accomplished by finding the difference between their logarithms. Recalling the manner in which the logarithmic scales are represented on the slide rule, the operation of division becomes self-evident.

Example 4: Divide 8 by 4.

Set the runner at 8 on the *D* scale. (The distance from the left index on the *D* scale to 8 on the *D* scale represents the mantissa of the logarithm of 8.) Set 4 on the *C* scale to the runner. (The distance from 4 to the left index of the *C* scale represents the mantissa of the logarithm of 4.) Set the runner to the left index on the *C* scale. Read the answer 2 on the *D* scale. (The difference between the mantissa of the logarithms of the two numbers is represented by the distance from 2 on the *D* scale to the left index on the *D* scale.)

In the further examples the following abbreviations will be used:
A, *B*, *C*, and *D* refer to corresponding scales.
R is the runner.
LI is left index.
MI is middle index.
RI is right index.

Example 5: $\frac{48 \times 63}{97 \times 15}$.

R to 48 *D*; 97 *C* to *R*; *R* to 63 *C*; 15 *C* to *R*; *R* to *LI C*; Answer, 2.08 *D*.

23. Scales *A* and *B* may be used for solving multiplications and divisions in exactly the same manner as are the *C* and *D* scales, but are more generally employed in conjunction with the *C* and *D* scales for finding the squares and square roots of numbers.

The *logarithmic* scale employed in the construction of the *A* and *B* scales (1 = 12½ cm.) is one-half of that of the *C* and *D* scales (1 = 25 cm.). Hence if the runner is set to a given number on the *D* scale, the square of the number is given by the runner reading on the *A* scale, for the effect has been graphically to multiply the mantissa by two.

Square root is performed by setting the runner to the number on the *A* (or *B*) scale and reading the root on the *D* (or *C*) scale. If the *integer* of the number contains an odd number of places (as 4.83; 125; 17,536) the runner is set on the left scale; if it contains an even number of places (as 16; 42.8; 1,174) the runner is set on the right scale. If the number is a decimal without ciphers between the decimal point and the first finite figure (as 0.428; 0.87) or with an even number of ciphers between the decimal point and the first finite figure (as 0.0087; 0.000064) the runner is set to the number on the right scale; if the number is a decimal with an odd number of ciphers between the decimal point and the first finite figure (as 0.0426; 0.0000065) the runner is set to the number on the left scale.

Example 6: Find the square root of 16.4.
Set *R* to 16.4 right *A*; read answer 4.05 *D*.

Example 7: $\frac{0.53(14.3)^2}{0.035\sqrt{1{,}178}}$.

Set *R* to 0.53 *D*; 0.035 *C* to *R*; *R* to 14.3 *C*; *LI* to *R*; *R* to 14.3 *C*; 1,178 right *B* to *R*; *R* to *RI*; read answer 90.4 *D*.

Example 8: $(12.2)^{5/2}$.

Set *R* to 12.2 *D*; *LI* to *R*; *R* to 12.2 *C*; *LI* to *R*; *R* to 12.2 *B*; read answer 164 *D*. Or set *R* to 12.2 *D*; *LI* to *R*; read mantissa on scale of equal parts as 0.087 (log of number is 1.087). Set *R* to 0.087 *D*; *RI* to *R*; *R* to 2.5 (= 5/2); read 0.217 *D*; set 0.217 on scale of equal parts to back index at right end; set *R* to *LI* on *C*; read answer 164 *D*.

24. On the back of the slide is a scale for sines (*S*) to the *logarithmic* scale of 1 = 12½ cm., and one for tangents (*T*) for which the logarithmic scale is 1 = 25 cm. If the slide is removed, turned over, and replaced so that the indices of the *S* and *T* scales coincide with those of the *A* and *D* scales, the values of the natural sines of angles between 0°34′ and 90° and of natural tangents between 0°34′ and 45° are read directly from the *A* and *D* scales respectively. Thus by readings on the *A* scale the sine of 0°34′ is seen to be 0.0100, the sine of 5°44′ is seen to be 0.100, the sine of 30° is seen to be 0.500, etc., and by reading on the *D* scale the tangent of 5°43′ is seen to be 0.100, the tangent of 30° is seen to be 0.577, etc. The tangent of any angle less than 5°43′ may be considered to equal the sine, within the precision of the rule. Sines of angles less than 0°34′ may be assumed to be proportional to the angle. On this assumption since the sine of 0°34′ is 0.0100, the sine of 0°05′ would be $^{05}/_{34} \times 0.0100 = 0.00147$. The true value is 0.00145.

The values of other trigonometric functions may be obtained by the simple relationships of trigonometry, as will be shown in the following examples.

The trigonometric scales are used principally with the slide in its normal position (*i.e.*, with the *S* and *T* scales underneath so that they are read by the back index at the right end of the rule). In the following examples the slide is assumed to be in this position. For the sake of brevity the letters *S* and *T* will refer to the sine and tangent scales respectively, and the reading of either of these scales will refer to the reading shown by the index on the back and at the right end of the rule.

Example 9: sin 20°45′; cos 69°15′.
Set *S* to 20°45′; set *R* to *RI* on *A*; read answer 0.354 *B*.

Example 10: sec 69°15′; cosec 20°45′

$$\sec 69°15' = \frac{1}{\cos 69°15'} = \frac{1}{\sin 20°45'}$$

Set *S* to 20°45′; set *R* to *LI* on *B*; read answer 2.82 *A*.

Example 11: tan 20°45′; cot 69°15′.
Set 20°45′ on *T*; set *R* to *RI* on *D*; read answer 0.379 *C*.

Example 12: tan 69°15′; cot 20°45′.

$$\tan 69°15' = \cot 20°45' = \frac{1}{\tan 20°45'}$$

Set 20°45′ on *T*; set *R* to *LI* on *C*; read answer 2.64 on *D*.

25. Other scales shown on many rules are an inverted *C* scale (sometimes designated as the *CI* scale) and one similar to the *A* scale but constructed to the logarithmic scale of $1 = 8\frac{1}{3}$ cm., its three parts occupying a total length of 25 cm., just as do the two parts of the *A* scale (sometimes designated as the *K* or cube scale). The use of the inverted *C* scale makes possible the solution of many of the problems of proportion with a fewer number of settings of the rule. It is employed in conjunction with the *D* scale. The distance from the left index to any number represents to *logarithmic scale* the mantissa of the reciprocal of the number. Hence, multiplications using the *CI* and *D* scales are made in the same manner as are divisions with the *C* and *D* scales.

The *K* or cube scale is used in calculating cubes and cube roots. The runner set at a given number on the *D* scale indicates the cube of the number on the *K* scale, or when set to a number on one of the

K scales indicates the cube root on the D scale. If the integer of the number of which the cube root is to be determined contains 1, 4, 7, or 10 places the runner is set on the left K scale; if it contains 2, 5, 8, or 11 places the runner is set on the middle K scale; if it contains 3, 6, 9, or 12 places the runner is set on the right K scale.

CHAPTER IV

DRAFTING

26. Remarks.—The drawings of *surveying* consist of maps, profiles, cross-sections, and, to some extent, graphical calculations, the usefulness of which is largely dependent upon the accuracy with which points and lines are projected upon paper. For the most part, few dimensions are shown, and the person who makes use of the drawings has to rely upon distances as measured with a scale or angles as measured with a protractor. Moreover, the drawings of surveying are so irregular and the data upon which the drawings are based are in such form that the use of the T-square and triangles (as in mechanical drawing) for the construction of parallel and right lines, is the exception rather than the rule.

Although a knowledge of mechanical drawing is helpful, much of the drafting with which we are concerned calls for a degree of skill and precision of execution quite unnecessary on dimensional plans.

27. Maps.—Maps may be divided into two classes: those that become a part of public records of land division, and those that form the basis of a study. The best examples of the former are the plats filed as parts of deeds in the county registry of deeds (in most states); and good examples of the latter are the preliminary maps along the proposed route of a railroad on which the location is to be projected. It is evident that the dividing line beween these two classes is very indistinct, since many maps might serve both purposes. The requirements of a map will depend upon its purpose.

In general the information that should appear on a map that is to become a part of a public record includes:

1. The lengths of all lines shown.
2. The bearings of all lines shown, or the angles between intersecting lines.
3. The location of the tract either with reference to principal meridian and base-line, if the tract is within the boundaries of the United States public land surveys; or with reference to coordinate axes, if within a city having a coordinate system.
4. The number of each block and lot, if the plat is of a city, town, subdivision, or addition.

5. The position and kind of each monument set, with distances to reference marks.

6. The positions and names of all roads, streams, landmarks, etc.

7. The names of adjacent property owners, and the bearings of all property lines intersecting the lines of the tract mapped.

8. The direction of the meridian (true or magnetic or both).

9. A legend or key to symbols shown on the map.

10. A graphic scale used with a corresponding note stating the scale. (The latter is usually shown in the title.)

11. A full and continuous description of the boundaries of the tract by bearing and length of sides; and the area of the tract.

12. The witnessed signatures of those possessing title to the tract mapped; and if the tract is an addition to town or city, a dedication of all streets and alleys to the use of the public.

13. A certification by the surveyor that the plat is correct to the best of his knowledge.

14. A neat and explicit title showing the name of the tract, or its owner's name, its location, the scale of the drawing, the surveyor's name, and the date.

Of maps made the basis of studies, there are so many varieties and the requirements are so varied that a definite statement of all that each should include would be impossible. In general, maps of this kind show very few dimensions (often not any), the value of the map depending upon the correct representation of the position of features of the land rather than directly upon field measurements or computed values. Maps of this class may be divided into two kinds:

1. Those that graphically represent in plan such natural and artificial features as streams, lakes, boundaries, condition and culture of land, and public and private works; and

2. Those that include some or all of the preceding features, but also represent the relief or contour of the ground.

Other things which should *always* appear on these two kinds of maps are:

1. The direction of the meridian.

2. A legend or key to symbols used, if they are other than the common conventional signs (Figs. 258*a* to *d*).

3. A graphic scale of the map with a corresponding note stating the scale.

4. A neat and appropriate title generally stating the kind or purpose of the map, the name of the tract mapped or the name of the project for which the map is to be used, the location of the tract,

the scale of the drawing, the contour interval, the name of the engineer or draftsman or both, and the date.

28. Maps of large areas, as of a state or country, which show the location of cities and towns, streams and lakes, and the boundary lines of the principal civil divisions are called *geographic maps.* Maps of this character which show also the general location of some kind of the works of man are designated by the name of the works represented. Thus we may have a *railroad map of the United States,* showing the names of railroads and the principal cities through which they pass; an *irrigation map of the state of California,* showing the location of tracts which have been placed under irrigation; or a *highway map of the state of New York,* indicating the location of highways under state control.

Cadastral maps are large-scale maps which show the natural features of the land but also show the position of land lines and reference monuments, the names of streets, the ownership of property, and in more or less detail the nature and extent of public and private improvements. Such maps usually represent relatively small areas, as a city, or portion thereof.

Topographic maps are maps which indicate the relief of the ground in such manner that elevations may be determined by inspection. The relief is usually shown by irregular lines, called contour lines, drawn through points of equal elevation.

Maps of this kind not made for a specific purpose are called *general topographic maps.* In addition to the topographic and geographic features, such maps usually represent public and, to some extent, private works. Usually the general topographic map is drawn to a small scale. The quadrangle maps of the U. S. Geological Survey are good examples.

Hydrographic maps show the shore lines, the location and depth of soundings or lines of equal depth, and often the topographic and other features of lands adjacent to the shores.

General hydrographic maps are those the use of which is intended to be general in character, for example, the maps of the U. S. Coast and Geodetic Survey.

Many maps are constructed for the specific purpose of aiding in determining the feasibility of engineering projects or of facilitating the location and construction of the works of man. The features included on such maps are entirely dependent upon the purpose for which the maps are made and may or may not represent the topography or hydrography of the area mapped. The maps are usually designated according to the purpose for which they are made. For example, the map made the basis for preliminary studies to determine the location of a railroad is termed the "preliminary map," the one showing the alignment of the located line is called the "location map," the one showing the boundaries of rights of way and intersecting land lines is designated as the "right-of-way map," etc.

29. Map Projection.—A map shows graphically the location of certain features on or close to the surface of the earth. Since the surface of the earth is curved and the surface of the map is a plane, no map can be made to represent a given territory without some distortion. If the area is small, the earth's surface may be regarded as plane, and the map constructed by orthographic projection, as in mechanical drawing, will represent the relative location of objects without measurable distortion. The maps of plane surveying are constructed in this manner, points being plotted either by rectangular coordinates or by horizontal angles and distances.

As the size of the territory increases, the above method becomes inadequate and various forms of projection are employed to minimize the effect of map distortion. The points of control are plotted by spherical coordinates through the use of elaborate geographic tables. Since the spherical coordinates of a point are its latitude and longitude, it is customary to show meridians and parallels on the finished map. The maps of states and countries, as well as those of much smaller areas, are constructed in this manner.

Frequently surveys employing the principles of plane surveying are primarily controlled by points established by a previous geodetic survey. If the area is sufficiently extensive to make it desirable, the primary points of control may be plotted by one or another of the special forms of projection, and the details of the map, which are referred to the primary points of control, may be assumed to lie in a plane surface and may be plotted by the methods of ordinary orthographic projection. The topographic maps of the U. S. Geological Survey are constructed in this manner.

30. Scales.—The common map scales are of three types:

1. One unit of length on the drawing, the *map distance*, represents a given number of units of length on the ground, the *ground distance*, the ratio of the former to the latter being called the *representative fraction*. For example, some of the maps of the U. S. Geological Survey have a scale of $\frac{1}{62,500}$ which indicates that 1 ft. (or 1 in.) on the drawing is the equivalent of 62,500 ft. (or inches) on the ground.

2. A given number of inches on the drawing represents 1 mile on the ground. Thus the scale 1 in. = 1 mile means that 1 in. on the drawing is the equivalent of 1 mile on the ground. The correspond-

ing representative fraction is $\frac{1}{63{,}360}$. Geographical, military, and land maps frequently exhibit the inch-mile relation.

3. One inch on the drawing represents some whole number of tens, hundreds, or thousands of feet on the ground. Thus the scale 1 in. = 200 ft. indicates that 1 in. on the paper is the equivalent of 200 ft. on the ground. Most maps directly employed by engineers in the design and construction of the works of man are drawn to scales of this type. It is called the *engineers' scale.*

The magnitude of the scale to which a map should be drawn, or in other words, the relation between the size of the map and the size of the area represented, depends upon the purpose of the map, and to some extent upon the character and extent of the tract shown. In general, it may be taken as a rule that the scale of a map should be no larger than is necessary to represent the location of details with the requisite precision. Maps constructed for engineering projects have scales generally ranging from 1 in. = 20 ft. to 1 in. = 800 ft. Maps of land subdivisions have scales ranging from 6 in. = 1 mile to 1 in. = 1 mile. Cadastral maps have scales ranging from 6 in. = 1 mile to 1 in. = 100 ft. General topographic maps have scales ranging from $\frac{1}{10{,}560}$ to $\frac{1}{250{,}000}$. Geographic maps have scales of 1 in. = 1 mile to 1 in. = 20 miles or smaller.

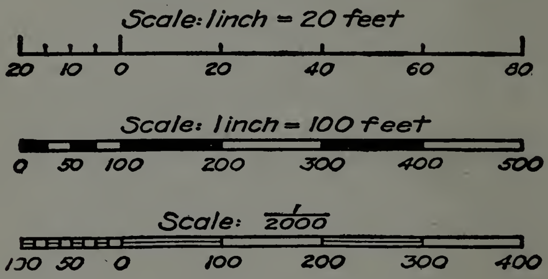


Fig. 30.—Scales.

For convenience in later discussions maps will be arbitrarily divided into those of

1. Large scale. Greater than 1 in. = 100 ft.
2. Intermediate scale. 1 in. = 100 ft. to 1 in. = 1,000 ft.
3. Small scale. Less than 1 in. = 1,000 ft.

The scale should be shown near the title of the map so that it will catch the eye readily. It should be expressed both numerically, as for example, 1 in. = 400 ft. and graphically in the form of a line subdivided into convenient units of length as illustrated in Fig. 30. Drawing papers shrink and swell more or less with changes in climatic conditions. The graphical scale makes it possible to determine map distances accurately either from drawings which have been thus affected or from photographs of the drawings.

31. Meridian Arrows.—The direction of the meridian is indicated by a needle or feathered arrow pointing north, of sufficient length to be transferred with reasonable accuracy to any part of the map. A meridian needle 4 or 5 in. long will be sufficient for large maps; 2 or 3 in. long for small maps. The true meridian is usually represented by an arrow with *full* head; the magnetic meridian by an arrow with *half* head. When both are shown the angle between them should be indicated. A simple design is shown in Fig. 31.

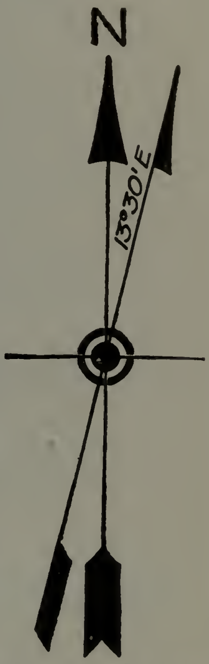


Fig. 31.—Meridian arrows.

32. Profiles.—Longitudinal sections made by projecting the ground line upon a vertical surface are known as profiles. In conjunction with maps, they are of assistance to the engineer in fixing the grades and alignment of such works as sewers, railroads, highways, and canals. They are also of value in estimating volumes of earthwork. The data from which profiles are plotted consist of ground elevations at known distances apart along some line, for example, the center line of a highway. The ground profile is formed by a continuous line drawn through the plotted points. In addition to the ground profile there is also shown one or more grade profiles and other pertinent information. For example, the profile along the center line of a canal would show the ground line and the canal bed and probably also the water surface and top of bank.

Profiles are usually made on specially ruled paper called *profile paper*, the rulings being shown either as orange or as green lines. The lines of the profile paper greatly facilitate plotting and make it possible to read elevations directly from the profile. The scales are so chosen that the spaces on the paper represent convenient distances. For example, with Plate *A* paper a horizontal scale of 1 in. = 400 ft. makes one horizontal space on the paper the equivalent of 100 ft. on the ground, and a vertical scale of 1 in. = 20 ft. makes one vertical space the equivalent of 1 ft. difference in elevation.

33. Cross-sections.—The calculations of volumes of earthwork are frequently facilitated by plotting cross-sections of the earthwork to scale. The area of a cross-section is then determined either by means of a planimeter or by dividing the figure into triangles and rectangles and arithmetically calculating the partial areas. The data for plotting consist of elevations and distances either calculated or measured in the field.

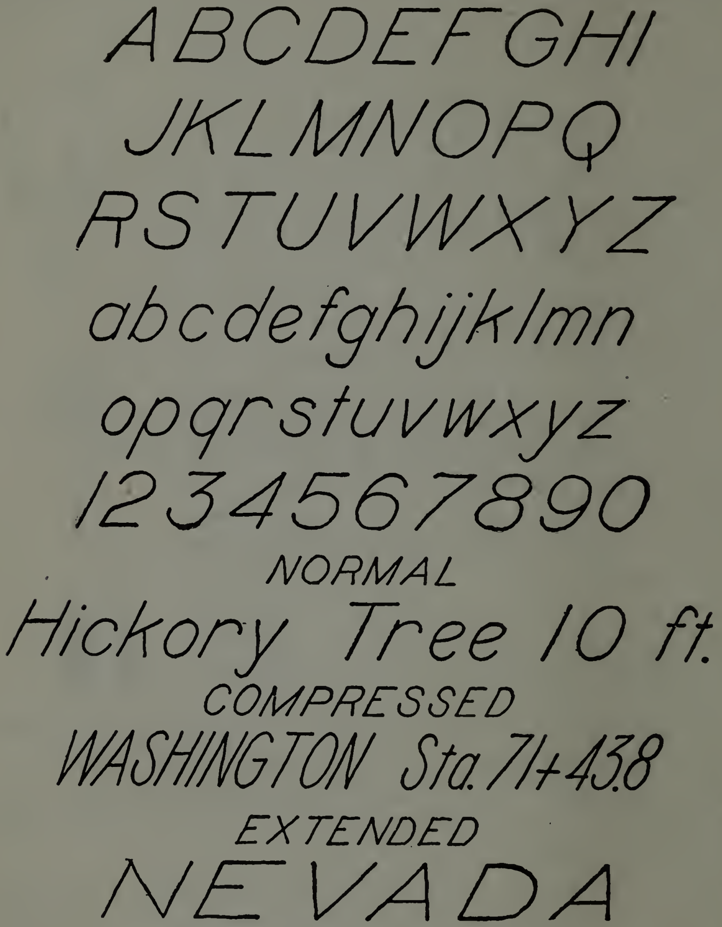

ABCDEFGHI
JKLMNOPQ
RSTUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890
NORMAL
Hickory Tree 10 ft.
COMPRESSED
WASHINGTON Sta.71+43.8
EXTENDED
NEVADA

Fig. 34*a*.—Reinhardt letters, slope form.

Cross-sections may be shown on ordinary paper, are sometimes drawn on profile paper, but more often are plotted on cross-ruled paper called *cross-section paper*.

34. Lettering.—Inasmuch as a drawing is likely to be judged by the quality of its lettering, it is important that the draftsman be able to form letters with at least a fair degree of skill and to assemble them in such form, size, and position as to make the drawing clear and of pleasing appearance. In machine and structural drawing simplicity and clearness are of primary importance; a considerable portion of the drawings made by the map draftsman requires in addition, a certain artistic quality. This is particularly true of maps which are to be largely used by the public, and on such work the draftsman is often justified in expending considerable time in adding the quality of beauty to that of utility.

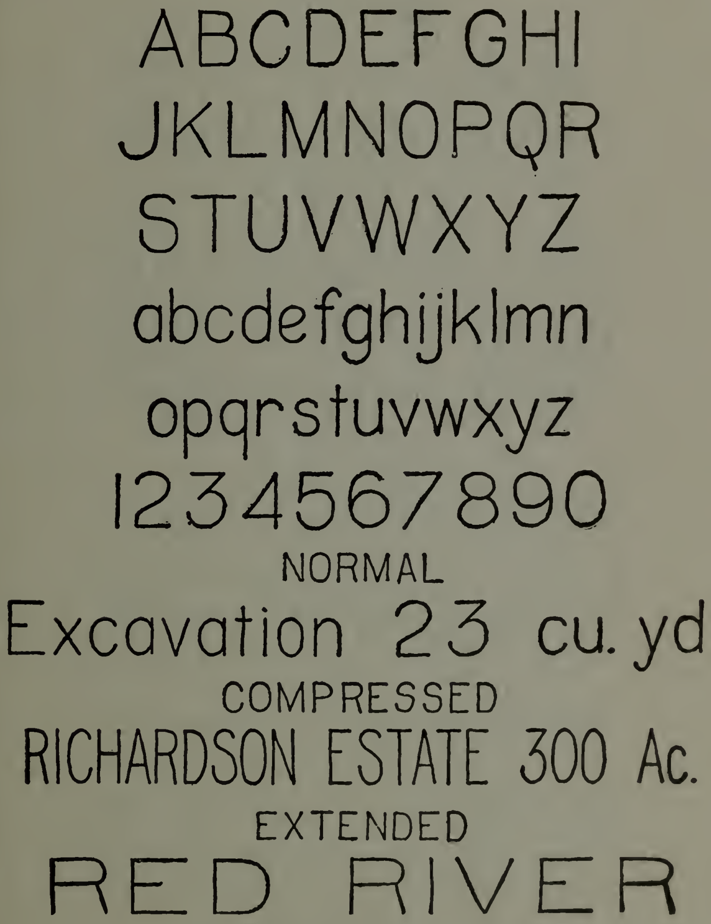

ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890
NORMAL
Excavation 23 cu. yd
COMPRESSED
RICHARDSON ESTATE 300 Ac.
EXTENDED
RED RIVER

Fig. 34*b*.—Reinhardt letters, vertical form.

In general, lettering should be free-hand, and it should be of a style in keeping with the purpose of the drawing.

For *office drawings*, or drawings which are not to be used by the general public, the *Reinhardt* style of single stroke lettering is almost universally employed in this country. Reinhardt letters are made in two forms, inclined and vertical, as illustrated in Figs. 34*a* and 34*b*.

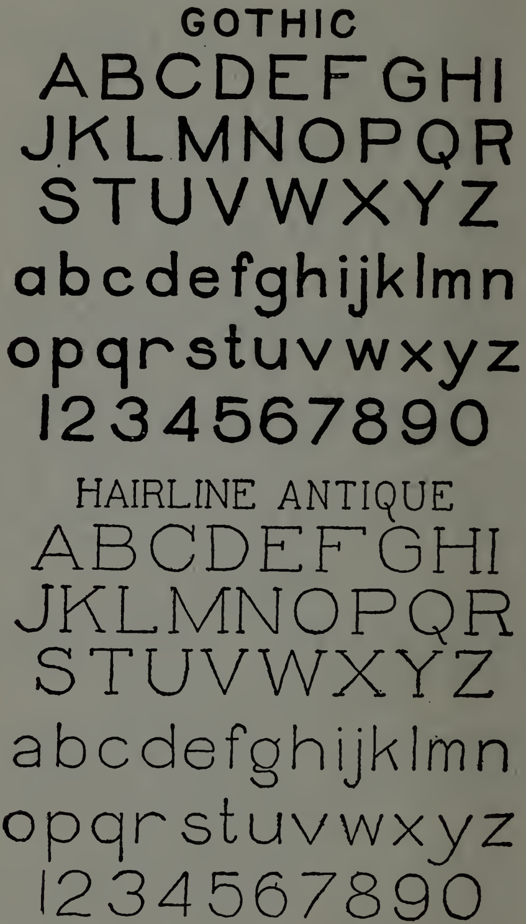

GOTHIC
ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890
HAIRLINE ANTIQUE
ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890

Fig. 34*c*.—Gothic lettering.

On drawings where their use is justified, *Italic*, *Roman*, and *Gothic* letters may be employed if the draftsman is sufficiently skilled in the execution of these styles.

The *Gothic* alphabet in vertical form is shown in Fig. 34*c*. The Gothic style may be employed when it is desired that the lettering stand out from the body of the drawing to catch the eye quickly.

Hair-line Antique lettering (Fig. 34*c*), a modification of the Gothic style, may be used when the letters are to be subdued so as not to interfere with the general clearness of the drawing. Generally only the capitals are employed.

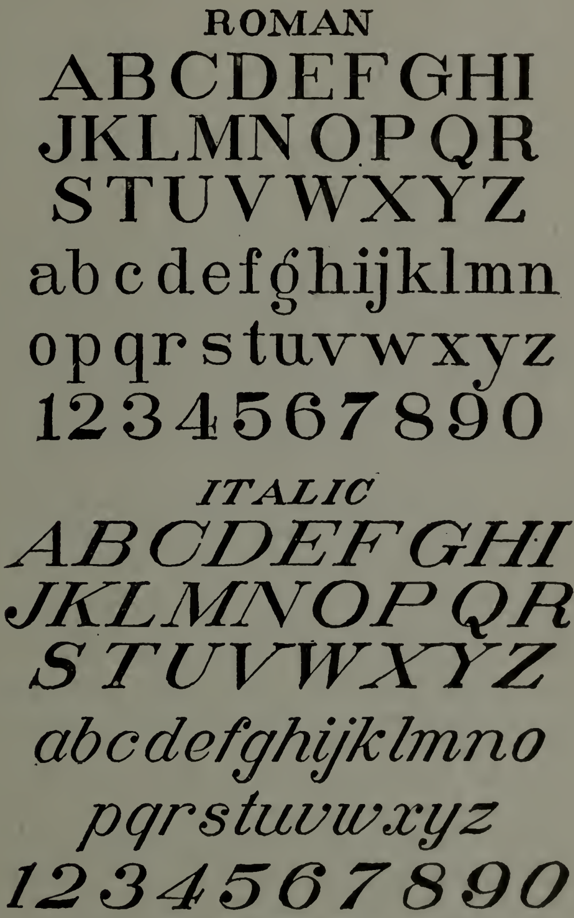

ROMAN
ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890
ITALIC
ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmno
pqrstuvwxyz
1234567890

Fig. 34*d*.—Italic and Roman lettering.

Italic and *Roman* letters (Fig. 34*d*) are shaded, and for this reason are difficult to construct. Relatively few draftsmen possess the skill to make Roman letters that look well on a drawing. *Italics* are very little different from Roman letters except that they are inclined, but slight deviations in their form are not as noticeable.

In map drafting, where the details to be shown are many and are varied in character, it often renders the map clearer if a particular style of lettering is employed for each class of objects shown. Not infrequently the several styles just described might be employed on a single map. For example, the topographic maps of the U. S. Geological Survey show the names of civil divisions in Roman letters, the names of streams and lakes and other hydrographic features in Italics, the names of mountains, valleys, and other land forms in vertical Gothics, public works in inclined Gothics, and marginal lettering in hair-line antique.

Detailed information concerning lettering is to be found in textbooks on drawing, but it is perhaps appropriate here to offer a few suggestions to the beginner.

1. Always use guide lines.
2. Be sufficiently familiar with the construction of each letter so that its form will always appear the same.
3. It is very important that the slope of letters in a word, sentence, or paragraph be uniform.
4. Never seek to improve lettering by making straight portions of free-hand letters by mechanical means.
5. Do all lettering in pencil as neatly as possible before inking in.

35. Titles.—Titles should be so constructed that they will readily catch the eye, and the sizes of different parts should be in proportion to their importance. The best position for the title is the lower right-hand corner of the sheet. In general, each line should be centered and the distance between lines should be such that the title as a whole will appear well balanced. The space occupied by the title should be in proportion to the size of the map. The different parts of the title should be in straight lines, and only the common styles of capital letters should be used. Few well-designed titles contain lower-case letters. It is best to construct the title of all Gothic or all Roman letters, but a change of style of lettering between different parts is permissible when such change will serve to accentuate the important parts of the title. Thus the principal part may be constructed in Roman letters, while the remainder is in Gothics. The use of Gothic capitals is most common. Slope and vertical letters should not be included in the same title. In general, titles should be constructed free-hand, as well as the notes on the body of the map. The use of Reinhardt letters for the important parts of titles, even those of office plans, is not prevalent, though they may be appropriately employed for the less important parts where small letters are used.

Generally the title should contain:

1. The name of the general class to which the drawing belongs.

2. The name of the area mapped, the main object of the drawing, or the purpose for which the survey was made.

3. Often the name of the individual or organization for whom the drawing is made or the name of the project with which the drawing is concerned.

4. The scale of the drawing.

5. The location of the office from which the drawing is issued.

6. The date of completion.

7. The name of the engineer or surveyor in charge, and frequently the name of the draftsman.

36. Notes and Legends.—Explanatory notes or legends are usually of assistance in interpreting a drawing. They should be as brief as circumstances will allow, but at the same time should include sufficient information as to leave no doubt in the mind of the person using the drawing. A key to the symbols representing various details ought to be shown even though the symbols be conventional in character. The nature and source of data upon which the drawing is based ought to be made known.

Notes ought to be in such a position on the drawing as to catch the eye readily, conditions allowing, a favorable position being the lower right-hand corner just to the left of the title.

37. Pencils, Inks, and Colors.—For drawings on hard, fine-grained papers the 6H pencil is quite widely used. For very thin lines and for permanent work the 8H is occasionally employed. Many drawings on the smooth papers are made with the 4H pencil. On the soft profile and cross-section papers, lines made with a 2H pencil show up well. For coarse tracings made on tracing paper a pencil as soft as 2B is sometimes necessary. Drawings made directly on the rough side of tracing cloth will show up sufficiently well for inking if a 2H pencil is used.

38. The bottled inks commonly used by the map draftsman are black, brown, orange, blue, green, and vermilion. For line drawings, the waterproof inks are satisfactory. Lines made with them will not smudge when rubbed with the moist hand and are not affected by the application of liquid tints or washes.

Water colors for tinting are obtainable in a variety of colors either in tubes or pans. The colors are mixed with water until the desired tint is produced and are then applied with a camel's-hair brush. Tracings may be tinted by rubbing the rough side with colored crayons, but cannot be tinted with a wash.

39. Drawing Instruments.—Besides the equipment commonly used in mechanical drawing there are several instruments and devices which are generally useful in the work of the surveyor.

Engineers' scales are divided into 10, 20, 30, 40, 50, 60, 80, and 100 parts to the inch. A scale divided into 10 parts to the inch is called a 10-ft. scale; one divided into 20 parts to the inch a 20-ft. scale; etc. Rules thus divided are flat or triangular in shape and are obtainable in lengths of 3, 6, 12, 18, and 24 in. The 12-in. triangular boxwood rule with 10-, 20-, 30-, 40-, 50-, and 60-ft. scales on its three faces is most commonly used. It has the advantage of compactness.

40. The Protractor.—This is a device for laying off and measuring angles on drawings. In its usual form it consists of a full circle or semicircular arc of metal, celluloid, or paper graduated in degrees or fractions of a degree. Protractors are obtainable in sizes from 3 in. to 14 in. in diameter and in a variety of designs. The smaller protractors are usually graduated to degrees or ½ degrees; the larger sizes are frequently graduated to ¼ degrees. Some are equipped with verniers reading to five minutes or to single minutes but this refinement adds little to either the precision with which angles can be laid off or to the facility with which the protractor can be used. Others have radial scales by means of which a distance and angle may be plotted in one operation.

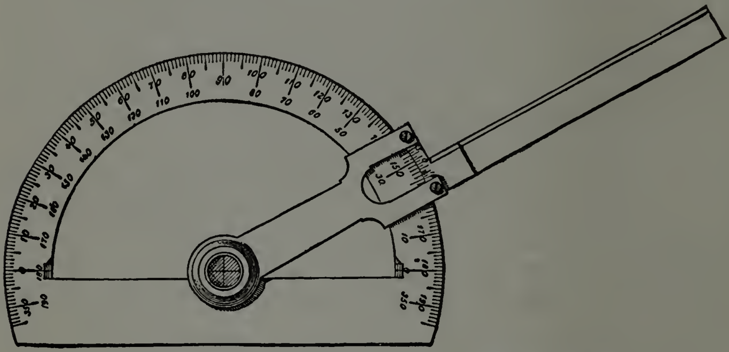


FIG. 40*a*.—Semicircular protractor with radial arm.

Figure 40*a* illustrates a form of semicircular protractor with radial arm which may or may not be graduated to form a scale for linear measurements. The protractor has a horn center, and the radial arm is held by friction.

Full-circle paper protractors are usually printed in 8- and 14-in. sizes on rectangular sheets of tough paper or bristol board, without the graduations being numbered. The 14-in. size is graduated to ¼ degrees; and the 8-in. size to ½ degrees. To prepare such a protractor for use, its graduations are numbered as desired and it is cut either on a circle passing through the outer ends of the graduations or on a circle of somewhat smaller radius. In the former case the outer portion is discarded and angles are laid off by using

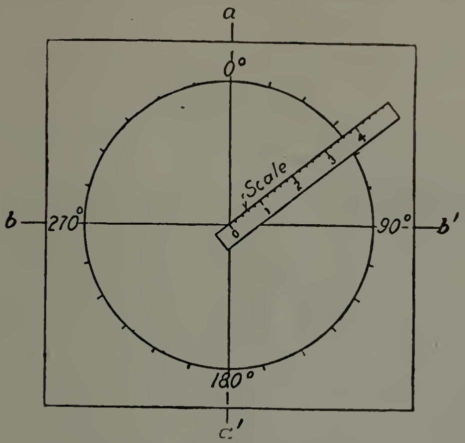


Fig. 40*b*.—Laying off an angle with paper protractor.

the inner portion in the manner described for the semicircular metal protractor. In the latter case the inner portion is discarded, and angles are laid off by means of a straight-edge passing through the center of the protractor. The protractor is centered and oriented by means of two intersecting lines on the drawing, one line being at right angles to the other. Figure 40*b* illustrates the method of laying off an angle with the latter form. The angle is to be laid off from the line *aa′* about *O* as a center. The line *bb′* is drawn at right angles to *aa′* through *O* and the 0°, 90°, 180°, and 270° graduations are made to coincide with these lines as shown. The protractor is then held in position by thumb tacks or weights, the straight-edge is passed through the point *O* at the desired angle, and the line forming the second leg of the angle is drawn. An angle and distance may be plotted in a single operation by employing a scale for a straight-edge, the zero of the scale being held at the point *O* and the distance being marked when the scale is at the given angle.

41. The Contour Pen (Fig. 41).—This pen is useful for drawing contours or other free-hand curves. The pen is rigidly connected to a shaft which freely turns in the handle. The point of the pen is eccentric with the axis of the shaft so that the pen will turn in whatever direction it is drawn on the paper. In use, the handle is held vertically, the fingernails of the third and fourth fingers of the right hand being in contact with the paper.

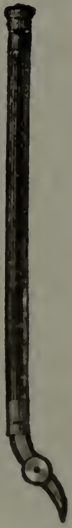


Fig. 41. Contour pen.

CHAPTER V

ERRORS

42. General Remarks.—In an earlier article reference was made to the necessity of the surveyor's appreciating the errors involved in measurements. Every observed or measured quantity contains errors of unknown magnitude due to a variety of causes, and hence a measurement is never exact. One of the important functions of the surveyor is to secure measurements which are correct within certain limits of error prescribed by the nature and purpose of the survey. This requires that he know the sources of errors, understand the effect of the various errors upon the observed quantities, and be familiar with the procedure necessary to maintain a required precision. Numerous instances could be cited where surveyors of considerable experience have displayed an ignorance of this phase of their work which was both ludicrous and lamentable.

43. Kinds of Errors.—The *resultant error* in a given quantity is the difference between the measurement and its true value. If the measurement is too large, the error is said to be positive; if too small, the error is said to be negative. The *resultant error* in a measurement is made up of individual errors from a variety of sources, some of the individual errors tending perhaps to make the measurement too large, and others to make it too small. For a single quantity which has been determined by observation, neither the resultant error nor any of its individual parts can ever be determined exactly, but can be fixed within certain probable limits.

A *systematic error* is an error that, so long as conditions remain unchanged, always has the same magnitude and the same algebraic sign. A change in conditions is always accompanied by a corresponding change in the magnitude of the error. A systematic error always follows some definite mathematical or physical law.

An *accidental error* is an error that is as likely to be positive as negative. It is due to causes beyond the control of the observer, and does not obey any mathematical or physical law.

A *constant error* is a systematic error which is always of the same amount and sign.

A *mistake* is an unintentional error of conduct arising from faulty judgment or confusion in the mind of the observer. It is quite distinct from the mathematical or physical meaning of error. Throughout this text the above distinction will be observed, and it need not be added that mistakes have no place in a discussion of the theory of errors. They are detected and eliminated by checking all work.

44. Sources of Error.—Errors may be said to arise from three sources:

1. From imperfections in the instruments or devices with which measurements are taken. For example, a tape may be too long, or a level may be out of adjustment. Such errors are termed *instrumental errors.*

2. From the limitation of the human senses of sight and touch. For example, in reading the angle on the graduated circle of a transit or in estimating the tension in a steel tape. Such errors are called *personal errors.*

3. From variations in the phenomena of nature, such as temperature, humidity, wind, gravity, refraction, and magnetic declination. For example, the length of tape will become greater or smaller according as the temperature increases or decreases. And readings of the magnetic needle are affected by variations in the magnetic declination. Such errors are called *natural errors.*

45. Systematic vs. Accidental Errors.—Since a systematic error, so long as conditions remain constant, always has the same magnitude and sign, it follows that the total systematic error in any given number of measurements is the algebraic sum of the individual errors of each of the individual measurements. Thus if a distance is measured with a tape which is too short, the systematic error due to the tape not being of the standard length would be directly proportional to the length of the line.

Example 1: The length of a line as measured with a 100-ft. tape at 60°F. is 1,000.00 ft. Later the tape is compared with the standard length and is found to be 100.021 ft. long. The error in the recorded length of the line is $-0.021 \times 10 = -0.21$ ft., and the true length of the line is 1,000.21 ft.

The above example illustrates the manner in which a *constant* systematic error increases with the number of observations. The example below illustrates the effect of a variable systematic error.

Example 2: A line measured with a 300-ft. tape is found to be 1,200.00 ft. long. Calculations based on observations of temperature of the tape indicate that its probable length was 300.018 ft. for the first tape length, 300.024 ft. for the second tape length, 300.020 ft. for the third tape length, and 300.016 ft. for the fourth tape length. The total systematic error due to variation in temperature would therefore be the sum of the above errors, or −0.018 −0.024 −0.020 −0.016 = −0.078 ft., and the length of the line would be 1,200.00 + 0.08 = 1,200.08 ft.

Quite often a systematic error from one source may be of opposite sign to that from another source so that the resultant systematic error is perhaps smaller than any of the errors from individual sources. Thus under a given tension and at a given temperature a tape might be of the standard length. Suppose that for the conditions under which the measurement of a line were made a variation from standard in tension in the tape produced an error of −0.022 ft. per tape length and a variation from standard in temperature produced an error of +0.018 ft., the resultant error due to variations in temperature and tension would be −0.004 ft.

Systematic errors may be either instrumental, personal, or natural in character. Their relative importance as compared to accidental errors depends upon the nature of the observations, the care exercised by the observer, and the instruments and methods of procedure employed. Generally speaking, the rougher the methods used, the larger the systematic errors as compared with the accidental errors; the more refined the methods of observing, the smaller the systematic errors as compared with the accidental errors. For many observations the order of procedure is such that systematic errors are eliminated, or at least reduced to a negligible quantity. Thus in chaining, the error due to temperature change in a steel tape may be nearly eliminated by observing the temperature and making correction; and errors in leveling, due to faulty adjustment of the level may be eliminated by balancing backsight and foresight distances.

Accidental errors, as the name signifies, are purely accidental in character, and there is no way of determining or eliminating them in the sense that we may determine or eliminate most systematic errors. Thus, while the effect of change of temperature on the length of a tape can be approximately eliminated by calculations based upon physical measurements, there is no corresponding method of eliminating the accidental error due to marking the ends of the tape on the ground or to reading the rod in leveling. While accidental errors are as likely to be positive as negative, the error of one observation of a quantity is not likely to be the same for the second observation.

According to the mathematical theory of probability, accidental errors tend to increase in proportion to the square root of the number of opportunities for error. Thus, if the accidental error in measuring one tape length were ±0.02 ft. the chances would be even that the total accidental error due to measuring 100 tape lengths would not exceed $\pm\sqrt{100} \times 0.02 = \pm 0.20$ ft. A systematic error of the same magnitude would produce a total error of 100 × 0.02 = 2.0 ft. It is thus seen that for any connected series of observations of independent but related quantities accidental errors, as compared with systematic errors of the same probable magnitude, are of relatively small importance. Though accidental errors cannot be eliminated, they may be reduced to a small quantity through the use of proper instruments and methods. By taking a series of like observations of a single quantity, an estimate of the accidental error may be made, as will later be shown; but its true magnitude can never be determined.

46. Discrepancies.—If a given quantity is measured twice, the difference between the two measurements is termed the discrepancy. Frequently quantities measured by the operations of surveying are verified or "checked" by a second measurement. If the discrepancy between two such measurements is small it is a good indication that no mistake has been made and also that the accidental errors are small. But attention is drawn to the fact that the magnitude of the discrepancy is in no way indicative of the magnitude of the systematic errors. For example, two measurements of a line a mile long made with a steel tape might show a discrepancy of 0.3 ft., but the systematic errors due to such causes as temperature, sag of tape, and slope of tape might perhaps be 3 ft.

47. Theory of Probability.—It has been stated that by employing proper methods, systematic errors may be largely eliminated. While this is true, it is also true that for certain kinds of surveys, particularly those of low precision, it is unnecessary and impracticable even approximately to eliminate such errors. For the surveys of higher precision special effort is made to eliminate, in general, systematic errors and the accuracy of a measured quantity is governed by the accidental error which it contains. To form a judgment of the probable value or the probable precision of a quantity, *from which systematic errors have been eliminated*, it is necessary to rely upon the law of probability, a law which deals with accidental errors of a series of like or related measurements. It is assumed that:

1. Small errors are more frequent than large ones.
2. Errors are as likely to be positive as negative.
3. Very large errors do not occur.
4. The mean of an infinite number of like observations of a quantity is its true value.

A thorough understanding of the law of probability may only be obtained by the study of a text on *Least Squares*, but a few of the rules for simpler cases of the adjustment of observations and determination of probable values and probable errors will here be given.

It should be clearly understood that the theory of probability applies to accidental errors only; it is useful in indicating the accuracy of results only in so far as they are affected by accidental errors and does not in any way determine the magnitude of systematic errors which may be present.

48. Probable Values.—In practice it is neither possible entirely to eliminate systematic errors nor to take an infinite number of observations, hence the value of a measured quantity is never exactly known. Assuming that systematic errors are so far eliminated as to be a negligible factor, the *most probable value* is a mathematical term used to designate that adjusted value of a quantity which, according to the principles of *Least Squares*, has more chances of being correct than has any other.

For a series of measurements of the same quantity made under identical conditions, the most probable value is the mean of the measurements.

Example 1: After all systematic errors have been eliminated the several measured lengths of a line are 1,012.36, 1,012.35, 1,012.38, 1,012.32, 1,012.33, and 1,012.30 ft. The most probable value is the mean of the measurements, or 1,012.34 ft.

For related measurements (all taken under like conditions) the sum of which should equal a mathematically exact quantity, the most probable values are the observed values corrected by the proportional part of the error between the sum of the observed values and the true quantity.

Example 2: The angles about a point have the following observed values: 130°15′20″, 142°37′30″, and 87°07′40″. The sum of the measurements is 360°00′30″. The total error is 30″. Since there are three angles, the proportional part of the error is 10″ for each measurement. The most probable values are:

$$\begin{array}{rcrcr}
130^\circ 15' 20'' & - & 10'' & = & 130^\circ 15' 10'' \\
142^\circ 37' 30'' & - & 10'' & = & 142^\circ 37' 20'' \\
\underline{87^\circ\ 07' 40''} & - & \underline{10''} & = & \underline{87^\circ 07' 30''} \\
360^\circ 00' 30'' & - & 30'' & = & 360^\circ 00' 00''
\end{array}$$

In the above example the discrepancy is divided equally among the several measured angles regardless of the relative sizes of the angles.

For related measurements (taken under identical conditions), the sum of which should equal a single measurement (also taken under like conditions), the most probable values of the several related quantities and also of the measured quantity representing their sum, are obtained by dividing the discrepancy equally between all the measurements. If the correction is added to each of the related measurements, it is subtracted from the measurement representing their sum; and *vice versa.*

Example 3: Measurements of three angles about a point O are AOB = 12°31′50″, BOC = 37°29′20″, and COD = 47°36′30″. The measurement of the single angle AOD is 97°37′00″. The discrepancy between the sum of the three measured angles and the measurement of the angle representing their sum is 40″. The proportional part of the discrepancy is $^{40}/_4 = 10''$. Since the size of the errors is independent of the size of the angle, this correction is to be subtracted from measurements of each of the angles AOB, BOC, and COD and is to be added to the measurement of the angle AOD. The most probable values are:

$$\begin{aligned}
AOB &= 12°31'50'' - 10'' = 12°31'40'' \\
BOC &= 37°29'20'' - 10'' = 37°29'10'' \\
COD &= 47°36'30'' - 10'' = \underline{47°36'20''} \\
&\qquad\qquad\qquad\qquad\quad 97°37'10'' \textit{ check} \\
AOD &= 97°37'00'' + 10'' = 97°37'10''
\end{aligned}$$

It is desired to emphasize the fact that the methods illustrated by the three preceding examples apply only to measurements each of which (in a given example) is made under the same conditions as are all the others, with the same instrument, observer, weather conditions, etc. In Example 2, the probabilities are that the error of measuring any one of the angles is the same as that of any other. It is true that the chances of the error in one measurement being the same as in the others is very small, but it is more probable that the errors will each be of the same magnitude than that they will be of any other assignable value.

In adjusting several measurements the sum of which should equal a single measurement it should be noted that there is a distinction between observations of the character of those cited in Example 3 and those for which several operations are involved in the measurement of a single quantity. For example, if distances between adjacent points along a line of considerable length were measured with a tape and then if the full length of the line were measured in the same manner, it is evident that the corrections should be made to depend upon some function of the number of applications of the tape, being greater for the full distance than for any of its parts, and being greater for long segments of the line than for short ones. This calls for weighting the several measurements. For a discussion of the relation between errors and weights see Art. 50.

Suppose that each of the values of the angles of Example 3 represented the mean of several measurements, but that for angle AOB

more measurements were taken than for the others. The probabilities are that the error in AOB would be less than the errors in the other given values. In adjusting observations of this and similar character one needs to determine what are termed the "probable errors" of each series of observations and then to correct each given value in the inverse ratio of its probable error. In the following pages the operation of determining probable error will be explained and a few of the simpler cases of its use in the adjustment of observations will be given.

49. Probable Error of a Single Measured Quantity.—If all systematic errors are assumed to have been eliminated from a given measurement, there is left an accidental error which can neither be eliminated nor exactly determined. If a series of like or related observations of a single quantity is made, a variety of values of the quantity is obtained. The variations between these values furnish data from which probable errors may be determined. The probable error of a measurement is a mathematical term giving an indication of precision and does not signify the true error nor the error most likely to occur.

Probable error is a plus or minus quantity within which limits the true error is as likely as not to fall. In other words if the probable error of a measurement is both added to and subtracted from the measurement, the chances are even that the true value of the observed quantity lies inside (or outside) the limits thus set. Thus if 6.23 represents the mean of several measurements and 0.11 represents the probable error of the mean value, the chances are even that the true value lies between the limits $6.23 - 0.11 = 6.12$ and $6.23 + 0.11 = 6.34$. In writing the probable error, its numerical value is usually preceded by the sign ± and is placed after the number to which it refers. For the example just cited the quantity would be written 6.23 ± 0.11.

As has already been stated, when a series of observations of a single quantity is made under uniform conditions and in such manner as to eliminate systematic errors, the mean of the series is the most probable value of the quantity sought. For the purpose of determining the probable error this mean value is mathematically regarded as being the true value, and the difference between each of the individual measurements and the mean value is determined. These differences are termed the *residuals*. The theory of *Least Squares* demonstrates that the probable error is a function of the square root of the sum of the squares of the residuals. The probable error is a valid measure of the precision of observations only after systematic errors have been reduced to a negligible quantity.

The probable error of a single observation is calculated by the expression

$$E_s = 0.6745\sqrt{\frac{\Sigma v^2}{n - 1}} \tag{1}$$

in which Σv^2 is the sum of the squares of the residuals, and n is the number of observations.

The probable error of the mean of a number of observations is determined by the equation

$$E_m = 0.6745\sqrt{\frac{\Sigma v^2}{n(n - 1)}} = \frac{E_s}{\sqrt{n}} \tag{2}$$

The probable error of a single observation may be determined approximately by the expression

$$E_s = \frac{0.8453\Sigma v}{\sqrt{n(n - 1)}} \text{ (approximate)} \tag{3}$$

in which Σv is the sum of the residuals without regard to signs.

The probable error of a single observation may be calculated with about the same degree of approximation by the formula

$$E_s = 0.8453\,\bar{v} \text{ (approximate)} \tag{4}$$

in which $\bar{v}$ is the mean value of the residuals without regard to signs.

Equations (3) and (4) are more convenient to apply than is Eq. (1). Whether or not they may be properly used will depend upon the number of observations, the distribution of the residuals, and the desired number of places in the probable error.

While no attempt to deduce the above expressions will here be made, it is well to remark that they are based upon the hypothesis that a large number of measurements of a single quantity has been taken. The results of experiment indicate, however, that they may be applied to a limited number of observations with good results. It seems doubtful if they can be consistently applied to a series of observations containing less than ten measurements.

The following example illustrates the methods of applying the preceding equations and also indicates the degree of approximation arising through the use of Eqs. (3) and (4).

Example: Following is a series of 10 rod readings which were taken with a 20-in. wye level. The day was calm and cloudy. The instrument was set up and the target rod was held on a point 600 ft. away. Before each reading the target was moved and the instrument was leveled.

Rod reading	v, feet	v^2
2.467	0.002	0.000004
2.460	0.005	0.000025
2.469	0.004	0.000016
2.465	0.000	0.000000
2.471	0.006	0.000036
2.461	0.004	0.000016
2.463	0.002	0.000004
2.466	0.001	0.000001
2.460	0.005	0.000025
2.468	0.003	0.000009
Mean value	Σv	Σv^2
2.465	0.032	0.000136

Following the same procedure as in the preceding example, the probable error of a single observation is

$$\text{(By Eq. (1))}\ E_s = \pm 0.6745\sqrt{\frac{0.000136}{9}} = \pm 0.00262\text{ ft.}$$

$$\text{(By Eq. (3))}\ E_s = \pm\frac{0.8453 \times 0.032}{\sqrt{10 \times 9}} = \pm 0.00284\text{ ft.}$$

$$\text{(By Eq. (4))}\ E_s = \pm 0.8453 \times 0.0032 = \pm 0.00271\text{ ft.}$$

Using the value of E_s determined by Eq. (1)

$$E_m = \pm\frac{0.00262}{\sqrt{10}} = \pm 0.00083\text{ ft.}$$

The preceding example is illuminating not only as showing the steps made in calculating probable errors but also as indicating in a measure the degree of approximation introduced by using the approximate expressions for the probable error of a single observation (Eqs. (3) and (4)).

In work involving many calculations of probable errors a decided saving in labor will be accomplished if one or another of the approximate formulas is used. The results of the preceding example have purposely been extended to more places than are consistent for the given data (see Art. 18), in order to make the above comparison. Except for the observations taken on surveys of very high precision, the approximate formulas are always sufficiently exact.

50. Weighted Observations.—It has been repeatedly stated in the preceding pages that all observations were assumed to have been taken under the same conditions and consequently, were equally reliable. Quite frequently in surveying, however, it is required to combine the results of measurements which were not made under similar conditions and which therefore, have different degrees of reliability. For example, suppose that the angles about a point have been measured under identical conditions with results as follows: $A = 121°46'00''$ (one measurement); $B = 179°14'27''$ (average of four measurements); and $C = 58°58'53''$ (average of nine measurements). The sum of these values is 359°59′20″, which shows a total error of 40″. Obviously, this error can not properly be equally divided among the three angles, and hence we must determine what proportion of the total error shall be assigned to each of the measurements.

If it is assumed that each single reading was made with equal care, then it is a logical assumption that angle B, the average result of four readings, has four times the reliability of angle A; and that angle C, the average of nine readings, has nine times the reliability of angle A. The numbers four and nine are called *weights* since they represent degrees of reliability. The weight of A is, of course, unity. These numbers are not absolute but merely relative. Thus, the numbers 2, 8, and 18 would represent the weights as well as the numbers 1, 4, and 9.

The mean or average value of measurements which have different weights is called the *weighted mean.*

For example, suppose that an angle has been measured at different times by different observers, with results as follows: 47°37′30″ (one measurement); 47°37′22″ (average of four measurements); and 47°37′40″ (average of nine measurements). To obtain the average value of this angle as determined by these three observers (assuming that all measurements were made with equal care) we must multiply each value by its weight, add the products, and divide by the sum of the weights, thus

$$
\begin{array}{rr}
47°37'30'' \times 1 = & 47°37'30'' \\
22'' \times 4 = & 88'' \\
40'' \times \underline{9} = & \underline{360''} \\
\overline{\underline{14|}} & \underline{478} \\
 & 47°37'34''
\end{array}
$$

The resultant value 47°37′34″ is the weighted mean.

Often weights will be assigned to observations, not according to the number of observations, but arbitrarily, according to the best judgment of the observer. Thus, by one line of levels the elevation of a point of reference might be determined as 537.64 ft. and by another line of levels run over the same route as 537.58 ft. But the levelman might assign a weight of 3 to the first elevation and a weight of 1 to the second, because the first line was run on a calm, temperate day and the second line was run on a very windy, cold day. In other words, he might judge the results secured on the first day as being equal in value to the average of 3 lines of levels run on the second day. Under these assumptions the weighted and accepted elevation would be $\frac{537.64 \times 3 + 537.58 \times 1}{4} = 537.62$ ft.

51. The Adjustment of Weighted Observations.—Since field measurements are frequently made under dissimilar conditions as to observers, instruments, and weather, it becomes necessary in the adjustment of related measurements to take into account the weights (as nearly as they may be determined) which apply to each of the separate measurments.

In the preceding article it was assumed that weights vary directly with the number of observations. It has been stated that probable errors vary inversely with the square root of the number of observations. It follows that, for observations made with equal care, weights are inversely proportional to the squares of the corresponding probable errors or if W_1 and W_2 are the weights to be assigned given measurements and E_1 and E_2 the corresponding probable errors, then $W_1 : W_2 :: \frac{1}{E_1^2} : \frac{1}{E_2^2}$. Accordingly, if the probable errors of given measured quantities have been computed the relative weights of these quantities can be determined, which weights may be used in any adjustment of values that may be required.

Applications of the principles stated above are shown by the following examples:

Example 1: Three angles about a point are each measured by a series of observations. The mean values with their probable errors are tabulated below. Their sum should equal 360°. It is desired to determine the most probable values of the angles.

a	121°46′37″	± 02″
b	179°14′35″	± 04″
c	58°58′34″	± 06″
Sum	359°59′46″	

The total error in the sum of the above angles is seen to be 14″. This is to be distributed among the three angles. Let W_a, W_b, and W_c be the weights of the angles a, b, and c respectively

$$\text{Then } \frac{W_a}{\left(\frac{1}{2}\right)^2} = \frac{W_b}{\left(\frac{1}{4}\right)^2} = \frac{W_c}{\left(\frac{1}{6}\right)^2}$$

Let C_a, C_b, and C_c be respectively the corrections to be applied to angles a, b, and c. Since the corrections are inversely proportional to the weights

$$\frac{C_a}{(2)^2} = \frac{C_b}{(4)^2} = \frac{C_c}{(6)^2} \text{ or}$$

$$C_a = \frac{C_b}{4} = \frac{C_c}{9}$$

Also $$C_a + C_b + C_c = 14$$

Therefore $$C_a = \frac{1}{14} \times 14 = 01''$$

$$C_b = \tfrac{4}{14} \times 14 = 04''$$
$$C_c = \tfrac{9}{14} \times 14 = 09''$$

The most probable values are therefore

$$\begin{array}{llcl}
a & 121^\circ 46' 37'' & + 01'' = & 121^\circ 46' 38'' \\
b & 179^\circ 14' 35'' & + 04'' = & 179^\circ 14' 39'' \\
c & \underline{58^\circ 58' 34''} & + \underline{09''} = & \underline{58^\circ 58' 43''} \\
 & 359^\circ 59' 46'' & + 14'' = & 360^\circ 00' 00'' \textit{ check}
\end{array}$$

A somewhat similar case arises when a quantity is determined by several independent series of measurements, as illustrated by the example below.

Example 2: Lines of levels to establish the elevation of a point are run over four different routes. The observed elevations of the point with probable errors are given below.

Line	Observed elevation, feet
a	721.05 ± 0.12
b	721.37 ± 0.24
c	720.62 ± 0.36
d	721.67 ± 0.48

Remembering that weights vary inversely as the squares of the probable errors

$$\frac{W_a}{(1/12)^2} = \frac{W_b}{(1/24)^2} = \frac{W_c}{(1/36)^2} = \frac{W_d}{(1/48)^2} \text{ or}$$
$$W_a = 4W_b = 9W_c = 16W_d$$

Let $W_a = 1$; then $W_b = \tfrac{1}{4}$; $W_c = \tfrac{1}{9}$; and $W_d = \tfrac{1}{16}$.
Multiplying the weights by each of the observed elevations, the weighted observations are obtained as shown in the tabulation:

Line	Observed elevation	Weight	Weighted observations
a.	721.05	1	721.05
b.	721.37	$\frac{1}{4}$	180.34
c.	720.62	$\frac{1}{9}$	80.07
d.	721.67	$\frac{1}{16}$	45.10
Total.	. . . . . .	$\frac{205}{144}$	1,026.56

The most probable value of the elevation is the sum of the weighted observations divided by the sum of the weights, or

$$\frac{1{,}026.56 \times 144}{205} = 721.10 \text{ ft.}$$

52. Interrelation of Errors.—If E_1 and E_2 represent respectively the probable errors of lengths L_1 and L_2 then the probable error of the area representing the product of these two lengths is

$$E_a = \sqrt{L_1^2E_2^2 + L_2^2E_1^2}$$

The probable error of the product of a constant or known quantity C and a measured quantity Q for which the probable error is E, is

$$E_p = CE$$

The probable error of the sum of independent measurements Q_1, Q_2, . . ., Q_n for which the probable errors are E_1, E_2 . . ., E_n, is

$$E_s = \sqrt{E_1^2 + E_2^2 \ . \ . \ . + E_n^2}$$

The probable error of the difference between two independent measurements Q_1 and Q_2 for which the probable errors are E_1 and E_2 is

$$E_d = \sqrt{E_1^2 + E_2^2}$$

CHAPTER VI

MEASUREMENT OF DISTANCE

GENERAL METHODS

53. Remarks.—When the distance between two points is referred to, it is always understood to mean the *horizontal distance*, regardless of the relative elevation of the points. In geodetic surveying horizontal distance is reduced to the equivalent at sea-level, but in plane surveying such reductions are unnecessary. Though slope distances are frequently measured, they are subsequently reduced to their equivalent on the horizontal projection, and in the latter form are used in plotting maps, calculating land areas, etc.

Distances may be measured by a variety of methods each of which will be found useful under certain conditions.

54. Pacing.—This method furnishes a rapid means of approximately checking more accurate measurements of distance. It is extensively employed in small-scale mapping. It is also used on exploratory or reconnaissance surveys, the paces of saddle animals frequently forming the basis of measurement.

Under average conditions a person of experience will have little difficulty in pacing with a precision of $\frac{1}{100}$.

Many land surveyors estimate distances by the 3-ft. pace, which is somewhat longer than the natural pace of the average person. Military sketchers and many topographers of government surveys maintain a pace that is the natural length or a little shorter. The writer favors the $2\frac{1}{2}$-ft. pace; it being a little less than the natural step, allowance can be made for uneven ground by lengthening the pace without tiring; and a convenient relation exists between the pace and the foot, *i.e.*, 40 paces = 100 ft. Each two paces or double step is sometimes called a stride. Paces or strides are usually counted by pressing a tally register, or by a pedometer or passometer which registers mechanically.

55. The Stadia.—This method offers a rapid means of determining distances. Two additional horizontal hairs are mounted on the cross-hair ring in the telescope of the transit, level, or plane-table alidade. The distance from the instrument to a given point is indicated by the intercept between the stadia hairs as shown on a rod held vertically at the point. The accuracy of the stadia depends upon the instrument, the observer, the atmospheric conditions, and the length of sight. Under average conditions it will yield a precision between $\frac{1}{300}$ and $\frac{1}{1,000}$. It is particularly useful in topographic surveying. The stadia method is described in detail in Chap. XVII.

56. Direct Measurement.—The most accurate and most common method of determining distance is by direct measurement. Formerly on surveys of ordinary precision it was the practice to measure the length of lines with the engineers' chain or the Gunter's chain; for measurements of the highest precision special bars were used. At the present time practically all direct linear measurements, as they occur on surveys, are made with tapes.

The *engineers' chain* is 100 ft. long and is composed of 100 links each 1 ft. long. At every 10 links brass tags are fastened, notches on the tags indicating the number of 10-link segments between the tag and the end of the tape. Distances measured with the engineers' chain are recorded in feet and decimals.

The surveyors' or Gunter's Chain is 66 ft. long and is divided into 100 links each 7.92 in. long. It was formerly much used in land surveying on account of the convenient relation between its length and the units of land measure.

1 (Gunter's) chain = 100 links = 4 rods
80 (Gunter's) chains = 1 mile
10 square (Gunter's) chains = 1 acre

Distances were recorded in chains and links.

Measuring with chains was called "chaining." The term has survived and is now generally used to refer to the operation of measuring lines with tapes.

The precision of distance measured with tapes depends upon the degree of refinement with which measurements are taken. On the one hand, rough chaining through broken country may be less accurate than the stadia. On the other hand, when extreme care is taken to eliminate all possible errors, measurements have been taken with a probable error of less than $\frac{1}{1,000,000}$.

57. Other Methods.—The revolutions of the wheel of a vehicle furnish a means of measuring distance along highways. The ordinary automobile speedometer registers distance to 0.1 mile. By driving over a course of known length, the speedometer may be standardized so that long distances can be determined with an accuracy considerably greater than by pacing.

The odometer, a device which registers the number of revolutions of the wheel, is more satisfactory than the speedometer, and it may be readily attached to any vehicle. By measuring the circumference of the wheel with a tape, the relation between distance and revolutions is fixed. On smooth roads the precision may be as great as that obtained with the stadia. The distance indicated by the odometer is somewhat greater than the true horizontal distance, of course, but under the conditions for which it is used the odometer reading requires no correction except in hilly country. The device is extensively used on plane-table traverses for small-scale maps.

58. Choice of Methods.—Practically all of the more important lines, including land boundaries, main traverses of horizontal control for extensive maps, and the lines for the location and construction of the works of man, are measured with tapes because no other practicable method furnishes the required precision. It is certain that much time has been wasted in chaining distances that could have been measured with all necessary precision by some less laborious method. The important point for the surveyor to bear in mind is that each of the methods mentioned in the preceding articles has a field of usefulness, and may properly be employed when it will furnish measurements of the required accuracy. Within the past few years the advantages of the stadia method have come to be more fully appreciated, and now practically all linear measurements for many surveys for maps are obtained through its use. On the surveys for a single enterprise the writer has found occasion to employ all the above methods to good advantage.

CHAINING

59. Remarks.—Chaining, as the term is customarily used, refers to the operation of measuring with the tape for the purpose of obtaining the horizontal distance between points on or near the surface of the earth. The persons who manipulate the tape are generally called "chainmen." To some extent the term "taping" is succeeding the term "chaining" and not infrequently the operators are called "tapemen."

60. Tapes.—Tapes are made in a variety of materials, lengths, and weights. Those more commonly used by the surveyor are the heavy steel tape, sometimes called the surveyors' tape or the chain tape, and the metallic tape. The ribbon of the metallic tape is of waterproofed fabric into which are woven small bronze wires to prevent its stretching. It is usually 50 ft. in length and is graduated to feet, tenths, and half-tenths. It is shown in Fig. 60. It is used principally in earthwork cross-sectioning and similar work where a light, flexible tape is desirable and small errors in length are of no consequence.

For very precise measurements the invar tape is coming to be used quite generally. Invar metal is a composition of nickel and steel with a very low coefficient of thermal expansion, sometimes as small as $\frac{1}{30}$ that of steel. It is a soft metal, and the tapes must be handled very carefully to avoid bends and kinks. This property and also the high cost make invar tapes impracticable for ordinary use.

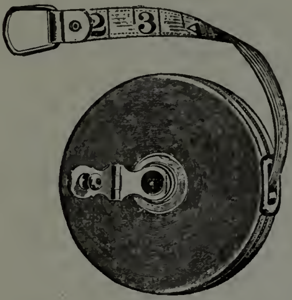


Fig. 60.—Metallic tape.

61. Steel Tapes, Their Care and Use. The steel tape is generally employed for the direct linear measurement of all important survey lines. In the United States and Canada the length most commonly used is 100 ft. The tapes used in surveying are usually graduated in feet and decimals. An exception is the 66-ft. tape which, corresponding to the old Gunter's chain, is graduated into 100 links each 7.92 in. long.

Small, lightweight box tapes usually have graduations etched every 0.01 ft. throughout their entire length. The heavy chain tapes usually have graduations with numbers every foot, but with only the end feet graduated to tenths or hundredths of feet. The graduations may be etched or may be stamped on babbitt metal or on brass sleeves. The chain tapes are also obtainable with etched graduations every 0.01 ft. throughout their length.

The chain tape may be wound on a reel but except in city and mining surveying the use of the reel is dispensed with. Rawhide thongs serving as handles are ordinarily fastened to the rings at each end.

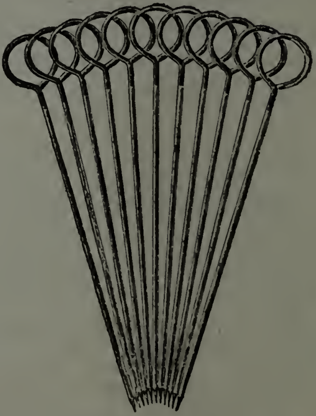


Fig. 62.—Chaining pins.

Some tapes have an extra graduated foot on one or both ends. Others have shoulders at zero and last graduation to assist in locating these points. Usually the ribbon extends about 6 in. beyond the graduated portion of the tape, but for some tapes the ends of the rings mark the zero and last graduation. The latter type is not well adapted to precise measurements. The common widths of tape are $\frac{3}{16}$, $\frac{1}{4}$, and $\frac{5}{16}$ in.

The steel tape, being elastic, stretches when a pull is applied. It also expands or contracts as the temperature changes. Tapes when received from the manufacturer are usually quite close to the standard length when subjected to a given pull and a given temperature. For the 100-ft. tape some manufacturers attempt to furnish the standard length at 62°F. under a pull of 12 lb., the tape being horizontal and supported throughout its entire length, but among manufacturers there is no uniformity of practice in this respect. It is well to have a standard of length available to which the tape can be referred occasionally. Many cities have such standards.

For a small fee the U. S. Bureau of Standards, Washington, D. C. will standardize a tape for any specified pull and will issue a certificate stating its length under the conditions attached to the standardization test.

62. Chaining Pins.—These pins, which are also called surveyors' arrows, are commonly employed to mark the ends of the tape during the process of chaining between two points more than a tape-length apart. They are of steel and are usually 10 to 14 in. long. A set consists of eleven pins (see Fig. 62).

63. Range poles, also called flags, flag poles, or lining-rods, are used as signals to indicate the position of points or the direction of lines. They are constructed of either steel or wood, are of octagonal or circular cross-section, and are pointed at the lower end. The common length is 8 ft. The wooden range poles are shod with a steel point. Usually they are painted with alternate bands of red and white, the length of band being 1 ft. (see Fig. 63).

64. Measurement of Lines with Tape; Smooth Level Ground.—The procedure followed in chaining distances with the tape depends to some extent upon the required precision and the purpose of the survey. The following represents the usual practice when the measurements are of ordinary precision $\left(\text{say } \frac{1}{5{,}000}\right)$: The tape is supported throughout its length and the only requirement is that the distance between two fixed points (as the corners of a parcel of land) be determined. The equipment will be assumed to consist of one or

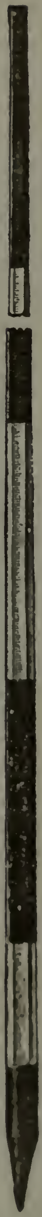


Fig. 63. Range pole.

more range poles, 11 chaining pins, and a 100-ft. heavy steel tape, with the intervals 0 to 1 ft. and 99 to 100 ft. graduated in tenths of feet and the remainder of the tape graduated in feet. One range pole will be placed behind the distant point to indicate its location.

The rear chainman with one pin stations himself at the point of beginning. The head chainman, with the zero end of the tape and 10 pins, advances towards the distant point. When the head chainman has gone nearly 100 ft. the rear chainman calls "*chain*" or "*tape,*" a signal for the head chainman to halt. The rear chainman holds the 100-ft. mark at the point of beginning and by hand signals or by speaking, lines in a chaining pin (held by the head chainman) with the range pole marking the distant point. During the *lining in* process the rear chainman is in a kneeling position on the line and facing the distant point; the head chainman is in a kneeling position to one side and facing the line so that the rear chainman will have a clear view of the signal marking the distant point. The head chainman with his right hand sets the pin vertically on line and a short distance to the rear of the zero mark. With his left hand he then pulls the tape taut and making sure that it is straight, brings it in contact with the pin. The rear chainman, when he observes that the 100-ft. mark is at the point of beginning, calls "*stick*" or "*all right.*" The head chainman pulls the pin and sticks it at the zero mark of the tape and in a position sloping away from the line. As a check he again pulls the tape taut and notes that the zero point coincides with the pin at its intersection with the ground. He then calls "*stuck*" or "*all right,*" the rear chainman releases the tape, the head chainman moves forward as before, and so the process is repeated. As the rear chainman leaves each intermediate point

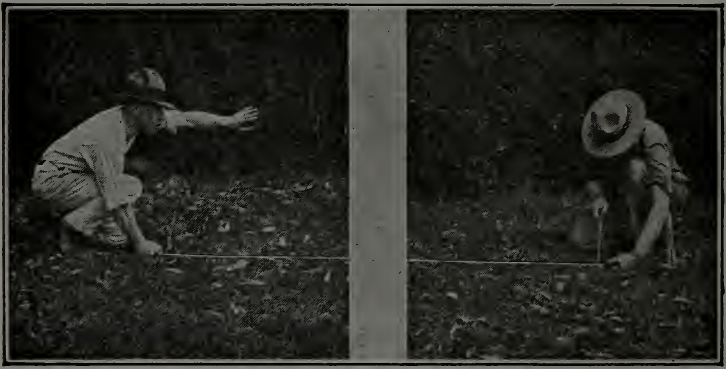


Fig. 64.—Chainmen taking a measurement with the tape and pins.

he pulls the pin. Thus there is always one pin in the ground and the number of pins held by the rear chainman at any time indicates the number of hundreds of feet from the point of beginning to the pin in the ground. Figure 64 shows chainmen in the act of taking a measurement.

At the end of each thousand feet the head chainman has placed his last pin in the ground. He signals for pins, the rear chainman comes forward with the ten pins which he has pulled, both chainmen count them to see that none is lost, and the tally is recorded by the head chainman. The head chainman takes the ten pins, and the procedure is repeated.

When the end of the course is reached, the head chainman halts and the rear chainman comes forward to the last pin set. The head chainman holds the zero mark at the terminal point. The rear chainman pulls the tape taut, and observes the number of feet between the last pin and the end of the line. He then holds the next larger foot mark at the pin and the head chainman pulls the tape taut and reads the decimal by means of the finer graduations of the end foot. The decimal is counted from the 1-ft. mark. Thus the distance in feet between the last pin and the end of the line is one less than that indicated by the foot mark held by the rear chainman, plus the decimal read by the head chainman. For example, with the rear chainman holding at 87 ft. and the head chainman reading 0.68 ft., the distance from the last pin is $87 - 1 + 0.68 = 86.68$ ft.

When the transit is set up on the line to be measured, the transitman usually directs the head chainman in placing the pins on line. The rear chainman maintains a position that will give the transitman an unobstructed view, and the head chainman kneels or stands on line facing the transitman.

Many surveys require stakes to set on line at short intervals, usually every 100 ft. Sometimes pins are used in chaining as already described, each stake being driven by the rear chainman after he has pulled the pin. On more accurate surveys, measurements are carried forward by setting a tack or small nail in the head of each stake. In setting the tack, the head chainman holds the pin (or range pole) on the head of the stake and places it on line as directed by the transitman. He pulls the tape taut, making sure that one edge is on line by bringing it in contact with the pin. He then uses the pin to mark the position of the tack at the zero point of the tape. When the tack has been driven, it is tested for line and distance.

65. Horizontal Measurements over Uneven or Sloping Ground.—The process of chaining over uneven or sloping ground is much the same as that just described for level ground. The measurements are carried forward by holding the tape in a horizontal position and the plumb-line is used by either or, at times, by both chainmen for projecting from tape to pin or *vice versa.* If the course is downhill, the head chainman must plumb from the zero (or other) point on the tape to the ground; if up hill, the rear chainman must plumb from the pin to the 100-ft. (or other) point on the tape; if uneven, both chainmen will find it necessary to use a plumb-bob.

To secure anything like the same precision as in chaining over level ground, considerable skill is required. Some experience is necessary to determine when the tape is nearly horizontal. The tape is unsupported for most of its length, and the pull must be increased to eliminate the effect of sag.

When the slope is less than 5 or 6 ft. in a hundred, the head chainman advances a full tape length at a time, and pins are set by him and collected by the rear chainman as described in the preceding article. If the course is downhill, the head chainman estimates when the tape is horizontal. Holding the plumb-line in position at the zero point of the tape and noting that the plumb-bob clears the ground by a few inches, he pulls the tape taut and is directed to the line by the rear chainman. When the plumb-bob comes to rest, he releases the string quickly, allowing the bob to fall and stick in the ground. He removes the bob and sets a pin in its place. As a check, the measurement is repeated. If the course is uphill, the head chainman holds the zero end of the tape firmly on the ground and on line. The rear chainman, with plumb-line suspended from the 100-ft. mark, signals the head chainman to give or take until the bob comes to rest over the pin. The head chainman sets the pin, and the measurement is repeated.

When the course is steeper and is downhill, the head chainman advances a full tape length and then returns to an intermediate point from which he can hold the tape horizontally. He suspends the plumb-line at a foot mark and is lined in by the rear chainman. A pin is set at the indicated point by the head chainman. The rear chainman comes forward, gives the head chainman a pin, and holds the tape at the foot mark from which the plumb-line was previously suspended. The head chainman proceeds to another point from which he can hold the tape horizontally. The rear chainman holds the foot mark to the pin. The head chainman sets a second pin. And so the process is repeated until the head chainman reaches the zero mark on the tape. At each *intermediate* point of a tape length the rear chainman gives the head chainman a pin, but not at the point marking the full tape length.

To illustrate, Fig. 65 represents the profile of a line to be measured in the direction of *A* to *D*, and *A* is a pin marking the end of a 100-ft. interval from the point of beginning. The head chainman goes forward until the 100-ft. mark is at *A*, where the rear chainman is stationed. The head chainman then returns to *B* where he holds the tape horizontally and plumbs from the 83-ft. mark to set a pin at *B*. The rear chainman gives the head chainman a pin, and holds the 83-ft. mark at *B*. The head chainman plumbs from the 51-ft. mark and sets a pin at *C*. The rear chainman gives the head chainman a pin, and holds the 51-ft. mark at *C*. The head chainman plumbs from the zero mark to set a pin at *D* at the end of the full tape length. The rear chainman goes forward but retains the pin which he pulled at *C*.

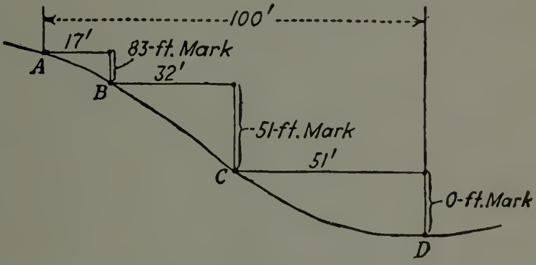


Fig. 65.—Chaining on steep slope.

In this manner the tape is always advanced a full length at a time and the number of pins held by the rear chainman at each 100-ft. point indicates the number of hundreds of feet from the last tally. The tape is usually estimated to be horizontal by eye. This quite commonly results in the downhill end being too low, sometimes causing a large error in horizontal measurement. The writer has seen inexperienced chainmen in rough country introduce an error as great as 8 ft. per mile in this manner. The safe procedure is to insist on the chainman using a hand level in country of this character.

The operation of carrying the horizontal distance forward by plumbing down at an intermediate foot mark, as just described, is frequently termed "breaking chain" or "breaking tape."

66. Measurements on the Slope.—Where the ground is fairly smooth, measurements on the slope may be made more accurately and quickly than horizontal measurements, but some means of determining the slope or difference in elevation between successive 100-ft. points is required. For surveys of ordinary precision the clinometer (for measuring slopes) or the hand level (for measuring difference in elevation) may be used to good advantage. If all that is desired is the distance between the ends of the line, the usual procedure, so far as chaining is concerned, is exactly the same as on level ground; but a record is kept either of the difference of elevation, or of the slope of each 100-ft length. The horizontal distances are then calculated from the distances measured on the slope, and the horizontal length of the line is determined.

When stakes are to be placed every 100 ft., corrections to the slope distances may be applied as the chaining progresses, either by use of the slide rule or by mental calculation. Corrections are much more readily calculated than are the horizontal distances themselves, as will shortly be demonstrated. Unless the correction is greater than 1 ft. the 100-ft. tape with an extra foot on the zero end is particularly useful for measuring on slopes in that the head chainman can first determine the correction for slope and in one operation can then lay off the true slope distance to give a horizontal distance of 100 ft.

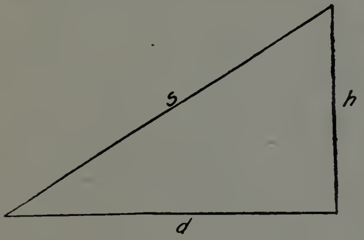


Fig. 67.

67. Corrections for Slope.—For measurements of ordinary precision when the slope is not greater than about 20 in 100, the correction to slope distance to give horizontal distance may be calculated by the approximate formula developed below.

In Fig. 67 let s represent the slope distance, h the difference in elevation between two points, and d the horizontal distance. Then

$$h^2 = s^2 - d^2 = (s + d)(s - d).$$

When the slope is not large $s + d = 2s$ (approximate). Making this substitution

$$h^2 = 2s(s - d) \text{ (approximate)}$$

and the correction is

$$C_h = s - d = \frac{h^2}{2s} \text{ (approximate)} \tag{1}$$

For the usual case, when $s = 100$ ft., the above expression may be solved mentally, and the opportunities for its use are so frequent as to make it worth remembering. The error introduced through its use is negligible for ordinary slopes.

Where the angle of the slope is determined, as when using the clinometer, formula (1) may still be used quite readily if it is remembered that for angles less than 20° the slope per 100 ft. is about 1.75 ft. times the slope angle in degrees.

When vertical angles are measured with sufficient precision to so warrant, the horizontal distance may be calculated by exact trigonometric formula.

Having given the slope distance, to find its horizontal projection, the correction is *subtracted.*

68. Errors in Chaining.—These errors may be attributed to the following causes:

1. *Tape Not Standard Length.*—This produces a constant error which may be practically eliminated by standardizing the tape.

2. *Imperfect Alignment.*—The head chainman is likely to set the pin sometimes on one side and sometimes on the other side of the true line. This produces a variable systematic error since the horizontal angle which the tape makes with the line is not the same for one tape length as for the next. The error cannot be eliminated but can be reduced to a negligible quantity by care in lining. Generally it is the least important of the errors of chaining. The linear error due to one end of the tape's being off line by a given amount can be calculated by formula (1), Art. 67. For a 100-ft. tape, the error amounts to 0.005 ft. when one end with respect to the other is off line 1 ft. Many surveyors use unnecessary care in securing good alignment without paying much attention to other more important sources of error. Errors in alignment tend to make the measured length greater than the true length.

3. *Tape Not Horizontal or Slope of Tape Not Correctly Determined.*—The effect is to produce an error similar to that due to imperfect alignment. With the eye it is difficult to estimate slopes or to tell when the tape is horizontal. It is not uncommon to see chainmen of considerable experience chaining on slopes as steep as 4 ft. in 100 ft. without realizing that slope corrections should be made. The error per tape length, due to the tape's not being level would be 0.08 ft. In ordinary chaining this is one of the largest of contributing errors. It will not be eliminated by repeated measurements, but may be reduced to a negligible amount by leveling the tape by means of the hand level or clinometer.

4. *Plumbing; Imperfections of Observing; Inability to Set Pins at Exact Distance.*—These are accidental errors, hence the probable error tends to vary as the square root of the number of tape lengths. The error due to plumbing is the only one that is of real importance. In ordinary chaining through rough country, where it is necessary to break chain frequently, the probable error per tape length may perhaps amount to ± 0.1 ft. for each tape length chained or ± 0.7 ft. per mile. While these errors cannot be eliminated, their effect on the resultant error is usually not large. When the required preci-

sion is high, errors of plumbing may be eliminated by chaining on the slope.

5. *Variable Tension in the Tape.*—The tape, being of an elastic material, is elongated when tension is applied. If the pull is greater than that for which the tape is standard, the tape is too long; if the pull is less, the tape is too short. The error is systematic and of a magnitude depending upon the methods employed and the individuals who are chaining. The error is negligible except in precise work. For the heavy chain tape an increase in tension of 3 lb. changes the length of the tape about 0.001 ft.

6. *Variations in Temperature.*—If the tape is standardized at a given temperature and measurements are taken at a higher temperature, the tape is too long. For a change of temperature of 10°F. a 100-ft. steel tape will undergo a change in length of about 0.006 ft., introducing an error of about 0.3 ft. per mile. Under a change of 50°F. the error per mile would be 1.5 ft. It is seen that, even for measurements of ordinary precision, the error due to thermal expansion becomes of consequence when the measurements are taken during extremely cold or extremely hot weather. Some tapes have a temperature scale at one end by means of which the correction for variation in temperature may be made without calculation.

7. *Tape Not Straight.*—In chaining through grass and brush or when the wind is blowing it is impossible to have all parts of the tape in perfect alignment with its ends. The error arising from this source is systematic and has the same effect as measuring with a tape that is too short. If the head chainman is careful to stretch the tape taut and to observe that it is straight by sighting over it, the error is of no consequence.

8. *Sag in Tape.*—This occurs whenever the tape is supported at intervals rather than throughout its full length. If standardized flat, *i.e.*, supported for full length, then with a heavy 100-ft. tape weighing 3 lb. supported at the ends under a tension of 10 lb. the systematic error due to sag alone is about 0.35 ft., per tape length. If the pull is increased to 20 lb. the error is about 0.09 ft.; if increased to 30 lb. the error is approximately 0.04 ft.

Normal Tension.—The tape stretches and this partially offsets the effect of sag. For the heavy tape the resultant error for a 30-lb. pull is perhaps 0.03 ft. per 100 ft. or 1.5 ft. per mile. With the lighter tapes a pull which may be applied without undue exertion may be determined, either by calculation or by experiment, at which the effect of sag will just offset the effect of increase in tension. This is usually called the *normal* tension.

69. It is seen that in ordinary chaining the systematic errors are likely to be of much greater magnitude than the accidental errors. Hence the resultant error varies as the number of tape lengths or as the length measured.

When every possible device is employed to detect and to eliminate these systematic errors, as in precise base-line measurement, the accidental errors of observation become of relatively great importance, and for this reason long tapes are usually employed. To make corrections for, or to eliminate, errors due to sag, or to elongation of tape by tension, the pull is observed by spring balances; to make corrections for thermal expansion, the temperature of the tape is observed by the use of thermometers.

It should be noted that, to obtain the true horizontal length of a measured line, the corrections are to be added to or subtracted from the measured length according to whether the errors tend to make the horizontal projection of the length of the tape greater or less than its normal length. If, on the other hand, it is desired to establish a given horizontal distance, the corresponding distance to be measured is found by applying corrections to the given distance in the opposite sense. Thus if a distance is measured with a tape that is too long, the correction is added; if a given distance is to be laid off with a tape that is too long, the correction is subtracted.

70. Correction for Sag.—When the tape sags between points of support it takes the form of a catenary. The correction to be applied is the difference in length between the arc and the subtended chord. For the purpose of determining the correction the arc may be assumed to be a parabola, and the correction is then given by the expression

$$C_s = \frac{w^2L^3}{24P^2} = \frac{W^2L}{24P^2} \qquad (1)$$

in which

w is the weight of tape in pounds per foot,
W is the total weight of the tape between supports in pounds,
L is the distance between supports in feet,
P is the applied tension in pounds, and
C_s is the correction between points of support in feet.

71. Correction for Tension.—If a tension greater or less than that for which the tape is standardized is used, the tape is elongated or shortened accordingly. The correction for variation in tension in a steel tape is given by the expression

$$C_p = \frac{(P - P_o)L}{AE} \qquad (2)$$

in which

P is the applied tension in pounds,
P_o is the tension for which the tape is standardized in pounds,
L is the length in feet,
A is the cross-sectional area in square inches,
C_p is the correction per tape length in feet, and
E is the modulus of elasticity of the steel, usually between 28,000,000 and 30,000,000 lb. per square inch.

72. Elimination of Effect of Sag.—By equating the right-hand members of equations (1) and (2) (Arts. 70, 71) the elongation due to increase in tension is made equal to the shortening due to sag. Normal tension, P_n, the pull which will produce this condition is given by the expression

$$P_n = \frac{0.204W\sqrt{AE}}{\sqrt{P_n - P_o}} \tag{3}$$

73. Correction for Temperature.—The coefficient of thermal expansion of steel is about 0.0000065 per 1°F. If the tape is standard at a temperature of T_o degrees and measurements are taken at a temperature of T degrees the correction for change in length is given by the formula

$$C_x = 0.0000065\, l(T - T_o) \tag{4}$$

in which l is the measured length.

74. Precision of Measurements with the Tape.—It would be valuable if a definite outline of procedure could be established to produce any desired degree of accuracy in chaining. Unfortunately the conditions are so varied and so much depends upon the skill of the individual that the surveyor must largely be guided by his own experience and his knowledge of the errors involved.

The usual practice in rough chaining through broken country is to take measurements with the tape horizontal, plumbing from the down hill end, breaking tape when necessary, applying tension by estimation, and making no corrections for sag, temperature, etc. The discussion of the preceding articles makes it evident that the large errors are likely to arise owing to (1) tape not level, (2) sag in tape, (3) variation in temperature, and (4) poor plumbing. Under normal conditions the precision may be expected to range between $\frac{1}{1{,}000}$ and $\frac{1}{3{,}000}$. If the chaining is done during extremely cold weather with inexperienced chainmen who fail to keep the tape taut and do not use reasonable care in keeping the tape horizontal the precision may be less than $\frac{1}{500}$.

In ordinary chaining over fairly flat, smooth ground the principal errors are those due to variation in temperature and inclination of tape. Usually $\frac{1}{5{,}000}$ is considered good chaining for the conditions stated, but to attain this precision a rough correction for variation in temperature must frequently be made. Under extreme weather conditions, change in temperature alone might introduce an error of $\frac{1}{2{,}000}$ or even greater.

In chaining along a smooth surface as a paved highway, if slope measurements are accurately taken, the principal error is that due to variation in temperature. For measurements of moderate precision when a rough correction for thermal expansion is made without actual observations of temperature a precision of $\frac{1}{10{,}000}$ is about the maximum that can be obtained.

For measurements of higher precision the temperature is determined by a thermometer attached to the tape, and the tension is regulated through the use of a spring balance. When measurements are corrected for the accumulative effects of sag, slope, etc., the remaining errors are largely accidental in character, though for measurements in sunlight there is likely to be an appreciable difference between the observed temperature and the actual temperature even though the bulb of the thermometer is in contact with the tape. Experience indicates that lines of considerable length may be measured under the conditions stated above with an accuracy as high as $\frac{1}{30{,}000}$.

When steel tapes are used, measurements calling for a very high precision, say $\frac{1}{100{,}000}$ to $\frac{1}{500{,}000}$ or higher, are made at night or on cloudy days so that uncertainties regarding the temperature of the tape are practically eliminated. For work of this character the tapes employed are usually 300 ft. or more in length and are very carefully standardized. The successive positions of the forward end of the tape are marked by lines scratched on zinc or copper strips fastened to substantial posts. Errors due to variations in temperature are greatly reduced by using an invar tape.

75. Mistakes.—Some of the mistakes commonly made by inexperienced chainmen are:

1. Adding or dropping a full tape length. This is not likely to occur if both chainmen count the pins, or when numbered stakes are used, if the rear chainman calls out the station number of the rear stake in response to which the head chainman calls out the number of

the forward stake as he marks it. The addition of one or more tape lengths may occur through failure of the rear chainman to give the head chainman a pin at breaks marking fractional tape lengths. A tape length may be dropped by the rear chainman not taking a pin at the point of beginning.

2. Adding a foot. This usually happens in measuring the fractional part of a tape length at the end of the line. This distance should be checked by the head chainman holding the zero mark on the tape at the terminal point and the rear chainman noting the number of feet and approximate fraction at the last pin set.

3. Wrong points taken as zero, or 100-ft. marks on tape. The chainman should note whether these marks are at end of rings or on the tape itself, also whether there is an extra graduated foot at one end of the tape.

4. Reading numbers wrong. It is good practice to observe the numbers of foot marks on either side of the one indicating the measurement, especially if the numbers are dirty or are worn. Also the tape should be read with the numbers right side up.

5. Calling numbers incorrectly or so that they are not clearly understood. For example, 50.3 might be called "fifty-three" and recorded as 53.0. When numbers are called to a recorder, he should repeat them as they are recorded. Whenever a decimal point or a zero occurs in a number it should be indicated by the person calling.

76. Numerical problems.

1. A line is measured with a 100-ft. steel tape and its measured length is 1,012.3 ft. Afterward the tape is compared with the standard and is found to be 0.03 ft. too long. Calculate the length of the line.

2. A building 80 by 160 ft. is to be laid out with a 50-ft. tape which is 0.016 ft. too long. What ground measurements should be made?

3. The slope measurement of a line is 800 ft. The differences in elevation between successive 100-ft. points, as measured with a hand level, are 1.0, 1.5, 2.5, 3.8, 4.6, 5.0, 7.5, and 6.2 ft. Determine the horizontal distance.

4. A line measured on the slope is 1,246.5 ft. Slope angles measured with a clinometer are as shown below. Calculate the horizontal distance by exact and by approximate methods.

Chainage in feet.....	0		300		800		1,000		1,246.5
Slope angle in degrees		½		1¼		2½		4	

5. Two points at a slope distance of 100 ft. apart have a difference in elevation of 12 ft. What is the slope distance to be laid off to establish a horizontal distance of 100 ft.? Calculate by exact and by approximate methods.

6. Calculate the effect of sag per tape length for the two tapes of Example 1, Art. 70, using tensions of 20 and 40 lb., and distances between supports of 25 and 100 ft.

7. A 100-ft. tape weighing 2 lb. is standard length under a tension of 12 lb. A line is measured with the tape under a tension of 35 lb. and found to be 4,863.5 ft. long. E = 29 000,000 lb. per square inch; 3.56 cu. in. of steel weigh 1 lb. Make the correction for increase in tension.

8. A second line is measured with the tape of problem 7, the tape being supported at intervals of 50 ft. and the pull being 35 lb. The measured length is 1,823.6 ft. Calculate the corrections for sag and variation in tension, and determine the corrected length of the line.

9. Calculate the pull for which the elongation of the tape of problem 7 is equal to the shortening due to sag, the tape being supported at its ends.

10. Chainmen made two independent measurements of a line 10,000 ft. long. The ground was sloping and measurements were taken with the tape horizontal. The tape was 100 ft. long, and weighed 3 lb. One of the measurements of the line was made by two chainmen supporting the tape at the zero and 100-ft. marks; the second measurement was made by three chainmen supporting the tape at the zero, 100-ft., and 50-ft. points. The discrepancy between the two measurements was 11.7 ft. Several tests with a spring balance indicated that the average pull exerted by the chainmen was 20 lb. How much of the above discrepancy might be attributed to the different modes of supporting the tape?

11. For the purpose of establishing monuments in a city, a line along a paved street is measured on the slope and observations of temperature are made at each application of the tape. The measured length on the slope is 1,320.64 ft., the street grade is 2.5 per cent, the applied tension is 12 lb., and the mean of the observed temperatures is 87.4°F. The 100-ft. steel tape used for the measurements is standardized at 70°F. and is found to be 0.004 ft. too short under a tension of 12 lb. Determine the horizontal length of the line.

12. A line through rough country is chained by horizontal measurements and found to be 2,450 ft. long. On the average, it was necessary to use the plumb-line every 50 ft. If the probable error of plumbing from the end of the tape to the ground is ±0.03 ft. parallel with the line, calculate the probable error due to inaccurate plumbing.

13. If, in problem 12, the average slope of the tape when measurements are taken is 2 in 100, what error is introduced in the length of the line?

14. A line roughly 2 miles long along a railroad track is measured with a steel tape and corrections are made for observed temperatures. What error will be introduced if the actual temperature of the tape was 2°F. warmer than the observed temperature? State the error in fractional form with numerator as one.

15. Assume that an invar tape is used under the conditions of problem 14. Calculate the error introduced.

16. A hedge along the line AB makes direct measurement impossible. A point C is established at an offset distance of 20 ft. from the line AB and roughly equidistant from the points A and B. The distances AC and CB are then chained and the following measurements obtained: $AC = 1287.2$ ft., $CB = 1353.0$ ft. By approximate method calculate the length of the line AB.

17. It is desired to measure a distance of approximately 10 miles with a maximum permissible error of $\frac{1}{10{,}000}$. The country is rolling (average slopes, 5 per cent) and wooded, so that for perhaps half the distance the tape must be held level, unsupported. Make any other necessary assumptions and write specifications for this work.

CHAPTER VII

MEASUREMENT OF DIFFERENCE IN ELEVATION

GENERAL METHODS

77. Remarks.—The difference in elevation between two points on the surface of the earth is the vertical distance between the two level surfaces (imaginary or real) in which the points lie. In Fig. 77 the irregular line represents the profile of the earth's surface in which are the points *A* and *B*. The curved lines are level lines representing the profile of imaginary level surfaces in which the points are located.

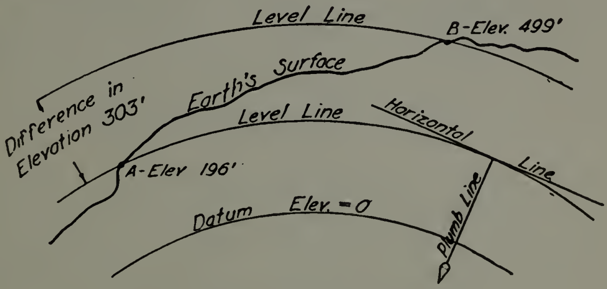


FIG. 77.—Difference in elevation.

The elevation of a point is its vertical distance above or below some arbitrarily assumed level surface called the "*datum*" or the "*datum plane*." In the figure the datum is represented by the lower curved line. If the elevation of point *A* with respect to the datum is 196 ft. and the difference in elevation between *A* and *B* is 303 ft. then the elevation of point *B* is 499 ft. The measurement of differences in elevation therefore deals either directly or indirectly with the determination of vertical distances. The operation is called *leveling*.

The datum most widely used is mean sea level, but most cities have a "city datum" which may or may not agree closely with mean sea level. Frequently elevations for a particular survey are referred to some datum which bears no known relation to sea level. For example, the initial point in a survey may be assumed to have an elevation of 100 ft. and the elevations of all succeeding points may be calculated accordingly. If the relative elevation of points is all that is desired, the relation between the assumed datum and sea level or any other datum in common use is of no consequence.

Difference in elevation may be measured by the following methods:

1. By measuring the variation in the pressure of the earth's atmosphere with a barometer, called barometric leveling.
2. By measuring a vertical angle and horizontal distance, called indirect or trigonometric leveling.
3. By measuring vertical distances directly, called direct or spirit leveling.

78. Curvature of the Earth.—A *level surface* has been defined as a surface every element of which is perpendicular to the action of gravity. A line tangent to a level surface is called a *horizontal line* and is perpendicular to the action of gravity (to the plumb-line) at the point of tangency. In Fig. 78 a horizontal line tangent to a level surface representing the mean surface of the earth is shown. The vertical distance between the horizontal line and mean surface is a measure of the earth's curvature. It varies approximately as the square of the distance from the point of tangency. A ray of light transmitted in the direction of the horizontal line at the point of tangency is refracted or bent downward slightly so that at some distance from the point of tangency it is below the horizontal line, and in the opposite sense a ray emanating from a given distant point below the horizontal line becomes horizontal. This phenomenon is called atmospheric refraction. Thus, as viewed from point A (Fig. 78), an object actually at B would apparently be at C. The actual line of sight is along the path of a curve AB.

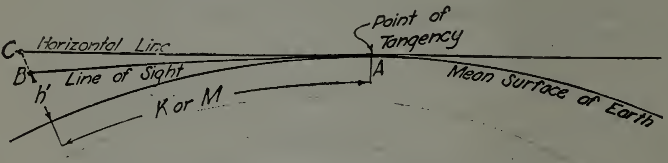


Fig. 78.—Earth's curvature and refraction.

The combined effect of the earth's curvature and atmospheric refraction is given by the expression:

$$h' = 0.57K^2 = 0.021M^2 \text{ (approximate)[1]} \qquad (1)$$

---

[1] The effect of the earth's curvature alone is about $0.66K^2$. The effect of the refraction is about $0.09K^2$ in the opposite direction.

in which

K is the distance from the point of tangency in miles,

M is the distance from the point of tangency in thousands of feet, and

h' is the combined effect of the earth's curvature and atmospheric refraction in feet.

79. Barometric Leveling.—Since the pressure of the earth's atmosphere varies inversely with the elevation, either the mercurial or aneroid barometer may be employed for making rough observations of difference in elevation. If at a given elevation the atmospheric pressure always remained constant, or even approximately so, the barometric method would be one of considerable precision, but the pressure in the course of a day or even in the course of an hour is likely to vary over a considerable range. The aneroid barometer is light in weight and is easily transported. The mercurial barometer is cumbersome and is only suitable for observations at a fixed station.

Usually, though not always, barometric observations are taken at a fixed station over the same period that observations are made on a second barometer which is carried from point to point in the field. This makes it possible to correct for atmospheric disturbances which could not be readily detected if a single barometer were used.

Barometric leveling is employed principally on exploratory or reconnaissance surveys where differences in elevation are large, as in hilly or mountainous country. Under the ordinary conditions, elevations determined by barometric leveling are likely to be several per cent in error. A single aneroid barometer is frequently used by topographers on small-scale mapping when the contour interval is large.

80. Indirect Leveling.—In Fig. 80, A represents a point of known elevation and B a point whose elevation is desired. In employing the method of indirect or trigonometric leveling, α, the vertical angle at A, is measured and the distance AD is known or is determined by some method of measurement. Within the limits of ordinary practice $AD = AC$ and $\angle BCA = 90°$. Therefore

$$h_b = AC \tan \alpha \qquad (2)$$

The correction for curvature and refraction is by Eq. (1), Art. 78

$$h' = 0.02M^2 = 0.02\left(\frac{AC}{1{,}000}\right)^2, \text{ in which } AC \text{ is in feet.}$$

The difference in elevation is therefore

$$H = h' + h_b = AC \tan \alpha + 0.02\left(\frac{AC}{1{,}000}\right)^2 \qquad (3)$$

If the vertical angle β is now taken from B to A, by a similar course of reasoning $h_a = EB \tan \beta$.

For the horizontal distances employed on any ordinary survey EB is the equivalent of AC. Therefore

$$h_a = AC \tan \beta \tag{4}$$

and the difference in elevation is

$$H = h_a - h' = AC \tan \beta - 0.02\left(\frac{AC}{1{,}000}\right)^2 \tag{5}$$

By looking at Eqs. (3) and (5) it will be noted that when the vertical angle is upward or positive the curvature and refraction correction is added; and when downward or negative, the curvature and refraction correction is subtracted.

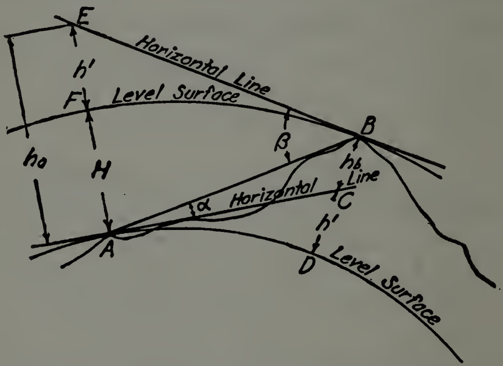


Fig. 80.—Indirect or trigonometric leveling.

Adding Eqs. (3) and (5),

$$2H = (h_a - h') + (h_b + h') = h_a + h_b,$$

or

$$H = \frac{h_a + h_b}{2} = \frac{AC}{2}(\tan \alpha + \tan \beta) \tag{6}$$

From Eq. (6) may be deduced the general rule that *when vertical angles are measured from (and to) each of two points whose difference in elevation is desired, the difference in elevation is one-half the horizontal distance between them multiplied by the sum of the tangents of the angles,* and the effect of the earth's curvature and atmospheric refraction is thereby eliminated. In precise trigonometric leveling this is the procedure employed.

In ordinary surveying, indirect leveling furnishes a rapid means of determining the elevations of points in rolling or rough country.

On reconnaissance surveys the angles may be measured with the clinometer and the distances may be obtained by pacing. On more accurate surveys angles are measured with the transit and distances by the stadia.

81. On lines of indirect levels for which angles are measured with the transit the usual procedure is illustrated by Fig. 81*a*. *A* and *D* are two points whose difference in elevation is desired. The successive positions of the instrument are indicated by the symbols T_1, T_2, and T_3. With the transit at T_1 the distance and vertical angle to *A* are determined by a *backsight*, and similar quantities are measured by taking a *foresight* to *B*. The instrument is then moved ahead to T_2, and observations are taken to *B* and *C*. And so the process is repeated until the end of the line is reached. If the transit is equidistant from the points on either side of it to which sights are taken, the effect of curvature and refraction will be eliminated, as is shown by Fig. 81*b*. The difference in elevation between *A* and *B* is

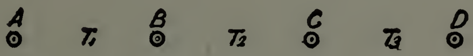


Fig. 81*a*.

$$H = H_a + H_b = h_a - h' + h_b + h' = h_a + h_b$$

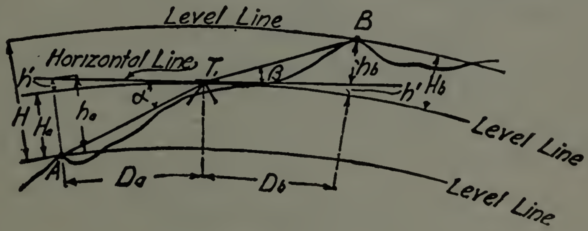


Fig. 81*b*.

And if D_a and D_b are the respective horizontal distances from the transit to points *A* and *B*, then

$$H = -D_a \tan \alpha + D_b \tan \beta \qquad (7)$$

in which $\tan \alpha$ and $\tan \beta$ are positive or negative according to whether the vertical angle is above or below the horizontal.

In practice usually very little attention is paid to keeping the backsight and foresight distances balanced since the effect of curvature and refraction is negligible (0.02 ft. for a distance of 1,000 ft.) as compared with the accuracy with which elevations can be determined by this method (generally not closer than tenths of feet). The error from such practice is accidental in that while one backsight distance may be greater than the corresponding foresight distance, normally the next is just as likely to be too small. Generally the transit is set at some convenient place where good sights can be obtained in both directions, and which is about the same distance from adjacent points on which sights are to be taken.

In small-scale mapping the indirect method of leveling is employed to determine the difference in elevation between the plane table and a point sometimes at a distance of several miles. In such cases the effect of curvature and refraction becomes large and the correction must be applied. For example, if the horizontal distance from the plane table to the point sighted were 10 miles, the correction would be 57 ft.

The errors incidental to indirect leveling are chiefly accidental. The precision attainable depends upon the length of sight, the instrument used, and the magnitude of the vertical angles. With the transit, under average conditions, the error may be expected to be not greater than 0.4 ft. times the square root of the distance in miles.

82. Direct Leveling.—In Fig. 82*a*, *A* represents a point of known elevation and *B* represents a point whose elevation is desired. In employing the method of direct or spirit leveling, the level is set up at some intermediate point as *L*, and the vertical distances *AC* and *BD* are observed by holding a leveling rod first at *A* and then at *B*, the line of sight of the instrument being horizontal.

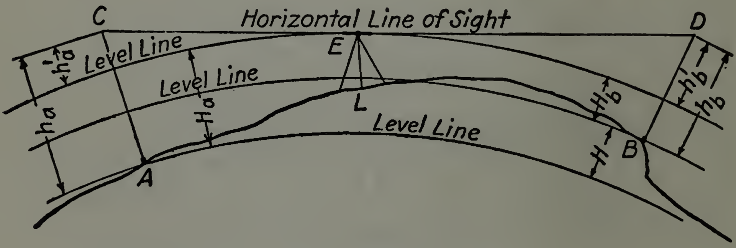


Fig. 82*a*.—Direct leveling

If the difference in elevation between the points *A* and *E* is designated as H_a and the difference in elevation between *E* and point *B* is designated as H_b,

$H_a = h_a - h'_a$ and $H_b = h_b - h'_b$ in which h_a and h_b are the vertical distances read at *A* and *B* respectively, and h'_a and h'_b are the effects of the curvature of the earth and refraction of the earth's atmosphere for the horizontal distances *LA* and *LB* respectively.

If H is the difference in elevation between A and B, then

$$H = H_a - H_b = (h_a - h'_a) - (h_b - h'_b)$$
$$H = h_a - h_b - h'_a + h'_b$$

If the backsight distance LA is equal to the foresight distance LB then $h'_a = h'_b$ and

$$H = h_a - h_b \tag{9}$$

Thus, by keeping backsight and foresight distances balanced the difference in elevation between two points is equal to the difference between the rod readings taken to the two points, and no correction for earth's curvature is necessary. In direct leveling, except in rare instances, the work is so conducted that the effect of the earth's curvature is eliminated, or nearly so.

On lines of direct levels the usual procedure is as indicated by Fig. 82*b*. A and D are two established points some distance apart, whose difference in elevation is desired. The symbols L_1, L_2, and L_3 indicate successive positions of the level, not necessarily on a line joining A and D. With the level in some convenient position as L_1, a backsight is taken to point A, and a foresight is taken to some convenient point B. The level is then moved ahead to L_2, and a backsight reading is taken to B and then a foresight reading is taken to some accessible point as C. And so the process is repeated until the terminal point is reached.

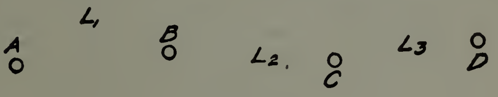


Fig. 82*b*.

In accurate leveling the backsight and foresight distances are measured by pacing or by stadia. Usually no special attempt is made to equalize each backsight distance with its corresponding foresight distance. The general rule is that their sums should balance, or nearly so, between established points that may be used as future references. Ordinarily the length of sight is so short that the usual procedure of leveling gives results practically free from errors due to the earth's curvature.

In ordinary leveling, *with an instrument in good adjustment*, the backsight and foresight distances are not measured, and no special attempt is made to equalize them. Normally, they will tend to balance in the long run. Even in leveling of moderate precision, however, a succession of very long backsight distances and correspondingly short foresight distances, or *vice versa*, would produce a systematic error of appreciable magnitude unless some correction were made for the earth's curvature. Keeping the backsight and foresight distances approximately balanced tends to eliminate certain instrumental errors, as will be seen later.

The operation of direct leveling to determine the elevations of points some distance apart is called *differential leveling*.

The operation of determining elevations of points at short measured intervals along a definitely located line (such as the center line for a railroad, sewer, or highway) is called *profile leveling*.

Precise leveling is a form of differential leveling for which the instruments used and the methods employed are such as to produce a high precision.

Direct leveling is also employed for determining elevations for cross-sections, grades, and contours. Direct leveling is the most precise method of determining elevations, and is the method most frequently used.

INSTRUMENTS FOR DIRECT LEVELING

83. Remarks.—Any instrument or device used for direct leveling has as its essential features a line of sight and a level tube or some other means of making the line of sight horizontal. The level tube is so mounted that its axis is parallel to the line of sight. The instrument used principally is the engineers' level. The hand level is a simple and useful device for roughly determining differences in elevation. Instruments which are frequently used for direct leveling, but which are not primarily and solely designed for this purpose, are the engineers' transit and the telescopic alidade.

The measurements of difference in elevation are determined by sighting upon graduated wooden rods, called leveling rods.

84. The Engineers' Level.—Figure 84*a* diagrammatically illustrates the principal parts of the engineers' level. The level consists of the telescope *A* mounted on the crossbar *B*, which is rigidly fastened to the spindle *C*. Attached to the telescope and parallel with it is the level tube *D*. The spindle fits into a cone-shaped bearing of the leveling head *E*, so that the level is free to revolve about the spindle as a vertical axis. The level is screwed to a wooden tripod *F*. In the tube of the telescope are cross-hairs at *G* which appear on the image viewed through the telescope as illustrated by Fig. 84*b*. The bubble of the level tube is centered by means of the leveling screws *H*.

The two distinct types are the *dumpy level*, for which the telescope tube is permanently fastened to the crossbar, and the *wye level*, for which the telescope is removable and rests in Y-shaped supports. Generally the leveling head is equipped with four leveling screws, but the three-screw type is favored by many engineers and surveyors, particularly for instruments of high precision. The level tube is usually under the telescope, but may be on top or at the side. A reflecting mirror or similar device, by means of which the level tube may be viewed while looking through the telescope, is sometimes employed. Sensitive instruments are often provided with a micrometer altitude screw, at one end of the crossbar, by means of which screw fine settings of the telescope may be made without moving the leveling screws.

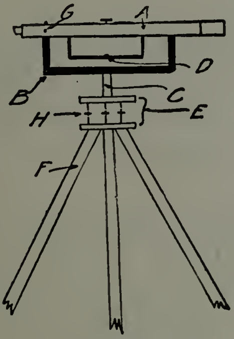


Fig. 84*a*.

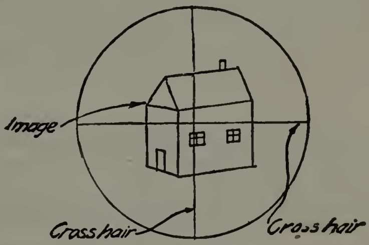


Fig. 84*b*.

85. The Level Tube.—A level tube is a glass vial whose inside upper surface (sometimes also its lower surface) is ground to the arc of a circle in a longitudinal direction. The tube is nearly filled with sulphuric ether or alcohol. The remaining space is occupied by a bubble of air which takes up a position at the high point in the tube. The tube is usually graduated in either direction from the middle, so that, by observing its ends, the center of the bubble may be brought to the mid-point of the tube. This operation is called "centering" the bubble.

A line tangent to the level tube at its mid- or zero-point is called the *axis of the level tube*. When the bubble is centered, the axis of the level tube is a *horizontal line*.

86. Sensitiveness of Level Tube.—If the radius of the circle to which the tube is ground is large, a small vertical movement of one end of the tube will cause a large displacement of the bubble; if small, the reverse is true. The radius of the level tube is thus seen to be a measure of its sensitiveness. The tubes of the better grades of engineers' levels are ground to radii varying roughly from 75 to 150 ft., the average being about 85 ft. The tubes of precise levels have a radius of 300 to 1,000 ft.

The sensitiveness of the level tube is generally expressed in seconds of the central angle whose arc is one division of the tube. The sensitiveness expressed in this manner is inversely proportional to the number of seconds. Thus a 10-sec. level tube (radius about 170 ft.) is twice as sensitive as a 20-sec. level tube (radius about 85 ft.).

87. The Telescope.—Figure 87 shows the principal parts of the telescope as it is commonly constructed. Rays of light emanating from an object within the field of view of the telescope are caught by the objective lens *A* and are brought to a focus and form an image in the plane of the cross-hairs *B*. The lenses of the eyepiece *C* form a microscope which is focused on the image at the cross-hairs. The objective lens is screwed in the outer end of the objective slide *D* which fits in the telescope tube *E*. The objective lens is focused by the screw *F* at the inner end of which is a pinion that engages the teeth of a rack fixed to the objective slide. The eyepiece slide is held in position laterally by rings *H*, *J*, through which it may be moved in a longitudinal direction for focusing. By means of screws the ring *J* may be moved laterally so that the intersection of the cross-hairs will appear in the center of the field of view.

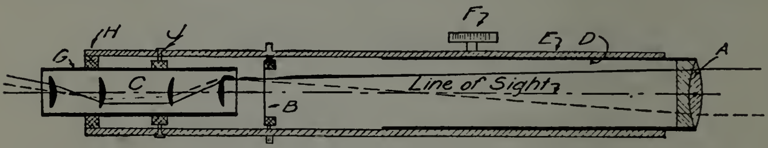


FIG. 87.—Longitudinal section of telescope.

The line of sight is defined by the intersection of the cross-hairs and the optical center of the objective lens. The cross-hairs may be so adjusted that the line of sight and the optical axis coincide.

88. Focusing.—In using the telescope, the eyepiece is first moved in or out until the cross-hairs appear sharp and distinct, after which no further focusing is necessary. When an object is sighted, the objective slide is moved in out or until the image appears clear, when it *should* be in the plane of the cross-hairs.

If a slight movement of the eye from side to side produces an apparent movement of the cross-hairs over the image, the plane of the image and the plane of the cross-hairs do not coincide and *parallax* is said to exist. Since parallax is a source of error in all observations it should be eliminated by refocusing the objective until further trial shows no apparent movement. The objective lens must be focused for each distance sighted. The nearer the object sighted, the greater must be the distance between the objective and the cross-hairs.

89. The Objective.—The principal function of the objective is to form an image for sighting purposes. For accuracy of measurements it should also produce an image which is well lighted, accurate in form, distinct in its outlines, and free from discolorations.

These desirable features of the image are obtained in most surveying instruments by providing an objective consisting of an outer double-convex lens of crown glass and an inner concavo-convex lens of flint glass. The two lenses are usually cemented together with balsam.

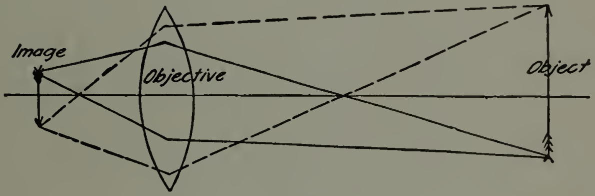


Fig. 89.

Figure 89 illustrates the manner in which rays from an object are deviated by the objective and brought to a focus to form the image. It will be noted that the image is inverted.

90. The *optical center* of the objective may be defined as that point in the lens through which a ray of light will pass without permanent deviation, regardless of the direction of the object from which the light emanates.

The *optical axis* is defined as the line taken by a light ray that experiences no deviation either on entering or on leaving the objective. It passes through the optical center and the centers of curvature of the lens.

The *principal focus* is a point on the optical axis back of the objective where rays entering the telescope parallel with the optical axis are brought to a focus; or it is a point in front of the objective from which diverging light rays entering the lens emerge from it parallel with the optical axis. Stated in another form, the image of a point on the optical axis and an infinite distance away is at the principal focus back of the objective. If a point is at the principal focus in front of the objective, however, it will have no image.

The *focal length* of the objective is the distance from its optical center to the principal focus. When the telescope is focused on a distant point, the focal length is very nearly the distance from the plane of the cross-hairs to the optical center of the objective, for reasons which the preceding paragraph makes clear.

91. Objective Slide.—Any lateral movement of the objective causes a deviation in the position of the optical axis and also in the line of sight, thereby introducing errors in measurements. The objective slide should therefore fit as neatly as possible and still admit readily of longitudinal movement for focusing. The workmanship on any good instrument is sufficiently precise to insure practical elimination of errors of this sort when the telescope is new, but in the course of long use, wear develops between the sliding parts, and the slide becomes loose. This produces uncertainties in observation which no amount of adjustment can overcome.

Particular care should be taken to protect the objective slide from dust, water, and other foreign matter. Most instruments are equipped with a guard which affords at least partial protection. If the slide is lubricated at all, only a drop of the finest watch oil should be used and all excess oil should be removed with a soft cloth. If the objective lens is removed it should be replaced as nearly as possible in its original position.

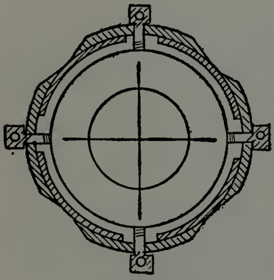


Fig. 92.

92. The Cross-hairs.—Threads from the cocoon of the brown spider are usually used as cross-hairs. Two threads, one vertical, the other horizontal, are fastened to a metal ring called the cross-hair ring or reticule, as shown in Fig. 92. The ring is held in position by four capstan-headed screws which pass through the telescope tube and tap into the ring. The holes in the telescope tube are slotted so that when the screws are loosened, the ring may be rotated through a small angle about its own axis. The ring is smaller than the inside of the tube, and it may be moved either horizontally or vertically by means of the screws. Thus, to move it to the left, the right-hand screw is loosened, and the left-hand one is tightened.

93. Stadia Hairs.—Two additional horizontal threads that are placed one on each side of the horizontal cross-hair are called stadia hairs. Usually they are mounted in the same plane with the cross-hairs and hence when the eyepiece is in focus all four hairs appear in the field of view.

94. The Eyepiece.—Attention has previously been drawn to the fact that the image formed by the objective is inverted. Eyepieces are of two general types.

The erecting or terrestrial eyepiece, the more common of the two types, reinverts the image so that the object appears to the eye in its normal position. In the common form it consists of four plano-convex lenses placed in a metal tube, called the eyepiece slide.

The inverting or astronomical eyepiece simply magnifies the image without reinverting it. It is composed of two plano-convex lenses.

For either eyepiece the ratio of the angle at the eye subtended by the magnified image, to that subtended by the object itself, is the magnifying power of the telescope.

The single advantage of the erecting eyepiece is that objects appear in their natural position, and this is the reason why its use is so common. Each lens which is interposed between the object and the eye absorbs some of the light which strikes it. Hence, other things being equal, the object is more brilliantly illuminated when viewed through the inverting eyepiece, and this is a great advantage, particularly when observations are made during cloudy days or near nightfall.

The slide of the erecting eyepiece is usually held in position by rings the inner ring being adjustable laterally. In most instruments the slide is held tightly by spring friction and it is focused by a screw-like motion.

The slide of the inverting eyepiece is usually held by a single wide ring which is fixed in the end of the telescope tube and which admits of no lateral adjustment.

The magnifying power of the telescope may be determined by observation. For the better grade of engineers' levels it is about 30 diameters. Some precise levels have a magnifying power as high as 50 diameters. The U. S. Coast Survey type of precise level has a magnification of 43 diameters. The magnification of transit telescopes ranges from 18 to 25 diameters.

95. Relation between Magnifying Power and Sensitiveness of Level Tube.—It is desirable that the sensitiveness of the level tube be such that for the smallest noticeable movement of the bubble there is an apparent movement of the cross-hairs on a level rod held at an average distance from the instrument; and likewise for the smallest noticeable movement of the cross-hairs there should be an observable movement of the bubble.

96. The Dumpy Level.—Figure 96 shows the details of a dumpy level with inverting eyepiece. The telescope is rigidly attached to the crossbar and the instrument is so constructed that the optical axis of the telescope is perpendicular to the axis of the spindle. The

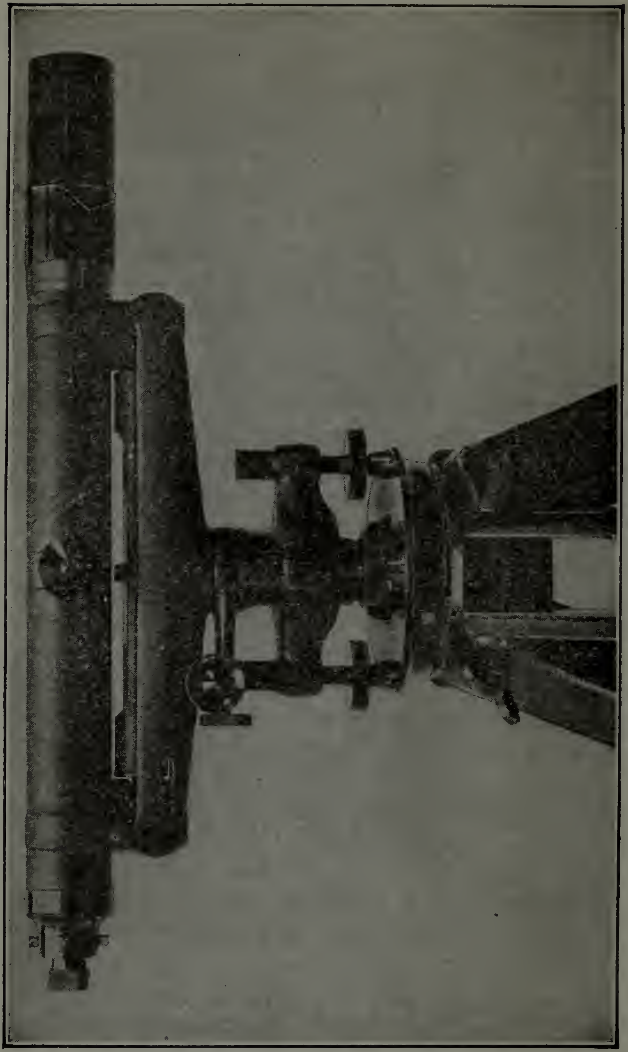


Fig. 96.—Engineers' dumpy level (inverting eyepiece).

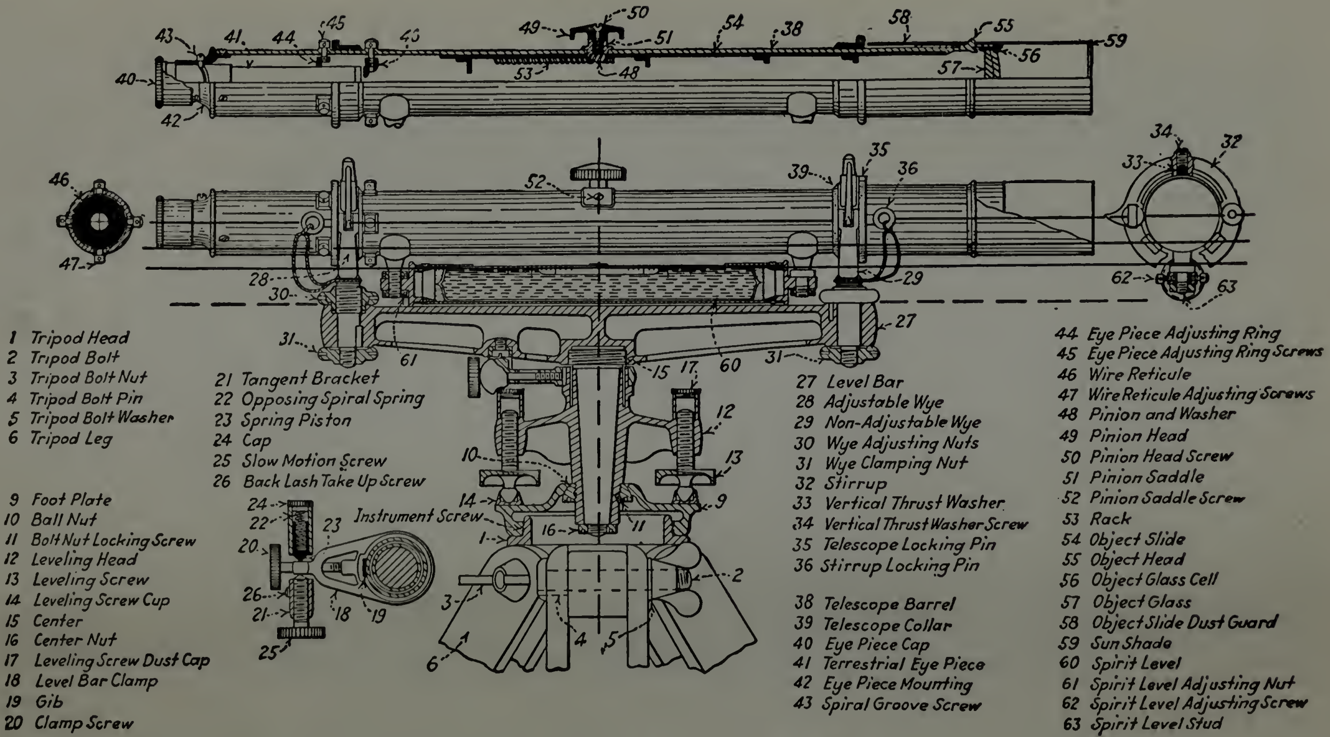
level tube is permanently placed so that its axis lies in the same vertical plane as the optical axis, but it is adjustable in altitude by means of a capstan-headed screw at one end. The spindle revolves in the socket of the leveling head which is controlled by the leveling screws. At the lower end of the spindle is a ball and socket joint which makes a flexible connection between the instrument proper and the foot plate. When the leveling screws are turned, the level is moved about this joint as a center. Above the leveling head is a collar and a clamp screw by means of which the spindle may be clamped to the leveling head. The tangent screw controls small movements of the level about its vertical axis after the clamp screw is tightened. The sunshade protects the objective from the direct rays of the sun. The adjusting screws for the cross-hair ring are near the eyepiece end of the telescope.

The dumpy level is more often equipped with an inverting eyepiece, and therefore for the same magnifying power it is considerably shorter than the wye level which is usually supplied with an erecting eyepiece. Its advantages over the wye level are that it is simpler in construction, has fewer parts that are subjected to wear, requires a lesser number of adjustments, and stays in adjustment better. It is pertinent to add that the most highly refined instruments used in precise leveling are modified forms of the dumpy level.

97. The Wye Level.—Figure 97 shows the details of a wye level with erecting eyepiece. The telescope rests in Y-shaped bearings called the wyes. The leg of each wye is threaded. It passes through the crossbar and is secured in position by capstan-headed nuts. By means of these nuts, one of the wyes may be raised or lowered. The telescope is secured in position by the wye clips. When the clips are raised the telescope may be revolved in the wyes or it may be lifted from the wyes and turned end for end. The enlarged portions of the telescope barrel which rest in the wyes are cylindrical in shape and are called the *rings*. The line joining the centers of the rings is defined as the *axis of the rings* or the *axis of the wyes*. The telescope is held longitudinally by flanges on the rings which bear against the wyes. When the clips are fastened, the telescope is held from turning about its axis by a lug on one of the clips engaging in a groove in the flange of one of the rings.

The level tube is attached to the telescope and is adjustable in both azimuth and altitude. Other details are much the same as for the dumpy level just described.

The distinguishing characteristics of the wye level are that the telescope may be revolved in its bearings and may be lifted from its bearings and turned end for end. These features considerably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1	Tripod Head
2	Tripod Bolt
3	Tripod Bolt Nut
4	Tripod Bolt Pin
5	Tripod Bolt Washer
6	Tripod Leg
9	Foot Plate
10	Ball Nut
11	Bolt Nut Locking Screw
12	Leveling Head
13	Leveling Screw
14	Leveling Screw Cup
15	Center
16	Center Nut
17	Leveling Screw Dust Cap
18	Level Bar Clamp
19	Gib
20	Clamp Screw
21	Tangent Bracket
22	Opposing Spiral Spring
23	Spring Piston
24	Cap
25	Slow Motion Screw
26	Back Lash Take Up Screw
27	Level Bar
28	Adjustable Wye
29	Non-Adjustable Wye
30	Wye Adjusting Nuts
31	Wye Clamping Nut
32	Stirrup
33	Vertical Thrust Washer
34	Vertical Thrust Washer Screw
35	Telescope Locking Pin
36	Stirrup Locking Pin
38	Telescope Barrel
39	Telescope Collar
40	Eye Piece Cap
41	Terrestrial Eye Piece
42	Eye Piece Mounting
43	Spiral Groove Screw
44	Eye Piece Adjusting Ring
45	Eye Piece Adjusting Ring Screws
46	Wire Reticule
47	Wire Reticule Adjusting Screws
48	Pinion and Washer
49	Pinion Head
50	Pinion Head Screw
51	Pinion Saddle
52	Pinion Saddle Screw
53	Rack
54	Object Slide
55	Object Head
56	Object Glass Cell
57	Object Glass
58	Object Slide Dust Guard
59	Sun Shade
60	Spirit Level
61	Spirit Level Adjusting Nut
62	Spirit Level Adjusting Screw
63	Spirit Level Stud

Fig. 97.—Cross-section of wye level.

facilitate the making of adjustments provided the bearings are not worn. When the bearings are worn it is impossible to adjust the instrument by the usual methods, and adjustments, if correctly made, are as difficult as for the dumpy level.

98. Hand Levels.—The Locke hand level is a serviceable device that is widely used for rough leveling. It consists of a brass tube about 6 in. long on which is mounted a level bubble as shown in Fig. 98. In the tube beneath the bubble is a prism. An opening is left in the tube, and the image of the bubble is reflected by the prism to the eye end of the level. Just beneath the bubble vial is a cross-wire which is adjustable by means of a screw, the head of which protrudes through the end of the case enclosing the vial. The eyepiece consists of a peep hole mounted in the end of a slide which fits inside the tube and is held in a given position by friction. Mounted on the right half of the inner end of the slide is a semicircular convex lens which magnifies the image of the bubble and cross-wire as reflected by the prism. Both the object and the eye ends of the tube are closed by discs of plain glass so that dust will not collect on the prism and lens. The magnifying lens is focused by moving the eyepiece slide in or out. In using the level the object is viewed directly through the left half of the tube, without magnification, while with the same eye and at the same time the position of the bubble with respect to the cross-wire is observed apparently in the right half of the field of view. The level is held with the bubble vial uppermost and is tipped up or down until the cross-wire bisects the bubble, with the line of sight is horizontal. After a little practice one may make observations with greater facility if he keeps both eyes open.

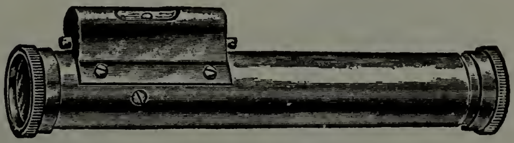


Fig. 98.—Locke hand level.

99. The Abney Level and Clinometer (Fig. 99).—As its name indicates, this level is suitable for both direct leveling and for measuring the angles of slopes. When it is employed as a level, the index of the vernier is set at zero, and it is then used in the same way as the Locke hand level. When it is used as a clinometer, the object is sighted and the level tube is moved by means of its control screw until the cross-wire bisects the bubble as viewed through the eyepiece.

100. Leveling Rods.—These are graduated wooden rods of rectangular cross-section by means of which difference in elevation is measured. In the United States, the rods are ordinarily graduated into feet and decimals. On some Government surveys the rods are graduated into decimals of the meter or the yard. The lower or ground end of the rod usually, though not always, is the point of zero measurement from which the graduations are numbered.

The rod is held vertically, and hence the reading of the rod as indicated by the horizontal cross-hair of the level is a measure of the vertical distance between the point on which the rod is held and the line of sight.

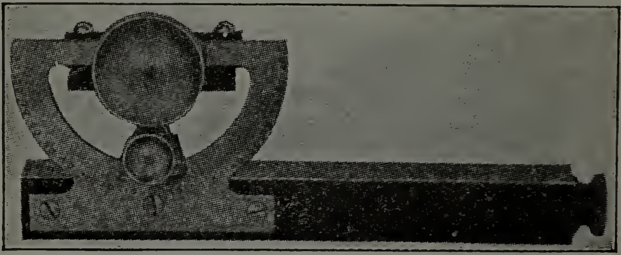


Fig. 99.—Abney level and clinometer.

Rods are obtainable in a variety of patterns and graduations, and are either in single pieces or in sections which are jointed together or slide past each other and are clamped together. A common length is 12 ft. The two general classes of leveling rods are: (1) self-reading rods, those that may be read directly by the observer as he looks through the telescope of the level; and (2) target rods, those for which a target, sliding on the rod, is set by the rodman as directed by the leveler.

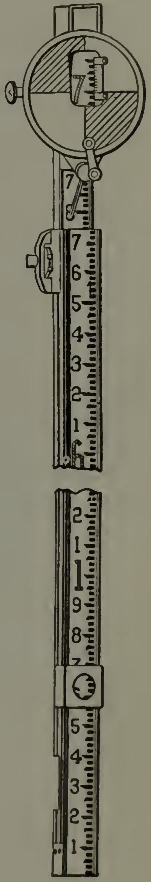
101. Self-reading Rods.—With the self-reading rods, the rodman simply holds the rod vertically; the leveler observes the graduation at which the line of sight intersects the rod, and records the reading. In most of the operations of leveling, the self-reading rod is much superior to the target rod in point of the speed with which observations may be made, and is nearly as accurate. It is the type most widely used at the present time, even for leveling of the highest precision. Observations closer than the smallest division on the rod are made by estimation.

The self-reading rod should be so marked that the graduations appear sharp and distinct for any normal distance between level and rod. Most commonly the background is white with graduations 0.01

ft. wide painted in black as shown in Fig. 101. The numbers indicating feet are in red and those indicating tenths of feet are in black. This makes a very satisfactory graduation when the length of sight is less than 400 or 500 ft. For sights of greater length rods with larger blocks of contrasting color are desirable.

The Philadelphia rod (Fig. 101) is the most widely used of any rod. It is equipped with a target which makes it possible to use it also as a target rod. It is usually in two sections, the parts being held in contact by two brass sleeves. By means of a screw attached to the upper sleeve the two parts of the rod may be clamped together in any relative position desired. For readings of less than 7 ft. the back strip is clamped in its normal position; for greater readings, the rod is extended its full length, when the graduations on the front face of the back strip are a continuation of those on the front strip.

102. Target Rods.—With the target rod, the leveler signals the rodman to slide the target up or down until it is bisected by the line of sight. With the target clamped in this position, the rodman, leveler, or both observe the indicated reading. Usually the target is equipped with a vernier or other device by means of which fractional measurements of the rod graduations may be read without estimation. The principal advantage of the target rod, in most leveling operations, is that mistakes are less likely to occur, particularly if both rodman and leveler read the rod. Under certain conditions its use materially facilitates the work; for example, when very long sights are taken, when the rod is partially obscured from view, or when it is necessary to establish a number of points all at the same elevation. When it is desired simply to secure readings on points of unknown elevation under the normal conditions of leveling, however, the use of the target rod greatly retards progress without adding much, if anything, to the precision obtained.



Note:- Cross hatched portions indicate red

Fig. 101.—Philadelphia rod.

The Philadelphia rod (Fig. 101) described in the preceding article, although employed as a self-reading rod, may also be used as a target rod. Lugs on the target engage in a groove on either side of the front strip. For readings on the lower half of the rod the target is moved in these grooves to the desired position. The reading is made by means of a vernier attached to the target. Graduations on the back of the rear strip are a continuation of those on the front strip and read downwards. On the back of the top sleeve is a vernier employed for observations with the rod extended. For readings greater than can be taken with the "short" rod, the target is set at the uppermost graduation on the face, and the rod is extended until the target is bisected by the line of sight. The vertical distance from foot of rod to target is then indicated by the back vernier reading. The verniers read to thousandths of feet.

The usual target (Fig. 103*a*) is a circular disc about 5 in. in diameter with horizontal and vertical lines formed by the junction of alternate quadrants of red and white. A rectangular space in the front of the target exposes a portion of the rod to view so that readings may be taken. The attached vernier fits closely to the rod, its zero point or index being at the horizontal line of the target.

103. Verniers.—The vernier is a small scale placed parallel and in contact with the main scale by means of which fractional parts of the least division on the rod may be accurately measured. Using the Philadelphia rod, as an example, the main scale is divided into hundredths of feet. If a plain target were used and it were set so that its index lay between, let us say, 6.15 ft. and 6.16 ft., a reading to thousandths of a foot would necessarily be estimated and would be uncertain in the last decimal place. Thus, one person might read 6.152 ft. and another might read 6.153 ft. If a vernier target were employed the reading could be made to the nearest thousandth of a foot without estimation and would be correct in the last decimal place.

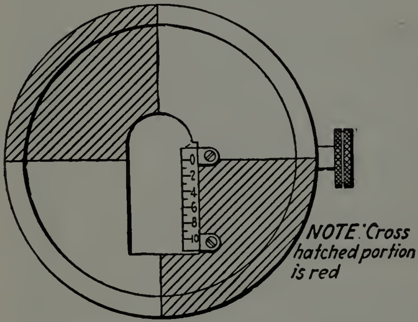


Fig. 103*a*.—Rod target.

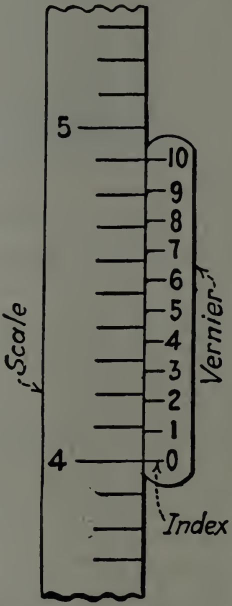


Fig. 103*b*.—Direct vernier.

The accuracy of the vernier depends upon the fact that the eye can more closely determine when two lines coincide than it can determine by estimation the distance between two parallel lines.

Verniers used on leveling rods are of two types: (1) *the direct vernier*, which is so graduated that there is one more space within its length than exists in the corresponding distance on the main scale (Fig. 103*b*); and (2) the *retrograde vernier*, which has *one less* space within its length than exists in the corresponding distance on the main scale (Fig. 104*b*).

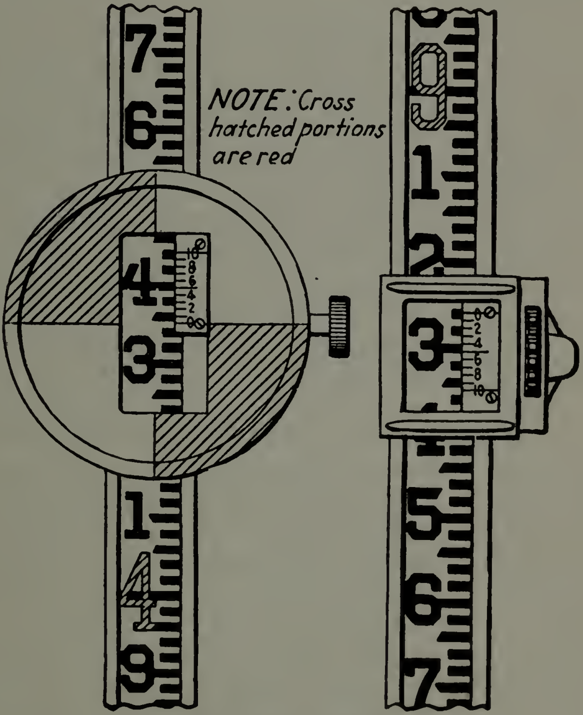


Fig. 104*a*.—Direct vernier settings.

104. The direct vernier is so graduated that 10 spaces are equal to 9 spaces or 0.09 ft. on the scale. Thus each space on the vernier is equal to 0.009 ft. In Fig. 103*b* the index of the vernier is set at 0.400 ft. If the vernier were moved upward 0.001 ft., its graduation numbered 1 would coincide with 0.41 ft. on the scale, and its index would be at 0.401 ft.; if it were moved upward 0.002 ft., its graduation numbered 2 would coincide with 0.42 ft. and the index would be at 0.402 ft. It is thus seen that the position of the index to the vernier is determined to thousandths of feet without estimation simply by noting where a graduation on the vernier coincides with one on the scale.

In general, if n is the number of spaces on the vernier whose total length is equal to that of $(n - 1)$ spaces on the scale, and if v is the length of a space on the scale, then the fineness of reading or *least count* of the vernier is $\frac{v}{n}$.

Figure 104*a* illustrates a setting of the vernier on the target and also on the back of a Philadelphia rod. The rod reading indicated by the target (4.347 ft. in figure) is determined by first observing the position of the vernier index on the scale to hundredths of feet (4.34 in figure), next by observing the number of spaces *on the vernier* from the index to the coinciding graduations (7 spaces in figure), and finally by adding the vernier reading (0.007 ft. in figure) to the scale reading (4.34 ft.). It should be noted that the coinciding graduation on the rod scale *does not* indicate the rod reading. Observations of the vernier on the back of the rod are made in the same manner except that readings of both scale and vernier are made by counting down the rod. Thus in Fig. 104*a* the scale reading is seen to be 9.26 ft. and the vernier reading is 0.004 ft.; hence the rod reading is 9.264 ft. It will be observed that the readings of the direct vernier increase in the same direction as the scale readings, that is, up the rod when the target is set on the "short rod" and down the rod when the "long rod" is used.

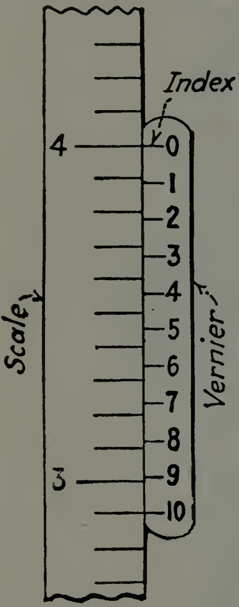


Fig. 104*b*.—Retrograde vernier.

104a. Figure 104*b* illustrates the manner in which the retrograde vernier is graduated. Ten spaces on the vernier are equal to 11 spaces or 0.11 ft. on the scale; hence each space on the vernier is equal to 0.011 ft. and the difference in length between one space on the vernier and one on the scale is 0.001 ft. The index is set at 0.400 ft. If the index is moved upward 0.001 ft. the vernier graduation numbered 1 will coincide with 0.39 ft.; if moved up 0.002 ft., the vernier graduation numbered 2 will coincide with 0.38 ft.; and so on. Thus the retrograde vernier is read in the same manner as the direct vernier.

In general, for the retrograde vernier, if n is the number of spaces on the vernier in a length of $(n + 1)$ spaces on the scale, and v is the length of one space on the scale, then the least count of the vernier is $\frac{v}{n}$.

The only advantage of the retrograde vernier is that its spaces are longer and hence it can be read somewhat more accurately than the direct vernier. For leveling rods, both types of vernier are in common use.

105. Turning Points.—A metal plate or pin which will serve temporarily as a solid or stable object on which the leveling rod may be held at turning points (the purpose of which will later be described) is a useful, if not necessary, part of the leveling equipment for careful lines of differential levels.

106. Rod Levels.—The rod level is an attachment for indicating the verticality of the leveling rod. The type in most common use consists of a hinged casting on each wing of which is mounted a level tube. The level is held so that the plane faces of its two wings are in full contact with adjacent sides of the rod. When both of the bubbles are centered, the rod is plumb. It is suitable for use with any type of rod. The hinge makes it possible to fold the level compactly when it is not in use.

Another type consists of a circular or "bull's-eye" level vial mounted on a metal angle or bracket which is attached by screws to the side of the rod.

107. Numerical Problems.

1. What is the combined effect of the earth's curvature and mean atmospheric refraction in a distance of 300 ft.? In a distance of 3,000 ft.? In a distance of 6 miles? In a distance of 60 miles?

2. Two points, A and B, are each distant 2,000 ft. from a third point from which vertical angles to A and B are taken. The vertical angle to A is $+3°21'$ and that to B is $+0°32'$. What is the difference in elevation between A and B?

3. Let A be a point of elevation of 100.00 ft. and B and C be points of unknown elevation. By means of an instrument set 4.00 ft. above B, vertical angles are observed, that to A being $-1°55'$ and that to C being $+3°36'$. If the horizontal distance AB is 1,500 ft. and the horizontal distance BC is 5,000 ft. what are the elevations of B and C, making due allowance for earth's curvature and atmospheric refraction?

4. Two points, A and B, are 1,000 ft. apart. The elevation of A is 615.03 ft. A level is set up on the line between A and B and at a distance of 250 ft. from A. The rod reading on A is 9.15 and that on B is 2.07. Making due allowance for curvature and refraction what is the elevation of B? What would be the magnitude and sign of the error introduced if the correction for curvature and refraction were omitted?

5. Two points, *A* and *B*, are 400 ft. apart and the elevation of *A* is 615.037 ft. A level is set up on line and distant 100 ft. from *A* in the direction of *B*. The rod reading on *A* is 5.812 and on *B* is 7.358. What is the elevation of *B*, neglecting the correction due to curvature and refraction? What is the magnitude and size of the error introduced by not considering the effect of curvature and refraction?

6. A sight is taken with an engineers' level at a rod held 300 ft. away and an initial reading of 6.323 ft. is observed. The bubble is then moved through five spaces on the level tube, when the rod reading is 6.589 ft. What is the sensitiveness of the level tube in seconds of arc? What is the radius of curvature of the level tube if one space is $\frac{1}{10}$ in.?

7. In viewing the rod through the telescope of a level the magnified image of the portion between 5.00 and 5.20 apparently covers the unmagnified image between 2.1 and 8.4. What is the magnifying power of the telescope?

8. Design a direct vernier reading to thousandths of feet, each space on the rod being equal to 0.025 ft.

9. Design a retrograde vernier, other conditions remaining as in Problem 8.

CHAPTER VIII

DIFFERENTIAL LEVELING

USE AND ADJUSTMENTS OF THE LEVEL

108. Setting up the Level.—When observations are to be made the level is placed in a desired position with tripod legs well spread and firmly pressed into the ground and with the tripod head somewhere near level. The telescope is brought over one pair of foot screws and the bubble is approximately centered; then the process is repeated with the telescope over the other pair. By repetition of this procedure the foot screws are manipulated until the bubble remains centered, or nearly so, for any direction in which the telescope is pointed. If the instrument is in adjustment the line of sight is then horizontal.

Following are items deserving attention:

1. The legs should be spread at such an angle that the tripod is stable and at the same time the object may be viewed through the telescope from a convenient and easy posture.

2. The friction bearings between tripod head and legs should be tightened by means of the wing nuts until each leg when held horizontally will barely fall from its own weight.

3. The foot screws should be tightened no more than is necessary to secure firm bearing. If the bubble is considerably off center it is best to leave all four screws rather loose until it is brought approximately level. Usually the final centering of the bubble will be facilitated by turning one screw rather than by attempting to manipulate two opposite screws at the same time.

4. If opposite foot screws are turned by bringing the thumbs toward each other, the bubble will move to the right; and *vice versa.*

109. Reading the Rod.—For accurate observations the rod is held vertically on some well-defined point of a stable object. The leveler revolves the telescope about the vertical axis until the rod is nearly in the middle of the field of view, focuses the objective for distinct vision, and carefully centers the bubble. If the self-reading rod is used, the leveler notes and records the reading indicated by the line of sight (apparent position of the horizontal cross-hair on rod). As a check he again observes the bubble and the rod. If the target rod is used, the procedure is identical except that the target is set by the rodman as directed by the leveler.

For less accurate leveling, as for example, when rod readings for points on the ground are determined to the nearest 0.1 ft., the observations usually are not checked and proportionally less care is exercised in keeping the rod vertical and the bubble centered, always bearing in mind the errors involved and the precision with which measurements are desired.

The following are pertinent suggestions:

1. No part of the body or clothing should come in contact with the level when an observation is being made, and care should be taken not to step near the feet of the tripod on soft or yielding ground. Between the times when the bubble is centered and the rod is read, the leveler should not move his feet except when the stability of the ground is assured.

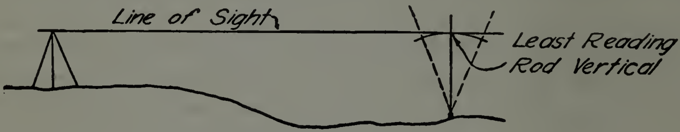


Fig. 109.—Waving the rod.

2. The eyepiece should be correctly focused on the cross-hairs before observations are begun, and the leveler should move his head up and down slightly while viewing the rod to assure himself that no parallax is present (Art. 88).

3. If the air is calm the rodman can plumb the rod quite accurately by balancing it upon the point on which it is held. By means of the vertical cross-hair the leveler can determine when the rod is held in a vertical plane passing through the instrument but has no way of telling whether it is not tipped forward or backward in this plane. If it is in either of these positions, the rod reading will be greater than the true vertical distance as illustrated by Fig. 109. To eliminate this error, the rodman slowly swings the rod forward and backward as shown in the figure, and the leveler takes the least reading, which occurs when the rod is vertical. This procedure is called *waving the rod.* The larger the rod reading, the larger the error due to the rod being held at a given inclination, and hence it is more important to wave the rod for large readings than for small ones.

110. Remarks on Adjustments.—Regardless of the precision of manufacture, all levels, as well as other surveying instruments, in process of use require certain field adjustments from time to time. In the interests of accuracy it becomes an important duty of the surveyor to test his instrument at short intervals and to make with facility such adjustments as are found necessary.

With a few exceptions the nuts and heads of screws are capstan shaped and hence are tightened with a pin. The instrumentman should possess an adjusting pin which fits the holes accurately, and should carry it at all times. To prevent damage to the instrument, it is imperative that the screws should not be unduly tightened. This is particularly true of those threading into the cross-hair ring; it requires no considerable force to strip their threads or to twist off the screw. When a screw has been set to a firm bearing, nothing is gained by further tightening.

The inverting eyepiece is incapable of lateral adjustment. The erecting eyepiece, when adjustable, is set so that the intersection of the cross-hairs appears in the center of the field of view. The ring through which the inner end of the eyepiece moves is controlled by screws which are manipulated in the same manner as those for adjusting the cross-hairs. On some levels these screws are slot-headed and are covered by a cylindrical sleeve. When the sleeve is unscrewed, the heads of the screws are exposed. This adjustment has no effect on the precision of the measurements, and ordinarily requires no attention except when the telescope is being assembled.

The cross-hair ring should be rotated about its axis by loosening two adjacent screws slightly and then tapping one of the screws lightly. The ring should be tightened by turning the same two screws in order that the position of the intersection of the cross-hairs will be disturbed a minimum amount.

The cross-hair ring is adjusted vertically or horizontally by first loosening one screw and then tightening the opposite one. The ring is moved in the direction of the screw which is being tightened. It is helpful to consider the line of sight as a lever whose fulcrum is the optical center of the objective; if the cross-hair ring is lowered, the line of sight for all points in front of the objective is raised.

In some instances one adjustment is likely to be altered by, or depends on, some other made subsequently. For example, the lateral movement of the cross-hair ring may likewise produce a small rotation. And the lateral adjustment of the level tube depends upon the vertical adjustment. Hence if an instrument is badly out of adjustment, related adjustments must be repeated until they are gradually perfected.

111. Adjustments of the Dumpy Level.—For a dumpy level in perfect adjustment the following relations should exist:

1. The axis of the level tube should be parallel with the optical axis and line of sight.

2. The horizontal cross-hair should lie in a horizontal plane when the instrument is leveled.

3. The vertical axis should be perpendicular to the axis of the level tube and the line of sight.[1]

The parts which are capable of, and require, adjustment are the cross-hairs and the level tube. The basis for adjustments is the vertical axis. The adjustments are as follows:

1. *To make the axis of the bubble tube perpendicular to the vertical axis (and hence parallel with the optical axis and the axis of the objective slide).* Set up the level and approximately center the bubble over each pair of foot screws; then bring the bubble carefully to center over one pair. Revolve the level 180° about its vertical axis. If the level tube is in adjustment, the bubble will retain its position. If the tube is not in adjustment the displacement of the bubble indicates double the actual error, as is shown by Fig. 111a. If $(90° - \alpha)$ represents the angle between the vertical axis and axis of level tube, then when the bubble is centered the vertical axis makes an angle of α with the true vertical. When the level is reversed, the bubble is displaced through the arc whose angle is 2α. Hence the correction is the arc whose angle is α. The correction is made by bringing the bubble halfway back to the center by means of the capstan-headed nuts at one end of the tube. The instrument is then releveled with the foot screws, and the process is repeated until the adjustment is perfected. Usually three or four trials are necessary. As a final check the bubble should remain centered over each pair of foot screws.

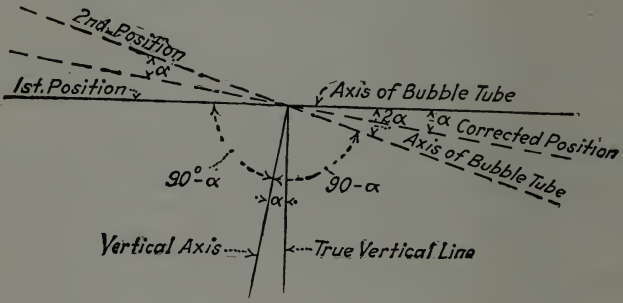


FIG. 111a.

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[1] Also the optical axis, the axis of the objective slide, and the line of sight should coincide, but for the type of level commonly used in the United States, the axis of the objective slide and the optical axis are permanently fixed perpendicular to the vertical axis by the manufacturer, and no provision for further adjustment is made.

2. *To make the horizontal cross-hair lie in a plane perpendicular to the vertical axis* (*and hence horizontal when the instrument is leveled*). Sight the horizontal cross-hair on some clearly defined point (as *A*, Fig. 111*b*) and rotate the instrument slowly about its vertical axis. If the point appears to travel along the cross-hair no adjustment is needed.

If the point departs from the cross-hair and takes some position as *A'* on the opposite side of the field of view, rotate the cross-hair ring until by further trial the above condition is satisfied. The instrument need not be level when the test is made.

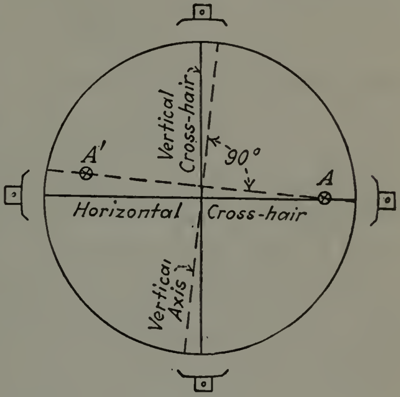

Fig. 111*b*.

3. *To make the line of sight parallel with the axis of the level tube* (*two-peg method*). Set two stakes 200 to 300 ft. apart on approximately level ground. Set and level the instrument in a position such that the eyepiece is ½ in. or less in front of the rod held on one of the stakes as at *A*, Fig. 111*c*. With the rod held at *A*, take a rod reading *a* by sighting through the objective end of the telescope (the eyepiece next to the rod). The cross-hairs will not be visible, but the field of view will be so small (one or two hundredths of a foot) that its center may be determined within one or two thousandths of a foot by holding the point of a pencil on the rod; and this may with sufficient accuracy be called the true rod reading.

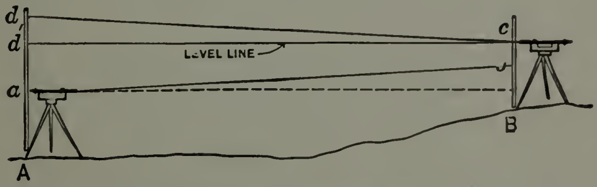

Fig. 111*c*.—The two-peg test.

Move the rod to the other stake *B*, and take a rod reading *b* in the usual manner.

Move the instrument to *B*, set up as before, and take rod readings *c* and *d*.

If $e = dd'$ represents the error in the line of sight for the distance A to B, then considering first the rod readings taken with the instrument at A, the true difference in elevation is

$$\text{Diff. El.} = a - (b - e) \tag{1}$$

and considering the rod readings with the level at B is

$$\text{Diff. El.} = (d - e) - c. \tag{2}$$

Adding (1) and (2) there results

$$\text{Diff. El.} = \frac{(a - b) + (d - c)}{2} \tag{3}$$

Equation (3) shows that the true difference in elevation is the mean of the difference between rod readings taken with the instrument at A and the difference between those taken with the instrument at B.

If the two differences in elevation thus determined are equal, that is, $(a - b) = (d - c)$, the line of sight is in adjustment. If not, then the correct rod reading at A for instrument with position unchanged at B is

$$d' = c + \frac{(a - b) + (d - c)^{1}}{2} \tag{4}$$

The adjustment is made by moving the cross-hair ring vertically until the line of sight cuts the rod at d'. The preceding steps are then repeated as a check on the accuracy of the adjustment.

It should be carefully noted that Eqs. (3) and (4) must be solved algebraically, *i.e.*, with due regard to signs; otherwise, if the error of adjustment should happen to be greater than the difference in elevation of the two points, the mean difference will not be the true difference in elevation.

112. Adjustments of the Wye Level.—For a level in perfect adjustment the following relations should exist:

1. The axis of the objective slide, the optical axis, the line of sight, and the axis of the wyes should coincide.

2. The axis of the level tube should lie in the same plane with and, further, should be parallel with the axis of the wyes.

3. The horizontal cross-hair should be truly horizontal when the instrument is level.

4. For convenience in leveling the vertical axis should be perpendicular to the axis of the level tube and the line of sight.

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[1] Strictly speaking, the effect of the earth's curvature and atmospheric refraction should be added (see Art. 100). For a length of sight of 200 ft. this correction would amount to 0.0008 ft. and for a distance of 300 ft. would be about 0.0018 ft. These quantities are so small as to be of no consequence in ordinary leveling and are negligible for that reason.

The axis of the wyes, the optical axis, and the axis of the objective slide are fixed in the proper relation by the manufacturer, which relation is presumed to be maintained without further attention. The remaining relations are established by the following adjustments:

1. *To make the axis of the level tube lie in the same plane with the axis of the wyes.* Set up and level the instrument. Raise the wye clips and rotate the telescope a few degrees in the wyes. If the relation exists, the bubble will remain centered.

If the bubble moves, bring it back to the center by means of the lateral adjusting screws at one end of the level tube.

2. *To make the axis of the level tube parallel with the axis of the wyes.* Level the instrument carefully, raise the wye clips, lift the telescope from the wyes, and turn it end for end. If the desired relation exists, the bubbles will remain centered.

If the bubble moves, the displacement is double the error (Fig. 111*a*). Hence bring it back halfway to the center by means of the vertical adjusting nuts at one end of the level tube. Level the instrument by means of the foot screws, and repeat the test until the adjustment is perfected.

3. *To make the horizontal cross-hair lie in a plane perpendicular to the vertical axis* (*and hence horizontal when the instrument is level*). This adjustment is exactly the same as for the dumpy level except that for some instruments the adjustment may be made by rotating the telescope in the wyes instead of the cross-hair ring being rotated in the barrel of the telescope. The telescope is fixed in the desired position by means of an adjustable stop attached to one of the wyes.

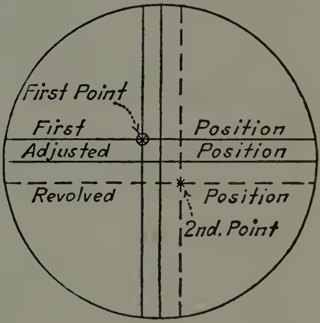


Fig. 112.

4. *To make the line of sight coincide with the axis of the wyes* (*and thus parallel with the axis of the level tube*). Raise the wye clips, sight the intersection of the cross-hairs on some well-defined point, and clamp the vertical axis. Revolve the telescope 180° in the wyes. If the line of sight still remains on the point, the desired relation exists.

If not, the cross-hair ring is adjusted until the line of sight takes a position midway between the point first sighted, and the one defined by the line of sight with the telescope in the revolved position (see Fig. 112). The above process is repeated until the proper relation is obtained. The adjustment is made by manipulating opposite screws, first bringing one cross-hair and then the other to what is estimated to be its correct position. If the required movement is large, however, it is best to loosen two adjacent screws slightly before attempting to align the cross-hairs.

5. *To make the axis of the wyes perpendicular to the vertical axis.* Inasmuch as the preceding adjustments have established parallelism or coincidence between the axis of the level tube, the line of sight, and the axis of the wyes, this adjustment in no wise adds to the precision of observations. The adjustment makes it possible, however, to level the instrument so that the bubble will remain centered for any direction in which the telescope may be pointed.

The test is made by leveling the instrument and revolving the telescope 180° about the vertical axis. If the bubble moves, one-half its displacement indicates the error (see Fig. 111*a*). The correction is made by means of the capstan nuts controlling the vertical position of one of the wyes. The adjustment is seen to be identical with that of the level tube of the dumpy level, with the exception that for the wye level, both the telescope and the level tube are moved in a vertical plane.

113. Regardless of the type of wye level, when the rings or the wyes become worn the adjustments described in Art. 112 are inadequate, and the two-peg test must be made as for the dumpy level. The common procedure is first to perform the adjustments for the cross-hairs in the usual manner, making the line of sight coincide as nearly as may be with the axis of the wyes. Then the true difference in elevation between two points having been determined by the "peg method," the line of sight is set to the proper rod reading for a horizontal line with the foot screws, and the bubble is centered by means of the capstan nuts at one end of the level tube.

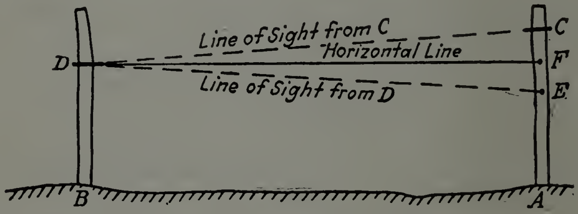


Fig. 114.—Adjustment of the Locke hand level.

114. Adjustment of the Hand Level.—The simplest procedure is to hold the hand level alongside an engineers' level which has been leveled and sighted at some well-defined point. The line of sight of the hand level should strike the same point when the bubble is centered.

The Locke level is adjusted by means of the screw at one end of the level tube, which screw moves the cross-wire defining the line of sight. The adjustment of the Abney level is made by raising or lowering one end of the level tube until the bubble is centered, the index having first been set at zero on the graduated arc.

The hand level may be used to establish a horizontal line by employing the principle of the two-peg test (see Art. 111). Let A and B (Fig. 114) be two posts, trees, or other convenient objects on nearly level ground. The level is held at C and with bubble centered is sighted to the point D. The level is then held at D, and the point E is established in a similar manner. The distance EC represents double the error. The point F established halfway between E and C is therefore on the horizontal line through D.

DIFFERENTIAL LEVELING

115. Differential Leveling.—Every construction enterprise of magnitude requires the establishment of more or less permanent monuments of known elevation, called *bench marks*, to which levels in a given locality may be referred. The nature, location, and interval between monuments of this character depend upon the uses to which they are likely to be put. Throughout the United States, in nearly all cities, and at scattered points in the less populated areas are the permanent bench marks established by the U. S. Geological Survey. They are of bronze set securely in stone or concrete and marked with the elevation above mean sea level. Though intended primarily as control points for the topographic mapping carried on by the Survey, they are very useful as points from which other bench marks may be established either for private or public enterprises. Similar monuments have been established by various other government, state, and municipal agencies, as well as by such private interests as railroads, water companies, and the like; so that generally speaking, one has not far to go before he can find some point of known elevation.

For a new project, levels are run from a bench mark of known or assumed elevation to scattered points in desirable locations for future reference. Such points may be either natural or artificial objects of a nature to be readily identified and of a sufficient permanence to serve the purpose for which they are intended. When the elevation of such points has been determined and their location has become a part of the record of the survey, the points become *bench marks*. The operation of leveling to establish bench marks is called *differential leveling*. Besides the permanent monuments already mentioned, objects frequently used as bench marks are stones, stakes, and pipes driven in the ground, nails in roots of trees, painted marks on street curbs, and spikes in telegraph poles. Differential leveling between two bench marks requires a series of "set-ups" of the instrument along the general route and, for each position of the level, rod readings back to a point of known elevation and forward to a point of unknown elevation are taken.

116. Definitions.—From the preceding article a *bench mark* (B.M.) is seen to be a definite point of more or less permanent character whose elevation and location are known.

A *turning point* (T.P.) is an intervening point between two bench marks upon which rod readings are taken. It may be a pin or plate (see Art. 105) which is carried forward by the rodman after observations have been made, or it may be any stable object which offers itself, as a street curb, railroad rail, stone, etc. The nature of the turning point is generally indicated in the notes, but no record is made of its position.

A *backsight* (B.S.) is a rod reading taken on a point of known elevation, as a bench mark or a turning point. Generally, though not always, it will be taken with the level sighting back along the line, hence the name.

A *foresight* (F.S.) is a rod reading taken on a point whose elevation is to be determined, as on a turning point or on a bench mark that is to be established.

The *height of instrument* (H.I.) is the elevation of the line of sight of the telescope at the instrument when the instrument is leveled.

117. Procedure in Differential Leveling.—In Fig. 117, B.M.1 represents a point of known elevation, as a bench mark, and B.M.2 represents a bench mark some distance away, the elevation of which it is desired to determine. The rod is held at B.M.1 and the level is set up in some convenient location, as L_1, along the general route, B.M.1 to B.M.2. The level is placed in such a location that a clear rod reading is obtainable, but no attempt is made to keep on the direct line joining B.M.1 and B.M.2 (unless, as is sometimes the case, it is desired to obtain rough measurements of distance between bench marks by stadia). A backsight is taken on B.M.1. The rodman then goes forward and, at the direction of the leveler, chooses a turning point T.P.1 at some convenient spot within the range of the telescope along the general route B.M.1 to B.M.2. (Generally no attempt is made to equalize each foresight distance with its corresponding backsight distance, that is, L_1-T.P.1 is not made equal to B.M.1$-L_1$; the chief requirement is that the turning point shall be a stable object at an elevation and in a location favorable to a rod reading of the required precision.) The rod is held on the turning point, and a foresight is taken. The leveler then sets up the instrument at some favorable point as L_2, takes a backsight to the rod held on the turning point, the rodman goes forward to establish a second turning point T.P.2; and so the process is repeated until finally a foresight is taken on the terminal point B.M.2.

Referring again to Fig. 117 it will be seen that the *backsight added to the elevation* of a point on which the backsight is taken gives the *height of instrument* and the *foresight subtracted from the height of instrument* determines the *elevation of the point* on which a foresight is taken. Thus if the elevation of B.M.1 is 721.05 ft. and the B.S. is 7.11 ft., then the H.I. with the instrument set up at L_1 is 721.05 + 7.11 = 728.16. And if the following F.S. is 1.24 ft., the elevation of T.P.1 is 728.16 − 1.24 = 726.92 ft. Also the difference between the backsight taken on a given point and the foresight taken on the following point is equal to the difference in elevation between the two points. It follows that the difference between the sum of all backsights and the sum of all foresights taken between two bench marks gives the difference in elevation between the bench marks.

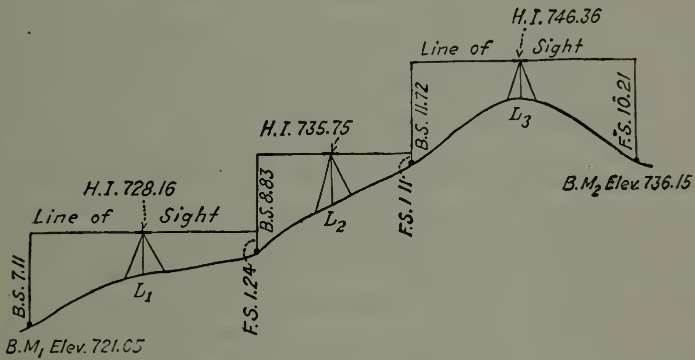


Fig. 117.—Differential leveling.

118. When several bench marks are to be established along a given route, each bench mark is made a turning point in the line of levels. Thus, when the elevation of a bench mark has been determined by taking a foresight on the point, the level is set up in some advanced location, a backsight is taken on the bench mark just established, and so the process is continued. Elevations of bench marks are checked sometimes by rerunning levels over the same route but more often by "tying on" to a previously established bench mark near the end of the line or by returning to the initial bench mark. A line of levels which ends at the point of beginning is called a *level* *circuit.* The final observation in a level circuit is therefore a foresight on the initial bench mark. If each bench mark in a level circuit is also a turning point, and the circuit checks within the prescribed limits of error, it is regarded as conclusive evidence that the elevations of all points in the circuit are correct within prescribed limits. But the fact that a level circuit checks is no indication that the elevation of a bench mark is correct unless the bench mark has been employed as a turning point.

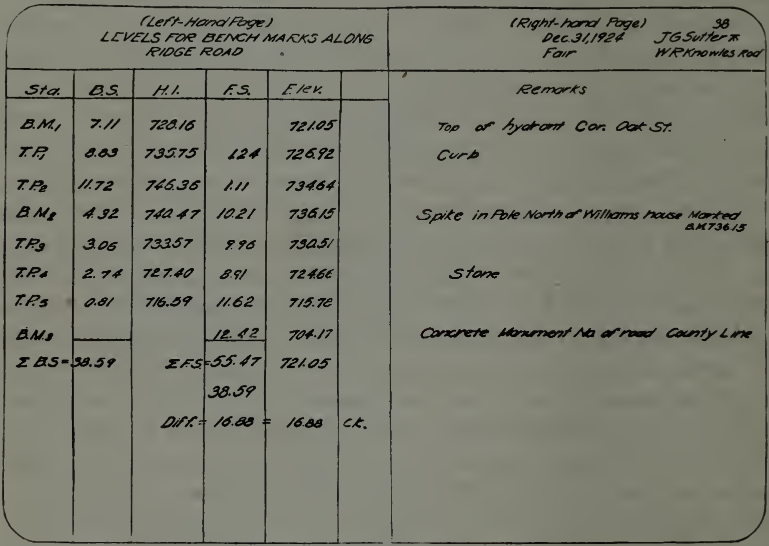
119. As was mentioned in an earlier article, usually no special attempt is made to balance each foresight distance against the preceding backsight distance. As to whether or not such distances are roughly determined and are approximately balanced between bench marks will depend upon the desired precision. It has already been pointed out (Art. 82) that the effect of the earth's curvature and atmospheric refraction is very slight unless there is an abnormal difference between the backsight and foresight distances. The effect of instrumental errors is likely to be of considerably greater consequence. No matter how carefully adjustments have been performed, the chances are that there is not absolute parallelism between the line of sight and the axis of the bubble tube, so that if the instrument were perfectly leveled, the line of sight would be inclined always slightly upward or always slightly downward. Evidently the error in a rod reading due to this imperfection of adjustment would be proportional to the distance from the instrument to the rod and for a given distance would be of the same magnitude and sign for a backsight as for a foresight. Since in the process of leveling backsights are added and foresights are subtracted, it is clear that instrumental errors of this nature are completely eliminated if, between bench marks, the sum of the backsight distances is made equal to the sum of the foresight distances.

In ordinary leveling no special attempt is made to equalize these distances if there is assurance that the instrument is in fair adjustment. Normally, for levels run over flat or gently rolling ground, the line of sight will fall within the length of the rod regardless of the position of the instrument and the distance between instrument and rod is governed by the optical qualities of the telescope. When the leveler moves forward to set up beyond the point where the rod is held he generally paces or estimates by eye a distance he assumes to be about the proper maximum length of sight, and when the rodman moves forward to set a turning point beyond the instrument, he does the same thing. In leveling uphill or downhill the length of sight is usually governed by the slope of the ground. In order that maximum distances between turning points may be obtained and hence progress be most rapid, the leveler sets up the instrument in a position such that the line of sight will intersect the rod near its top if the route is uphill, or near its bottom if the route is downhill. And he directs the rodman to a similarly favorable location for the turning point.

While between bench marks at the same (or nearly the same) elevation the backsight and foresight distances will tend to balance in the long run, regardless of the character of the terrain, it is instructive to note that a very small inclination of the line of sight may normally be expected to produce a marked error in levels which are run between two points having a large difference in elevation, unless some attempt is made to balance backsight and foresight distances. Consider running from sea level to a higher elevation a distance of 40 miles, the intervening ground averaging, say, 2½ ft. in 100 ft., so that, starting from the lower point the backsights would be near the top of the rod (say at 11.5 ft.) and the foresights would be near the bottom (say at 1 ft.). If the instrument were placed directly between turning points and the telescope were 4.5 ft. above the ground at each set up, the backsight distance would on the average be about twice as great as the foresight distance, hence the effect of any error in the line of sight would be twice as great for backsights as for foresights. Calculations show that the sum of foresights would be about 1⅓ ft. too large; hence the calculated elevation of the terminal bench mark would be approximately 1.3 ft. too great. This serves to illustrate how a relatively small error in adjustment may produce a large systematic error in elevation when backsight distances are consistently larger than foresight distances, or *vice versa;* and further, that under certain conditions, even for leveling of relatively low precision with a level in fair adjustment, balancing these distances, at least roughly, cannot be neglected.

Considering the fact that no instrument is likely to be in perfect adjustment and further that the effect of the earth's curvature is not a negligible quantity, it is clear that the equalization of backsight and foresight distances between bench marks is a necessity for accurate leveling, and hence these distances, as well as the rod readings, become a part of the record of leveling operations. In less refined leveling, distances are usually determined by pacing; in precise leveling, they are measured with the stadia.

120. Differential Level Notes.—For ordinary differential leveling when no special effort is made to equalize backsight and foresight distances between bench marks, the record of field work is usually kept in the form indicated by Fig. 120. In the same horizontal line with each turning point or bench mark shown in the first column, are all data concerning that point. The levels from B.M.1 to B.M. 2 are the same as shown by Fig. 117. The H.I.'s and elevations are calculated as the work progresses. Thus, when the backsight (7.11) has been taken on B.M. 1, it is added to the elevation (721.05) to determine the H.I. (728.16). The height of instrument is recorded on the same line with the backsight by means of which it is determined. When the first foresight (1.24) is observed, it is recorded on the line below and is subtracted from the preceding H.I. (728.16) to determine the elevation of T.P. 1 (726.92). And so the notes are continued. Usually at the foot of each page of level notes, the calculations are checked by taking the difference between the sum of the backsights and the sum of the foresights, and the difference between the initial and final elevation, as is illustrated by the numerical work at the bottom of Fig. 120. When these two differences agree it signifies that the additions and subtractions are correct but does not check against mistakes either in observing or recording. Bench marks should be briefly but accurately described and should be so marked in the field that they can be readily indentified. They are usually marked with paint or with crayon that will withstand the effects of the weather. When the bench mark is on stone or concrete the position is often indicated by a cross cut with a cold chisel. A bench mark may or may not be marked with its elevation. Whenever there might arise any question as to the exact position of the point on which the rod was held, its nature should be clearly indicated in the notes. A descrip-



(Left-Hand Page)
LEVELS FOR BENCH MARKS ALONG
RIDGE ROAD

(Right-hand Page) 38
Dec.31,1924 J.G.Sutter ⊼
Fair W.R.Knowles Rod

Sta.	B.S.	H.I.	F.S.	Elev.		Remarks
B.M.$_1$	7.11	728.16		721.05		Top of hydrant Cor. Oak St.
T.P.$_1$	8.83	735.75	1.24	726.92		Curb
T.P.$_2$	11.72	746.36	1.11	734.64		
B.M.$_2$	4.32	740.47	10.21	736.15		Spike in Pole North of Williams house Marked B.M.736.15
T.P.$_3$	3.06	733.57	9.96	730.51		
T.P.$_4$	2.74	727.40	8.91	724.66		Stone
T.P.$_5$	0.81	716.59	11.62	715.78		
B.M.$_3$			12.42	704.17		Concrete Monument No. of road County Line
Σ B.S.=	38.59	Σ F.S.=	55.47	721.05		
			38.59			
		Diff.=	16.88 =	16.88	ck.	

FIG. 120.—Differential level notes.

tion of turning points is of no particular importance unless the points are on objects that can be identified and might therefore become of some value in future leveling operations. Such points are usually marked with crayon, and very briefly described in the notes.

121.—When backsight and foresight distances are to be balanced the form of notes is the same, except that these distances are usually recorded in the last column of the left-hand page.

122. Precise Differential Leveling.—While the subject of precise leveling as practiced on government surveys is not to be considered here, it is appropriate to call attention to certain refinements by means of which a relatively high degree of precision may be obtained with the ordinary wye or dumpy level and the self-reading rod.

For work of this nature the rods should be treated in some manner to prevent expansion or contraction through change in moisture content and at intervals should be compared with a standard length. They should have attached rod levels for plumbing. It is particularly important that turning points be on solid objects with rounded tops so that the base of the rod may be held in the same position for both backsight and foresight. For example, the turning pin or turning plate described in a preceding article would be superior to a street curb.

The level should be equipped with stadia hairs in addition to the regular cross-hairs. Preferably it should be of the dumpy type with inverting eyepiece and reflecting mirror by means of which the bubble may be viewed at the instant the rod is read. To prevent unequal thermal expansion the level should be protected from the sun's rays by an umbrella. It should be set up very firmly so that no settlement will occur, and to eliminate as far as possible the effects of any change in atmospheric refraction, settlement of the tripod, or warping of the level, it is desirable that the shortest possible time elapse between a backsight and the succeeding foresight. The backsight and foresight distances are determined preferably by stadia but sometimes by pacing and are balanced very closely between bench marks.

Excellent results have been obtained by employing two rods and two rodmen, each occupying alternate turning points. If the instrument is not equipped with a reflecting mirror, an assistant should keep the bubble centered while the leveler is making the observations. The assistant should also act as a recorder. All three horizontal hairs should be read by estimation to thousandths of feet and the readings should be recorded. The mean of the readings for the three hairs is taken as the correct rod reading for each sight. The interval between the reading of the upper hair and that of the lower hair is a measure of the distance from instrument to rod. In order further to eliminate possible systematic errors the order of readings may be interchanged at alternate set-ups of the level; that is, at one set-up, the backsight may be observed before the foresight, and at the next set-up the foresight may be determined before the backsight. This would be practicable only when two rodmen are employed.

123. Reciprocal Leveling.—On occasion it becomes necessary to determine accurately the relative elevations of two intervisible points a considerable distance apart, between which points levels can not be run in the ordinary manner. For example, it may be desired to transfer levels from one side to the other of a deep canyon, or from bank to bank of a wide stream. With certain modifications the method employed in getting the true difference in elevation between two points prior to the adjustment of the line of sight of the dumpy level (*i.e.*, the two-peg test) may be utilized in situations of this kind.

If A and B are two such points, then the level is set up near A and one or more rod readings are taken on both A and B. Then the level is set up in a similar location near B and rod readings to near and distant points taken as before. The mean of the two differences in elevation thus determined is taken to be the true difference between the two points. Usually the distance between points is large (often a half-mile or more) so that it is necessary to use a target on the distant rod. If precise results are desired, a series of foresights is taken on the distant rod and sometimes also a series of backsights on the near rod, the bubble being recentered and the target reset after each observation. The difference in elevation is then computed by using the mean of the backsights and the mean of the foresights.

124. Errors.—In leveling, errors are due to some or all of the following causes:

1. *Imperfect Adjustment of the Instrument.*—Insofar as results are concerned there is but one adjustment that is essential, namely, that the line of sight shall be parallel to the axis of the level tube. Any inclination between these lines causes a systematic error, for then if the bubble were perfectly centered, the line of sight would be inclined always slightly upwards or always slightly downwards. Evidently the error in a rod reading due to this imperfection would be proportional to the distance from the instrument to the rod, and for a given distance would be of the same magnitude and sign for a backsight as for a foresight. Since backsights are added and foresights are subtracted, it is clear that the error in elevations will be completely eliminated to the extent that, between bench marks, the sum of the backsight distances is made equal to the sum of the foresight dis-

tances. And conversely, a systematic error will result to the extent that these distances are not equalized between any two bench marks.

The effect of the imperfect adjustment of the instrument is to be minimized (1) by adjusting the instrument and (2) by balancing backsight and foresight distances.

2. *Parallax.*—This condition produces an accidental error. It may be nearly eliminated by careful focusing.

3. *Earth's Curvature.*—This produces an error only when backsight and foresight distances are not balanced. Under ordinary conditions these distances do not tend to vary greatly and whatever error arises from this source is accidental in nature and in ordinary leveling is so small as to be of no consequence. When backsight distances are consistently made greater than foresight distances, or *vice versa,* it produces a systematic error of considerable magnitude, particularly when the sights are long. Its effect is the same as that due to the line of sight being inclined upward. It varies as the square of the distance from instrument to rod and hence will be eliminated not merely by equalizing the sum total of backsight and foresight distances between bench marks, but by balancing each length of foresight by a corresponding length of backsight.

4. *Atmospheric Refraction.*—This varies as the square of the distance, but is, under normal conditions, only about 1/7 that due to the earth's curvature and its effect is opposite in sign. It is usually considered together with the earth's curvature, but though the effect of the latter will be entirely eliminated if each backsight distance is made equal to the following foresight distance, the atmospheric refraction often changes rapidly and varies greatly in a short distance. It is particularly uncertain when the line of sight passes close to the ground. Hence it is impossible to eliminate entirely the effect of refraction even though the backsight and foresight distances are balanced. In ordinary leveling its effect is negligible. In accurate leveling the change in refraction may be minimized by keeping the line of sight well above the ground (say at least 2 ft.) and by taking, in quick succession, the backsight and foresight readings. In the long run the error is accidental, but over a short period, as a day, it may be systematic. So-called heat waves are evidences of a rapidly fluctuating refraction. Errors from this source may be reduced by shortening the length of sight until the rod appears steady.

5. *Variations in Temperature.*—The sun's rays falling on top of the telescope or on one end, and not on the other, will produce a warping or twisting of its parts and hence may influence rod readings through temporarily disturbing the adjustments. While this is not of much consequence in leveling of ordinary precision, it may produce

an appreciable error in more refined work. The error is usually accidental, but under certain conditions may become systematic. It is eliminated by shielding the instrument from the rays of the sun.

6. *Rod Not Standard Length.*—This produces a systematic error that varies directly as the difference in elevation and bears no relation to the length of the line over which levels are run. The error may be eliminated by comparing the rod with a standard length and applying the necessary corrections. The case is analagous to measurements of distance with a tape that is too long or too short. If the rod is too long the correction is added to a measured difference in elevation; if the rod is too short, the correction is subtracted.

Most manufactured rods are nearly standard length, but where large differences of elevation are to be determined, few rods are near enough the standard so that corrections can be ignored in accurate leveling.

7. *Expansion or Contraction of the Rod.*—Due to change in moisture content or change in temperature the leveling rod may expand or contract. The resultant error is systematic. Wood when well seasoned and painted will shrink or swell very little in the direction of the grain. Likewise its coefficient of thermal expansion is small. The error is of no particular consequence in ordinary leveling. For precise leveling, gage points may be established by inserting metal plugs in the rod, and corrections for shrinkage may be determined by observing any change in distance between the gage points. Corrections for thermal expansion may be based on observed temperatures of the rod, as indicated by an attached thermometer, the temperature being recorded in the notes.

8. *Rod Not Held Plumb.*—This condition produces rod readings that are too large. In running a line of levels uphill or downhill it becomes a systematic error, inasmuch as the backsights are larger than the foresights, or *vice versa*. Over rolling or level ground the resultant error is accidental since the backsights are, on the average, about equal to the foresights. The error varies directly with the magnitude of rod reading and directly as the square of the inclination. Thus, if a 10-ft. rod is 0.2 ft. out of plumb, the error amounts to 0.002 ft. for a 10-ft. reading and 0.0002 ft. for a 1-ft. reading; but if the rod were 0.4 ft. out of plumb, the corresponding errors would be 0.008 and 0.0008 ft. It is therefore evident that appreciable inclinations of the rod must be avoided. The error may be eliminated by using a rod level, or by waving the rod.

9. *Faulty Turning Points.*—This refers to turning points that are not well defined. A flat, rough stone, for example, does not make a good turning point for accurate leveling for the reason that no definite

point exists on which to hold the rod and consequently it is not likely to be held in the same position for both backsight and foresight. Errors from this source are accidental.

10. *Settlement of Turning Points or Tripod.*—If in the interval that elapses between taking a backsight and the following foresight, the tripod settles, the foresight will be too small and the observed elevation of the forward turning point will be too large. Similarly, if in the interval between foresight and backsight readings, a turning point settles, the height of instrument as calculated from the backsight reading will be too great. It is thus seen that by the normal leveling procedure, if either the level or the turning point settles, as will be the case to some extent when leveling over soft ground, the error will be systematic and the resulting elevations will always be too high. Few occasions arise when turning points may not be so selected or established as to eliminate the possibility of settlement, but care should be taken not to strike the bottom of the rod against the turning point between sights. On the other hand, some settlement of the instrument is nearly bound to occur when leveling over muddy, swampy, or thawing ground or melting snow. The errors due to such settlement may be greatly reduced by having one rodman set the turning point ahead, while the other remains at the turning point in the rear. Backsight and foresight readings may then be made in quick succession. Small errors remaining from this source may be made accidental by reversing the order of sights at alternate set-ups, as described in Art. 122.

11. *Bubble Not Exactly Centered at Instant of Sighting.*—This produces an accidental error which tends to vary as the distance from the instrument to the rod. Consequently the longer the sight, the greater the care that should be observed in leveling the instrument.

12. *Inability of Observer to Read the Rod or Set the Target Exactly on the Line of Sight.*—This causes an accidental error of a magnitude depending on the instrument, weather conditions, the length of sight, and the observer. It may be confined within reasonable limits through proper choice of length of sight.

125. Looking over the errors just listed it appears that under normal conditions the important ones are accidental, providing the proper leveling procedure is observed. Hence the resultant error may be expected to vary as the square root of the number of set-ups of the instrument or as the square root of the distance. Experience in general bears out this conclusion, and for this reason it is customary to express limiting errors of leveling in terms of the square root of the distance in miles, kilometers, or other unit of measure.

126. The precision of differential leveling depends perhaps upon more factors than any other operation of surveying. While it is influenced by the instrument employed, it depends chiefly upon the care and skill of the leveler and upon the degree of refinement with which the work is executed. Other conditions remaining the same, the error for a given length of line will tend to vary as the number of set-ups above a certain minimum, hence the precision may be expected to be lower in hilly country where the sights are limited to short distances than in flat country where normal backsight and foresight distances are employed. Above a certain length, however, the error of reading the rod increases very rapidly with the distance, therefore the precision will be lower for long sights than for those of normal length. Likewise, due to erroneous length of rod, unequal refraction, and other causes, the precision of leveling between two points of large difference in elevation is likely to be less than between two points, the same distance apart, at or near the same elevation. Atmospheric disturbances also bear an important relation to the accuracy attainable.

While conditions are so variable that no hard and fast rules can be laid down by means of which a desired precision can be maintained, practice indicates that under average conditions, with a level in good adjustment, the maximum errors may be kept within the limits shown below.

1. *Rough leveling*, such as is practised on rapid reconnaissance or preliminary surveys. Sights up to 1,000 ft. in length. Rod readings to tenths of feet. No particular attention paid to balancing backsight and foresight distances. Maximum error in feet $\pm 0.4 \sqrt{\text{distance in miles}}$.

2. *Ordinary leveling*, such as is necessary in connection with the location and construction of railroads, highways, and most other engineering works. Sights up to 500 ft. in length. Rod readings to hundredths of feet. Backsight and foresight distances roughly balanced when running for long distances uphill or downhill, but no attention paid to these distances when sights of normal length can be secured. Turning points on solid objects. Maximum error in feet, $\pm 0.1 \sqrt{\text{distance in miles}}$.

3. *Accurate leveling* for important city bench marks, or for the principal bench marks on extensive surveys. Sights up to 300 ft. in length. Rod readings to thousandths of feet either with the target or self-reading rod. Backsight and foresight distances measured by pacing and approximately balanced between bench marks. Rod waved for large rod readings. Bubble carefully centered before each sight. Turning points on metal pin or plate, or well-defined

points on solid objects. Tripod set on firm ground. Maximum error in feet $\pm 0.05\sqrt{\text{distance in miles}}$.

4. *Precise leveling* for establishing bench marks with great accuracy at widely distributed points. High-grade level equipped with stadia hairs and with sensitive bubble. Adjustments carefully tested daily. Rod standardized frequently. Sights up to 300 ft. in length. Rod readings of three horizontal hairs to thousandths of feet. Level protected from the sun. Turning points on metal pin or plate. Two rodmen. Backsights and following foresights taken in quick succession. Bubble very carefully centered and under observation at instant of taking sight. Rod plumbed with rod level. Backsight and foresight distances balanced between bench marks by stadia readings. Level set up securely on firm ground. Levels not run when the air is boiling badly nor during high winds. Maximum error in feet, $\pm 0.02\sqrt{\text{distance in miles}}$.

It should be borne in mind that the above limits of error for the several classes of leveling represent the maximum for average conditions. The *average* errors will usually be materially less.

127. Mistakes in Leveling.—Some of the mistakes that are commonly made in leveling are:

1. Confusion of numbers in reading the rod, as for example, reading and recording 4.92 when it should be 3.92. The mistake is not likely to occur if the numbers on either side of the observed reading are noted.

2. Recording backsights in foresight column, and *vice versa*.

3. Reduction of notes. Faulty additions and subtractions; adding foresights and subtracting backsights. Such mistakes will be detected if the difference between the sum of the backsights and the sum of the foresights is computed.

4. Rod held on wrong point. Unless the turning points are marked or otherwise clearly defined the rodman may not hold the rod on the same point for both foresight and backsight.

5. With the Philadelphia rod, not having the rod fully extended when reading the long rod. Before a reading on a turning point is taken the clamp should be inspected to see that it has not slipped.

6. Wrong reading of the vernier when the target rod is used.

7. Not having target set properly when the long rod is used. The vernier on the target should read the same as the vernier on the back of the rod when the rod is short.

128. Numerical Problems.

1. A line of differential levels was run between two bench marks 20 miles apart and the measured difference in elevation was found to be

2,163.4 ft. Later the rod whose nominal length is 13 ft. was found to be 0.003 ft. too short, the error being distributed over its full length. Correct the measured difference in elevation for erroneous length of rod.

2. Suppose the levels of problem 1 had been run by using a rod which was 0.003 ft. too short due to wear on the lower end. What would have been the error?

3. Suppose the line of levels of problem 1 were continued to form a circuit closing on the initial bench mark. Due to erroneous length of rod what error of closure would be expected?

4. Differential levels are run from B.M.1 (el. 470.07 ft.) to B.M.2, a distance of 100 miles. The backsight distances are 400 ft. in length and the foresight distances are 200 ft. in length. The elevation of B.M.2 as deduced from the level notes was 3,652.74 ft. Calculate the error due to earth's curvature and atmospheric refraction, and correct the elevation of B.M.2.

5. Later the levels of problem 4 were rerun using backsight distances of 200 ft. and foresight distances of 100 ft. The elevation of B.M.2 as deduced from the level notes was 3,651.38. Compute the error due to curvature and refraction and correct the elevation of B.M.2.

6. Suppose the instrument used in running the levels of problems 4 and 5 were out of adjustment so that when the bubble was centered the line of sight was inclined 0.001 ft. upward in a distance of 100 ft. Correct the results of problems 4 and 5 for inclination of line of sight.

7. A line of levels 10 miles long is run over thawing ground. Backsight and foresight distances average 300 ft. in length. What error would be introduced and what would be the sign of the correction to be applied to the elevation of the terminal bench mark if sights were taken in their normal order, and the average settlement of the instrument was 0.004 ft. between the instant of backsight reading and the instant of the following foresight reading? Suppose that at alternate set-ups the order of readings were reversed, what error would be introduced?

8. In looking through the telescope of a level a magnified tenth on the rod apparently covers 3.2 ft. of the unmagnified image. What is the magnifying power of the telescope?

9. The diameter of the field of view of a level is 5.25 ft. when the rod is 300 ft. from the objective. What is the angular width of the field of view?

10. A rod is held 300 ft. from the instrument and with the end of the bubble at a graduation a reading of 3.234 ft. is observed. The bubble is then displaced five spaces and a reading of 3 342 ft. is observed. If one space on the level tube is equal to 0.1 in., what is the radius of curvature of the level tube? What is the sensitiveness of the level tube expressed in seconds of arc per division?

11. If in running levels between two points, the rod were inclined 0.3 ft. in a height of 13 ft. what error would be introduced per set-up when backsight readings averaged 12 ft. and foresight readings averaged 1 ft.?

12. If levels are run from B.M.1 (el., 2,000.00 ft.) to B.M.2 (observed el., 3,000.00 ft.) and the rod is, on the average, 0.2 ft. out of plumb in a height of 12 ft. what error is introduced due to rod not being plumb? What is the correct elevation of B.M.2?

13. Suppose that in problem 12 both bench marks were at the same elevation. What error would be introduced?

14. The error of closure of a level circuit 100 miles long is 0.53 ft. The average length of sight is 250 ft. If all systematic errors have been eliminated, what is the probable error per set-up of the level? What is the probable error of a single observation of the rod?

15. If sights average 200 ft. in length and the probable error of a single observation is 0.004 ft. what is the probable error of running a line of levels 25 miles long? What is the probable error of running a line of levels 100 miles long?

16. In the peg test of a dumpy level the following observations are taken:

Instrument at *A*	Instrument at *B*
Rod reading on *A*, 4.937	3.077
Rod reading on *B*, 6.736	4.752

What is the true difference in elevation between the two points? With the instrument in position at *B*, to what rod reading on *A* should the line of sight be adjusted? What is the error in the line of sight for the distance *A* to *B*?

17. Complete the differential level notes shown below, performing the customary check.

Station	B.S.	H.I.	F.S.	El.
B.M.1	6.11	....	....	416.23
T.P.1	9.25	....	7.36	
T.P.2	11.48	....	3.12	
T.P.3	8.30	....	2.98	
B.M.2	12.29	....	4.37	
T.P.4	7.73	....	5.16	
T.P.5	8.24	....	3.38	
T.P.6	10.66	....	0.47	
B.M.3	.....	....	4.33	

CHAPTER IX

PROFILE LEVELING, GRADES, CROSS-SECTIONS

129. Profile Leveling.—The process of determining the elevations of points at short measured intervals along a fixed line is called profile leveling. In connection with the location and construction of railroads, highways, canals, sewers, drains, etc., stakes, or other marks, are placed at regular intervals along an established line, usually the center line. Ordinarily the interval between stakes is 100 ft. or some simple subdivision thereof, such as 50 ft. or 25 ft. The 100-ft. points, reckoned from the beginning of the line, are called *full stations* and all other points are designated as *plus stations.* Each stake is marked with its station and plus. Thus a stake set at 1,600 ft. from the point of beginning is numbered "16" or "16 + 00," and one set at 1,625 ft. from the point of beginning is numbered "16 + 25." Elevations by means of which the profile may be constructed are obtained by taking rod readings on the ground at each stake and at intermediate points where marked changes in the slope occur.

Figure 129 illustrates in plan and elevation the steps in leveling for profile. In this case stakes are set every 100 ft. as is the common practice in railway and highway location. The instrument is set up in some convenient location not necessarily on the line (as at L_1), the rod is held on a bench mark (B.M. 28, El. 564.31), a backsight (1.56) is taken, and the height of instrument (565.87) is obtained as in differential leveling. Readings are then taken with the rod held on the ground at successive stations along the line. These rod readings, being for points of unknown elevation, are foresights regardless of whether they are back or ahead of the level, in the forward direction of the line. They are frequently designated as intermediate foresights to distinguish them from foresights taken on turning points or bench marks. The intermediate foresights (0.7, 2.9, . . . 11.9) subtracted from the H.I. (567.87) give elevations of stations. When the rod has been advanced to a point beyond which further readings to ground points cannot be observed, a turning point (T.P.1) is selected, and a foresight (11.63) is taken to establish its elevation. The level is set up in an advanced position (L2), a backsight (0.41) is taken on the turning point just established (T.P.1).

Rod readings on ground points are then continued as before. The rodman observes where changes of slope occur (as 609 + 50, 610 + 40 . . . 610 + 65), and readings are taken to these intermediate stations. The "plus," or the distance from the preceding full station to the intermediate point, is measured by pacing or with a tape or the rod according to the precision required.

It is seen that elevations are carried forward in the same manner as in leveling to establish bench marks, that is, by a succession of turning points. The care exercised in taking observations on turning points depends upon the distance between bench marks the elevations of which have been determined previously, and also upon the required precision of the profile. For a ground profile usually the backsights and foresights are read to hundredths, and no particular attention is paid to balancing backsight and foresight distances; the intermediate foresights to ground points are read to tenths of feet only. Occasions arise when it is desirable or necessary to determine intermediate foresights to hundredths of feet, for example, in securing the profile of railroad track, or of the water grade in a canal; rod readings on turning points are then generally taken to thousandths of feet and backsight and foresight distances are often balanced.

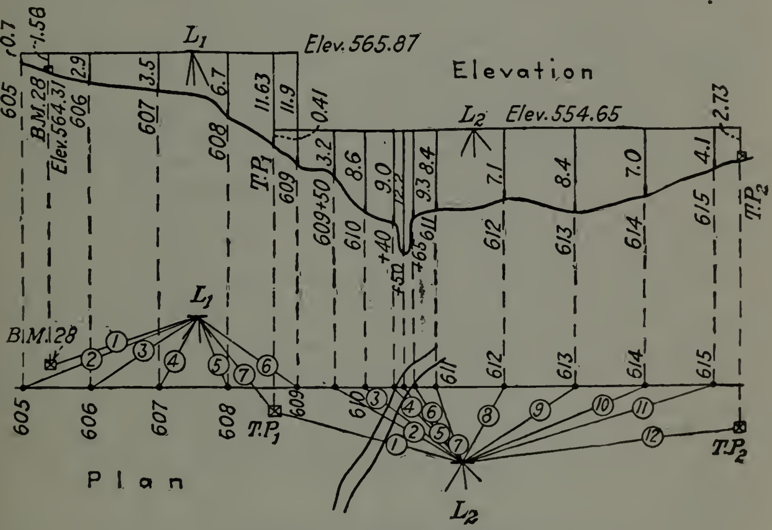

0.7
1.58
L1
Elev. 565.87
Elevation
2.9
3.5
6.7
11.63
11.9
0.41
605
B.M.28
Elev. 564.31
606
607
608
T.P.1
609
L2
Elev. 554.65
2.73
3.2
8.6
9.0
12.2
9.3
8.4
7.1
8.4
7.0
4.1
609+50
610
+40
+50
+65
611
612
613
614
615
T.P.2
B.M.128
1
2
3
4
5
6
7
605
606
607
608
T.P.1
609
610
611
612
613
614
615
8
9
10
11
12
T.P.2
L2
Plan

Fig. 129.—Profile leveling.

130. As the work of leveling the profile progresses bench marks are generally established to facilitate later work. These are made turning points wherever possible. To check the elevation of turning points, which is a desirable measure, it is sometimes necessary to run short lines of differential levels connecting the main line of profile levels with bench marks previously established by some other survey (for example, the bench marks of the U. S. Geological Survey, etc.); often the only means of checking is to run a line of differential levels back to the point of beginning. It is evident that the checking of turning points makes cumulative mistakes in the profile impossible, but does not detect mistakes in the individual intermediate foresight readings, and hence in the elevations of individual ground points. The only manner in which a profile can be absolutely checked is by rerunning profile levels over the line. Except on work of more than ordinary importance, the effect of occasional points on the profile being appreciably in error is not of sufficient moment to justify the additional work which this course would make necessary, and if turning points are checked it is regarded as sufficient.

131. Profile Level Notes.—The notes for profile leveling may be recorded as shown in Fig. 131, where foresights to turning points and bench marks are in a separate column from intermediate foresights to ground points. The values shown in the notes are the same as those illustrated in Fig. 130.

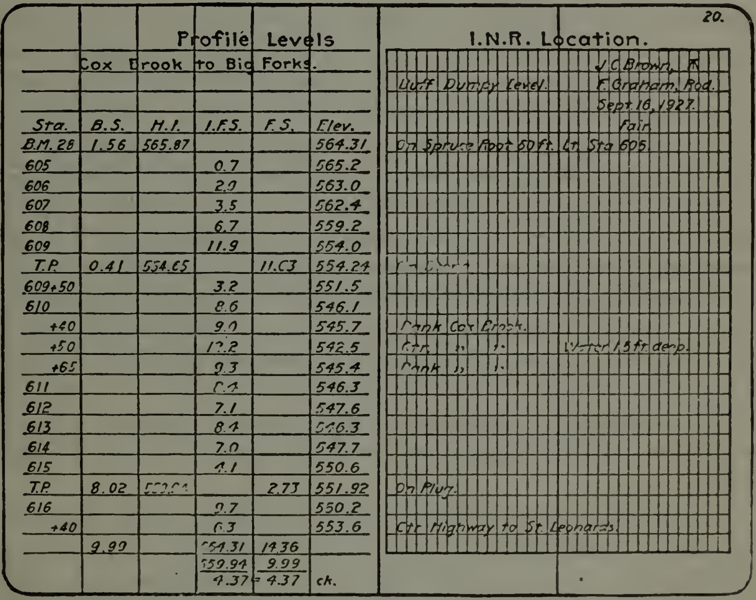
It will be observed that the notes for turning points are kept in the same manner as for differential leveling. H.I.'s and elevations of turning points are ordinarily computed as the work progresses. The difference between the sum of the backsights and the sum of the foresights taken between any two bench marks or turning points along the line should equal the difference in elevation between these points, which check is applied to each page of notes, as in differential leveling.

The calculations shown at the foot of the notes of Fig. 131 check all computations for H.I.'s and elevations of T.P.'s on the page and thus for the notes shown the difference between the sum of all backsights and the sum of all foresights is equal to the difference in elevation between B.M.28 and the last H.I.

The intermediate foresights taken from any given H.I. are recorded on lines below that on which the H.I. is shown and above that on which the succeeding H.I. is shown, so that elevations of ground points are always obtained by subtracting the corresponding intermediate foresights from the preceding H.I. For reasons that are obvious these elevations are recorded to the *number of decimal places contained in the intermediate foresights* regardless of the number of places in the H.I.

The right-hand page is reserved for concise descriptions of bench marks and for other items of moment. For example, in the notes of Fig. 131 the stream and highway crossings are noted, either of which might materially influence the elevation of the roadbed at these points. Occasionally simple sketches are employed in conjunction with the explanatory notes.

132. Cross-section Levels.—In connection with problems in drainage, irrigation, grading of earthwork, the location and construction of buildings, and similar enterprises, the shape of the surface of a piece of land is frequently desired. This may be obtained by dividing the given area into a system of squares and then determining the elevations of the corners and other points where changes in slope occur The lengths of the sides of the squares are usually 100 ft. or some simple subdivision thereof, such as 50 ft. or 25 ft. Directions of the lines may be obtained with the tape or transit, distances may be laid off with the tape or by stadia, and elevations may be determined either with the engineers' level or the hand level, depending upon the required precision. Stakes are set at the corners of the squares. The data secured by a survey of this character may be employed in the construction of a contour map.



Profile Levels

Cox Brook to Big Forks.

I.N.R. Location.

20.

Sta.	B.S.	H.I.	I.F.S.	F.S.	Elev.	
						J.C. Brown, ⊼
						Buff Dumpy Level. F. Graham, Rod.
						Sept. 16, 1927.
						Fair
B.M. 28	1.56	565.87			564.31	On Spruce Root 50 ft. Lt. Sta 605.
605			0.7		565.2	
606			2.9		563.0	
607			3.5		562.4	
608			6.7		559.2	
609			11.9		554.0	
T.P.	0.41	554.65		11.03	554.24	[illegible]
609+50			3.2		551.5	
610			8.6		546.1	
+40			9.0		545.7	Bank Cox Brook.
+50			12.2		542.5	Ctr. " " Water 1.5 ft. deep.
+65			9.3		545.4	Bank " "
611			8.4		546.3	
612			7.1		547.6	
613			8.4		546.3	
614			7.0		547.7	
615			4.1		550.6	
T.P.	8.02	559.94		2.73	551.92	On Plug.
616			9.7		550.2	
+40			6.3		553.6	Ctr. Highway to St. Leonards.
	9.99		564.31	14.36		
			559.94	9.99		
			4.37 =	4.37	ck.	

FIG. 131.—Profile level notes.

133. In rough, wooded country where long sights cannot be obtained and where approximate elevations (say to the nearest half-foot or foot) will answer the purpose, more rapid progress may be secured by using the hand level and topographers' rod. Usually the elevations of one or more stations on each cross-section line are accurately determined with the engineers' level. These stations then serve as vertical control points onto which cross-section levels run with the hand level may be tied.

134. Route Cross-sections.—Preliminary surveys for railroads, highways, and canals are often made by running a chained traverse line along the proposed route, stations being established by stakes set every 100 ft. The elevations of the stations are then determined by profile leveling, as already described. To furnish data for location studies and for estimating volumes of earthwork, the shape of the ground on either side of the traverse line may be determined by running levels over lines at right angles to the traverse. Usually these crosslines intersect the traverse at each station. The elevations may be determined with either the engineers' or the hand level, depending upon the desired precision and the length of the crosslines. Generally the hand level is used in rough country and the engineers' level is employed when the ground is comparatively flat. If the crosslines are short their direction is laid off by eye; if they are long their direction is fixed at 90° with the traverse by means of a compass, transit, or other suitable instrument. The rod is held at breaks in the surface slope, and distances are measured from the traverse line to these points.

135. Leveling for Earthwork.—Four general situations arise in connection with field measurements to determine volumes of earthwork.

1. A given area is to be cut or filled to a predetermined surface, for example, in excavating the basement for a building or in grading a piece of land. Cross-sections may be taken as described in Art. 132, though usually the sides of squares at the corners of which stakes are set will be less than 100 ft., sometimes as small as 10 ft. When the grade of the finished surface has been established the cut or fill at each station will be known and the volume of earthwork can be calculated.

2. A trench is to be excavated, as when a sewer or pipe line is to be laid. Profile levels are run along the proposed line. When the grade of the bottom of the trench has been fixed, the cut at each station can be determined. Knowing the necessary width of the trench at the top and bottom and also its depth at each station, the volume of excavation can be calculated.

3. An irregular mass of unknown volume is to be excavated at a given site. For example, earth is excavated from borrow pits to furnish material for railroad and highway fills and canal banks.

It becomes necessary to determine the shape of the surface at the site both before and after the material has been removed.

4. Earth must be cut or filled to a given grade line along some route as a highway, railroad, or canal, and further, must have a certain prescribed shape of cross-section.

136. Borrow-pit Cross-sections.—Sufficient data for the calculation of the volume of a borrow pit or similar excavation may be obtained by taking cross-sections of the site before and after the material has been removed. When the site for a borrow pit has been fixed, a base line is established in a position where it will remain undisturbed by the future excavation. At regular intervals (such as 10, 25, or 50 ft.) along the line stakes are set and crosslines through these points are established. Levels are then run over the crosslines, rod readings to tenths of feet being taken at frequent intervals (so that surface irregularities will be measured accurately), and distances being measured from the base line. When the material has been removed, the crosslines are reestablished and levels are rerun over such portions as are included within the lines of excavation. The difference between the original cross-section and the final cross-section shows the area cut at each crossline, from which the volume can be determined.

137. Road Cross-sections.—Figure 137 illustrates typical railroad or highway cross-sections in cut and in fill. The center line of the roadbed is located by center stakes set every 100 ft. (sometimes every 50 ft.), and profile levels are run as described in Art. 129. The profile is plotted and the grade line of the subgrade (*i.e.*, the roadbed upon which the pavement is placed in case of the highway or upon which the ballast is placed in case of the railroad) is established. The center cut or fill at each station is equal to the difference between the elevation of the ground line (as determined by the profile levels) and the elevation of the grade line (as established on the profile).

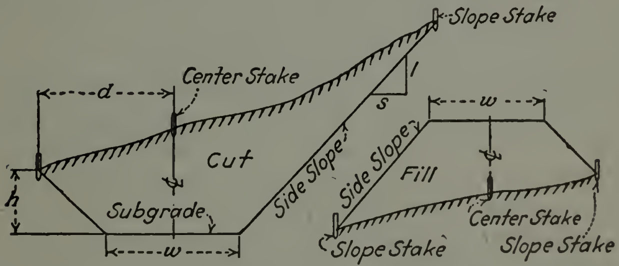


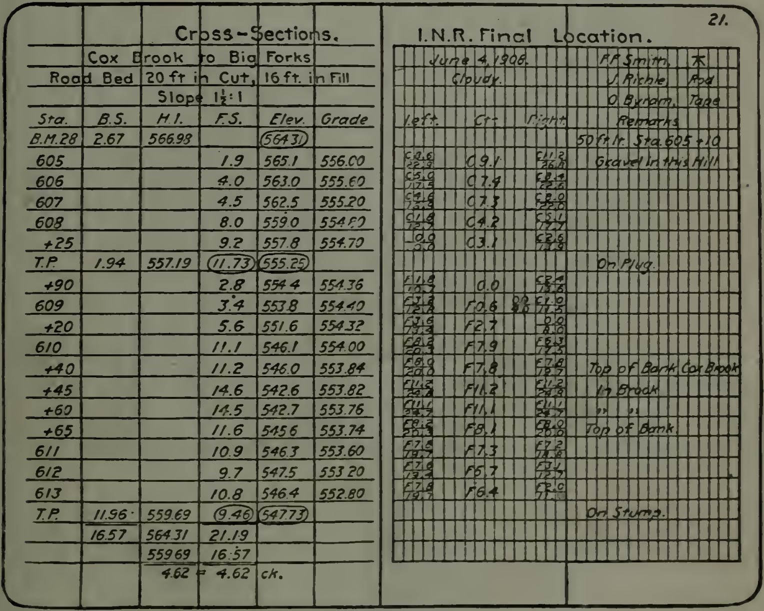
Fig. 137.—Road cross-sections.

Prior to actual construction, cross-sections are taken at full stations, at points where the line runs from cut into fill, and at such plusses as are necessary to provide reliable data upon which calculations of volume may be based; also, as a guide to those who are to do the grading, *slope stakes* are set opposite each center stake at the points marking the intersection of the side slopes with the natural ground surface, and both center and slope stakes are marked with the cut or fill (the distance above or below subgrade) at the point where the stake is driven.

The subgrade is usually a plane surface, transversely level, and on a given road is of uniform width in cut and a uniform but not usually the same width in fill. The side slopes are plane surfaces of constant slope for a given material of excavation. The rate of the side slope is stated in terms of the number of units measured horizontally to one unit measured vertically. Thus a 2 to 1 slope indicates a slope which rises (or falls) 1 ft. in a horizontal distance of 2 ft. The slope most commonly employed for cuts or fills through ordinary earth is 1½ to 1. For coarse gravel, the slope is often made 1 to 1; for loose rock, ½ to 1; for solid rock, ¼ to 1; for soft clay, 2 or 3 to 1.

137a. Level readings for cross-sections are usually taken with the engineers' level, and distances to the right or to the left of the center stake to points where observations are taken are measured with a metallic tape. Rod readings and distances are observed to tenths of feet. Prior to going to the field, the leveler secures a record of elevations of ground points as obtained from the profile levels, and also the elevation of the established grade at each station. In the field, the instrument is set up in any convenient location and the H.I. obtained by a backsight on a bench mark. As a check on the profile levels, the rod is held on the ground (sometimes on a short wooden peg driven flush with the ground) at a given station, a foresight is taken, and the cut or fill is calculated and marked on the back of the stake. A crossline through the station is established by estimation or by laying off a 90° angle. On each side of the center line, rod readings and distances from the center are taken to points of marked change in slope along the cross-section until the estimated location of the slope stake is reached. Here the rod is moved up or down the slope until by trial the measured distance from the center stake is made equal to the computed slope-stake distance for the particular cut or fill indicated by the rod reading; then the slope stake is set at this point.

If the ground is level in a direction transverse to the center line, the only rod reading necessary is that at the center stake, and the distance to the slope stake can be calculated once the center cut or fill has been determined; such a cross-section is called a *level section.* If the ground slopes in both directions from the center line, rod readings must be taken at each slope stake, in addition to the one taken at the center; the cross-section is called a *three-level section.* When rod readings are taken to the center stake, the slope stakes, and at points on each side of the center at a distance of half the width of the roadbed the cross-section is called a *five-level section.* A cross-section for which observations are taken to points between center and slope stakes at irregular intervals is called an *irregular section.*



Cross-Sections.

Cox Brook to Big Forks

Road Bed 20 ft in Cut, 16 ft. in Fill

Slope 1½:1

Sta.	B.S.	H.I.	F.S.	Elev.	Grade
B.M.28	2.67	566.98		564.31	
605			1.9	565.1	556.00
606			4.0	563.0	555.60
607			4.5	562.5	555.20
608			8.0	559.0	554.80
+25			9.2	557.8	554.70
T.P.	1.94	557.19	11.73	555.25	
+90			2.8	554.4	554.36
609			3.4	553.8	554.40
+20			5.6	551.6	554.32
610			11.1	546.1	554.00
+40			11.2	546.0	553.84
+45			14.6	542.6	553.82
+60			14.5	542.7	553.76
+65			11.6	545.6	553.74
611			10.9	546.3	553.60
612			9.7	547.5	553.20
613			10.8	546.4	552.80
T.P.	11.96	559.69	9.46	547.73	
	16.57	564.31	21.19		
		559.69	16.57		
		4.62	= 4.62	ck.	

I.N.R. Final Location. 21.

June 4, 1908.			F.F. Smith ⊼
Cloudy.			J. Richie Rod
			O. Byram. Tape
Left.	Ctr.	Right.	Remarks
			50 ft lt. Sta. 605+10
$\frac{C\,8.6}{22.9}$	C 9.1	$\frac{C\,11.2}{26.8}$	Gravel in this Hill
$\frac{C\,5.0}{17.5}$	C 7.4	$\frac{C\,8.4}{22.6}$	
$\frac{C\,4.6}{[illegible]}$	C 7.3	$\frac{C\,8.0}{22.0}$	
$\frac{C\,1.8}{12.7}$	C 4.2	$\frac{C\,5.1}{17.7}$	
$\frac{0.0}{10.0}$	C 3.1	$\frac{C\,2.6}{13.9}$	
			On Plug.
$\frac{F\,1.8}{10.7}$	0.0	$\frac{C\,2.4}{13.6}$	
$\frac{F\,3.2}{12.8}$	F 0.6 $\frac{0.0}{8.0}$	$\frac{C\,1.0}{11.5}$	
$\frac{F\,3.6}{[illegible]}$	F 2.7	$\frac{0.0}{8.0}$	
$\frac{F\,8.2}{20.3}$	F 7.9	$\frac{F\,6.3}{17.5}$	
$\frac{F\,8.0}{20.0}$	F 7.8	$\frac{F\,7.8}{19.7}$	Top of Bank Cox Brook
$\frac{F\,11.2}{[illegible]}$	F 11.2	$\frac{F\,11.2}{24.8}$	In Brook
$\frac{F\,11.1}{[illegible]}$	F 11.1	$\frac{F\,10.1}{[illegible]}$	″ ″
$\frac{F\,8.2}{20.3}$	F 8.1	$\frac{F\,8.0}{20.0}$	Top of Bank.
$\frac{F\,7.6}{[illegible]}$	F 7.3	$\frac{F\,7.2}{18.8}$	
$\frac{F\,7.6}{[illegible]}$	F 5.7	$\frac{F\,3.1}{12.7}$	
$\frac{F\,7.8}{19.7}$	F 6.4	$\frac{F\,2.0}{[illegible]}$	
			On Stump.

Fig. 138.—Railroad cross-section notes.

Where the cross-section passes from cut to fill, it is called a *side-hill section*, and an additional observation is made to determine the distance from center to the grade point, that is, the point where the subgrade will intersect the natural ground surface. A peg is usually driven to grade at this point and its position is indicated by a witness stake marked "grade."

138. Figure 138 illustrates a suitable form of cross-section notes. The left-hand page is seen to be essentially the same as for profile leveling with the exception that a column is added for grade elevations. The notes cover a portion of the same line for which profile level notes are shown in Fig. 131. The cross-section levels are seen to check the elevations of stations as determined by the profile levels within 0.1 or 0.2 ft., which is as close as can be expected. The cross-section notes on the right-hand page are in line with the station to which they refer. They illustrate a portion of a line passing from cut into fill where the slope is such that three-level sections are adequate. The cross-sections at 608 + 90 and 609 are side-hill sections. It is seen that cuts or fills are shown, rather than elevations, and that the cut or fill and the distance out from center to each point is expressed in fractional form.

The values in columns marked "right" and "left" are for points at which the slope stakes were driven.

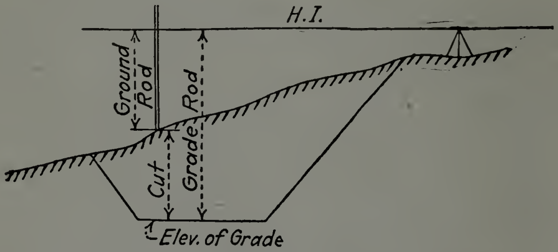


FIG. 139*a*.—Road cross-section in cut.

139. An explanation of the method usually employed to determine cuts and fills is appropriate. Figure 139*a* shows the level in position for taking rod readings at a section in cut. The height of instrument (H.I.) has been determined; the elevation of grade at the particular station is known. The leveler calculates the difference between the H.I. and grade elevation, which difference is known as the *grade rod;* that is, H.I. − El. of grade = grade rod. The rod is held at any point for which the cut is desired, and a reading is taken, which reading is called the *ground rod. The difference between the grade rod and the ground rod is equal to the cut.*

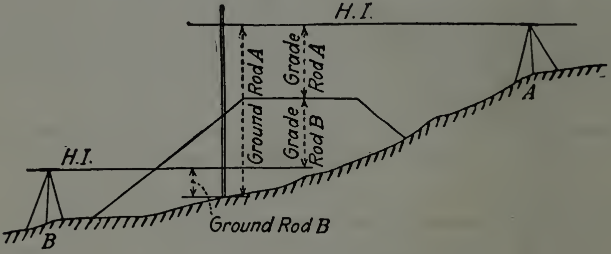


FIG. 139*b*.—Road cross-section in fill.

Figure 139*b* is a similar illustration for a cross-section *in fill.* It is clear that *if the H.I. is above grade, the fill is the difference between* *the ground rod and the grade rod; if the H.I. is below grade, the fill is the sum of the grade rod and the ground rod.*

140. Setting Slope Stakes.—The process of setting slope stakes requires some additional explanation.

If w is the width of roadbed, d is the measured distance from center to slope stake, s is the side-slope ratio (ratio of horizontal distance to drop or rise), and h is the cut (or fill) at the slope stake, then by Fig. 140 when the slope stake is in the correct position

$$d = \frac{w}{2} + hs \qquad (1)$$

A numerical example will best illustrate the steps involved in securing the equality designated in Eq. (1). Let $w = 20$ ft.; side slope, $1\frac{1}{2}$ to 1; grade rod, 15.2 ft. Suppose a slope stake is to be set on the left of the center stake (Fig 140). As a first trial the rod is held at A, ground rod = 6.6 ft.; h_1 = grade rod − ground rod = 15.2 − 6.6 = 8.6. The calculated distance for this value of h_1 is $\frac{w}{2} + h_1s = 10 + 8.6 \times \frac{3}{2} = 22.9$ ft. Measurement from the center stake shows d_1 to be 18.2. Hence the rodman should go farther out. For a second trial the rod is held at B; ground rod = 8.8; h_2 = grade rod − ground rod = 15.2 − 8.8 = 6.4 ft. $\frac{w}{2} + h_2s = 10.0 + 9.6 = 19.6$ ft. The measured value of d_2 is 22.5 ft. Hence the rod is too far out. As a third trial the rod is held at C, ground rod = 7.8; h = 15.2 − 7.8 = 7.4 ft. $\frac{w}{2} + hs = 10.0 + 7.4 \times \frac{3}{2} = 21.1$ ft. The measured value of d is 21.1 ft., hence this is the correct position of the slope stake. In the notes the coordinates of the slope stake are given by the fractional expression $\frac{c7.4}{21.1}$, but the trial observations are

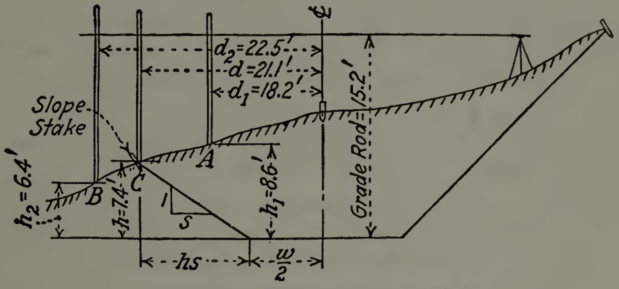


FIG. 140.—Setting slope stakes.

not recorded. While the above example is for a cut, the same procedure is followed in fill.

Slope stakes are set side to the line. On the back of the stake is marked the station number. On the front (side nearest the center line) is marked the cut or fill at the stake, and sometimes the distance from center to slope stake. The numbers read down the stake.

141. Setting Grades.—Typical works of construction where grades are required to be established in the field are as follows:

1. *Buildings.*—For the excavation, batter boards may be set some whole number of feet above the bottom. Grades for footings are given by ground pegs driven to the elevation of the top of the footing, or to the elevation of the top of the floor. Column bases and wall plates are set to grade directly by the leveler.

2. *Sewers and Pipe Lines.*—Grades may be given either by batter boards set some whole number of feet above the bottom of the trench or top of pipe, or by ground pegs or other suitable reference marks set just outside the trench. For example, on a paved street nails or crayon marks may be used. Distances below such points to the actual grade are marked on stakes or pavement. On account of the fact that many foremen are unfamiliar with tenths of feet, these distances are usually given in feet, inches, and eighths of inches.

3. *Pavements.*—For city streets ground pegs are usually driven to the grade of the top of the curb and are placed just outside the curb line on both sides of the street. The procedure is the same for highways except that the pegs are driven to the grade of the edge of the pavement. When the pavement is wide and is crowned similar stakes may be set on center line.

4. *Railroad Track.*—Track is usually laid on the subgrade and is lifted in position after the ballast has been dumped. On one side of the track and perhaps 3 ft. from the rail, stakes are usually driven to the elevation of the grade of top of rail.

141a. The operation of leveling for grades is similar to profile leveling. After the grade line has been established on the profile, the grade elevation for each station is known. Leveling for grades is started from a bench mark and is carried forward by turning points. The grade rod to be employed in setting a given grade stake to grade is calculated by subtracting the grade elevation from the H.I. The top of the stake is usually driven to grade and marked with crayon to indicate that it is at grade. Sometimes the rod is moved up or down the side of the stake until the grade rod is read, when the position of grade is indicated by a crayon mark or a nail driven into the stake at the foot of the rod. When points are established a given distance above or below grade the process is the same, except that the distance to grade is indicated either on the grade stake or on a witness stake nearby. Usually grade elevations are determined to hundredths of feet.

The notes are kept as in profile leveling except that the right-hand column of the left-hand page is for grade elevations. Also when the stakes are not driven to grade a record is kept of the cuts or fills (that is, the distances from established points to the actual grade line). The intermediate foresights recorded are the calculated grade rods, if the stakes are driven to grade; in any case the last reading taken on the grade stake at each station is shown in the foresight column opposite the station to which it refers.

The distance between points at which grade is established depends upon the character of the work and on whether the grade is uniform (straight line in profile) or on a vertical curve. Usually on track construction, grades are given at each 100-ft. station but sometimes every 50 ft. on vertical curves. For road pavements and sewers, grade is generally established every 50 ft. if the grade is uniform and every 25 ft. or even every 10 ft. if the grade is on a vertical curve.

142. Shooting in Grade.—Unless the grade is level the methods described in the preceding article necessitate calculating the grade rod for every station which is established at grade. Likewise if grade is established at any intermediate point the chainage of which has not been previously determined, the plus of the new station must be measured before the grade rod can be calculated.

When the line is tangent (straight in plan) and the grade is uniform for a considerable distance, the work of setting grade stakes may be facilitated by "shooting in" the grade in the manner now to be described. Let *A* and *B* be two stations some distance apart (say 800 to 1,000 ft.) on tangent, between which stations a uniform grade is to be established. A line of differential levels is run to include both *A* and *B*, and stakes are driven to grade at these points. The level is then set up close to *B* (with one pair of foot screws in line with *A*), and with the rod held on the stake, a reading is obtained by sighting through the objective end of the telescope (as in the peg test of the dumpy level). The rodman next holds the rod on *A*, and the leveler moves the telescope in a vertical plane (by means of the foot screws if a level is used, or by means of the altitude tangent screw if the transit is used) until the horizontal cross-hair appears to cut the rod at the reading previously obtained with rod at *B*. Neglecting the effect of the earth's curvature and atmospheric refraction (which will be of no consequence for the distances suggested above), the line of sight is now a uniform distance above the uniform grade for all points between *A* and *B*. Hence the grade at any intermediate station is established by observing the same rod reading, the instrument, of course, remaining undisturbed for the interval during which the intermediate stakes are being set. As a check, it is well to sight again to *A* before the instrument is moved, in order to detect any displacement of the line of sight.

143. The Gradienter.—Some levels are equipped with a micrometer screw at one end of the crossbar, by means of which the telescope may be turned through small vertical angles without moving the foot screws. When the screw has such a pitch that one full turn (sometimes two full turns) will cause a change of 1 per cent in the inclination of the line of sight, and further, the screw has an attached drum the rim of which is divided into 100 parts (sometimes into 50 parts), it is called a *gradienter*. When grades are to be established with a level equipped with gradienter the elevations are carried forward by direct leveling, as described in the preceding article, the instrument always being set up near a station on the line. After the H.I. for a given set-up has been determined, the gradienter drum is set to read zero when the instrument is level. The line of sight is then made parallel with the grade line by turning the gradienter until it reads the desired grade. (Thus, if one turn of the screw inclines the line of sight 1 per cent and the gradienter drum is divided into 100 parts, a 1.2 per cent grade would be laid off by one full turn and 20 spaces as indicated by the drum graduations.) Since the line of sight is parallel with the grade line, the grade rod for the station at which the level is set up is also the grade rod of any other station in the direction in which the instrument is pointed. Attention is called to the obvious fact that for points in advance of the level, the gradienter must be set to a reading equal and opposite to that for points in the rear.

The altitude tangent screw of the engineers' transit is often designed to be used as a gradienter. It is employed in the manner just described.

144. Contour Leveling.—In topographic surveying the engineers' level, in conjunction with other instruments, is sometimes employed for the direct location of contours.[1] Also in connection with the surveys for reservoir sites, levels are usually run to establish the proposed shore line. The process of establishing lines of this character consists of carrying a line of levels forward by turning points and of finding, by trial, a series of ground points at the required elevation.

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[1] A contour is an imaginary line connecting points of equal elevation on the surface of the earth.

145. Establishing Grade Contours.—In connection with the preliminary surveys for highways and canals in hilly or mountainous country where the general route lies along a side hill, levels are frequently run to establish points along some required grade. The irregular line joining such points is termed a grade contour. If a level (or transit) with gradienter is available, the simplest method is to follow the same procedure as in contour leveling (Art. 144), except that for both turning points and ground points the gradienter is set so that the line of sight is at the required grade.

146. If the level is not equipped with a gradienter, unless the grade is nearly level so that the required slope can be laid off with the bubble, the grade contour must be established by the more laborious method of direct leveling, distances being measured by stadia or by pacing and a new grade rod being calculated for each point at which a stake is driven, as described in Art. 141.

147. Numerical Problems.

1. Complete the profile level notes shown below.

Sta.	B.S.	H.I.	F.S.	El.
B.M. 10	6.32	....	....	836.76
179	....	....	10.1	
180	....	....	7.8	
+35	....	....	12.6	
181	....	....	4.7	
182	....	....	3.4	
T.P. 38	7.32	....	2.11	
183	....	....	8.5	
+40	....	....	4.6	
184	....	....	7.2	
185	....	....	10.6	
T.P. 39	5.93	....	11.49	
186	....	....	4.2	

2. The width of roadbed of a proposed railroad is 24 ft. in cut and the side slopes are to be 1½ to 1. At a given station the elevation of grade is 515.75. For obtaining the cross-section at the station the H.I. of the level is 528.32. The ground rod at the center stake is 6.5. Compute the grade rod and the center cut. The rod reading at the right slope stake is 1.2 and at the left slope stake is 10.9. Compute the cut and the distance out to each slope stake.

3. Make a page of cross-section notes for a highway running from cut into fill. The grade of the highway is 4.0 per cent, width of roadbed 24 ft. in cut and 18 ft. in fill, side slopes 1½ to 1. Show observations at center and slope stakes and at grade points.

4. Make a page of notes for establishing grades of top of rail from station 750 to station 762. Elevation of grade at station 750, 381.6; −0.6 per cent grade, stations 750 to 758; −0.4 per cent grade, stations 758 to 762; bench mark near station 750, el. 378.47.

5. The radius of curvature of the level tube of a level is 75 ft. and one space on the tube is equal to 1/10 in. How many spaces will the bubble have to be displaced from the center to make the line of sight parallel with a grade rising 1.5 ft. per mile?

CHAPTER X

PLOTTING PROFILES AND CROSS-SECTIONS; VOLUMES OF EARTHWORK

PROFILE AND CROSS-SECTIONS

148. The Profile.—Usually the profile is plotted on regular profile paper. The general practice is to begin the profile at the left end of the sheet; the station numbers thus increase from left to right. The horizontal and vertical scales to be employed depend upon the purpose of the profile. If it is for fixing grades, as for a railroad or highway, a scale of 1 in. = 400 ft. (horizontal) and 1 in. = 20 ft. (vertical) is frequently used; if it is to be made the basis of earthwork calculations, as for a sewer or other pipe line a scale as large as 1 in. = 40 ft. (horizontal) and 1 in. = 4 ft. (vertical) may be required.

The profile is plotted from the profile level notes. The ground line is formed by drawing a line through the plotted points. The profile should not be a succession of straight lines between adjacent points, for this will not represent the actual variation in the ground; on the other hand the profile at the summits and depressions should not be unduly rounded for such points will naturally be greatly exaggerated in sharpness on account of the exaggerated relation between horizontal and vertical scales.

Notes on the profile show the station and plus of important objects, as streams, roads, etc., crossed by the line. Such notes are placed directly above the points on the profile to which they refer. Generally an alignment diagram near the bottom of the sheet is so placed that points on the diagram are directly below corresponding points on the profile. In this way a ready comparison between profile and plan may be made without referring to the map. The alignment diagram indicates the location of tangents, curves, and changes in direction of the line, but is not a true plan view, except when the line is straight. Sometimes the directions of land lines, streams, and other objects crossed by the line are indicated on the alignment diagram.

Sometimes general drawings show the profile on the same sheet with the map or plan of the line. Such an arrangement is convenient for purposes of comparison in that the general relation between plan and profile may be seen at a glance.

149. Fixing Grades.—While the factors influencing the choice of grades will not be discussed here in detail, it is pertinent to mention some of them. In the location of highways, or similar routes, maximum permissible rates of grade are usually established. Likewise the elevation of grade is fixed within narrow limits at certain governing points as at terminals and at stream, highway, and railway crossings. Other things being equal, between such governing points the grade is fitted to the ground until the volume of earthwork in cuts will balance that in adjacent fills. For sewers and drains certain minimum permissible grades are established and these with the profile of the ground and elevations of governing points, as connections with mains, depths of basements, etc., operate largely to fix the grade. In any case the ground profile furnishes the basis for the study of economic grade elevation.

Between the points at which the elevation of grade is fixed, grade lines are established on the profile until by trial a satisfactory solution is obtained. Rates of grade and stations of points of change in grade are then fixed, and the corresponding elevations of points of change are calculated. The grade line may be a succession of straight lines abruptly changing direction at grade intersections, or if the changes in grade are considerable, it may be a series of straight lines connected by vertical curves at the summits and depressions. The points of change are marked by small circles, and on vertical lines through these points are shown their elevations (and plusses if they do not fall at full stations). The rates of grade are shown just above the grade line.

150. Finishing the Profile.—The profile is finished in ink. If it is to be blueprinted all lines are shown in black. Otherwise the grade line and the numerical notes pertaining to it are generally shown in red and the remainder of the sheet is inked in black. Sometimes the alignment diagram with its accompanying notes is shown in a contrasting color, as blue or orange. The station numbers of the heavy vertical lines are placed at the bottom of the sheet. The elevations of the heaviest horizontal lines are written at each end of the sheet, and at intermediate points if the profile is long. Numbers and explanatory notes which are written vertically on the profile are ordinarily placed so as to be read from the right end, though some engineers prefer them to read from the opposite hand so that as the profile is unrolled with the zero end towards the body the notes will appear right side up.

151. Plotting Cross-sections.—Cross-sections for earthwork, particularly those of irregular shape, are frequently drawn to scale. Generally they are plotted on regular cross-section paper, the use of "10 by 10 to the inch" paper being most common. Governing points of the cross-section are plotted from the cross-section notes. The surface may be indicated either by an irregular line drawn through the points or by a broken line connecting the several points. The scale to be used depends upon the accuracy with which it is desired to calculate the cross-sectional area. For large cross-sections a scale of 1 in. = 10 ft. both horizontally and vertically is appropriate. If the cross-sections are shallow, sometimes the vertical scale is exaggerated, as for profiles.

The cross-section for the first station on the base or center line is placed in the upper left-hand corner of the sheet, and successive cross-sections are placed one below the other. Below each cross-section is shown its station number. Areas of cross-sections in square feet and volumes of earthwork between successive cross-sections in cubic yards are also recorded on the sheet.

EARTHWORK CALCULATIONS

152. Areas of Regular Cross-sections.—Areas of regular cross-sections[1] are readily determined by numerical calculations without plotting. For purposes of computing earthwork the areas are calculated in square feet. For a trench the cross-sectional area at any point is determined by multiplying the average of the top and bottom widths by the depth. The same method of calculation may be applied to level cross-sections for railroads and highways, or if d is the distance to either slope-stake from the center, w is the width of the roadbed, and c is the center cut or fill, then the area is

$$A = c\left(d + \frac{w}{2}\right) \tag{1}$$

153. Area of Three-level Sections.—In Fig. 153 the three-level section may be divided into four triangles, as shown. Then from the figure the area is

$$A = \frac{1}{2} \cdot \frac{w}{2}(h_l + h_r) + \frac{1}{2}c(d_l + d_r),$$

or

$$A = \frac{w}{4}(h_l + h_r) + \frac{c}{2}(d_l + d_r) \tag{2}$$

---

[1] Regular cross-sections are cross-sections for which levels are taken at corresponding points on either side of the center-line. Level and three-level sections are regular.

Rule.—Multiply the sum of the distances to the slope stakes by one-half the center cut or fill; to this add the product of one-fourth the width of the roadbed and the sum of the side heights. The result is the cross-sectional area.

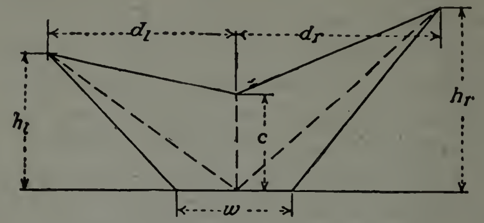


Fig. 153.—Three-level section.

154. Area of Irregular Road Cross-sections.—Figure 154 represents an irregular road cross-section. The cross-section notes give C the cut (or fill) at the center stake A and distances from center to, and cuts at points E, F, G, and H. The method indicated here is an adaptation of the general method for computing areas by means of coordinates (Art. 357). In this case the cross-section notes provide the x and y coordinates for each vertex of the section (the origin being at 0) if the fraction $\frac{0}{w/2}$ is supplied for the two vertices M and N, and if the algebraic signs, plus and minus, are used to designate directions to the right and left of the origin, respectively.

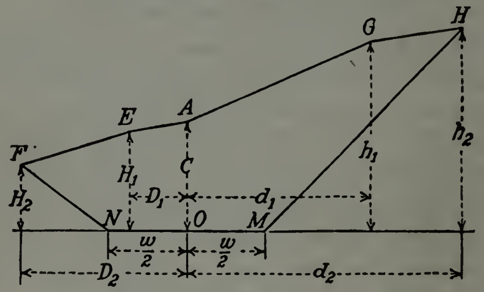


Fig. 154.—Irregular section.

In the usual form the notes are recorded as follows:

$$\frac{H_2}{D_2} \quad \frac{H_1}{D_1} \quad \frac{C}{0} \quad \frac{h_1}{d_1} \quad \frac{h_2}{d_2}$$

Then, as stated above, if the fractions and algebraic signs are supplied, the coordinates of the section appear as follows:

$$\frac{0}{-\frac{w}{2}} \quad \frac{H_2}{-D_2} \quad \frac{H_1}{-D_1} \quad \frac{C}{0} \quad \frac{h_1}{+d_1} \quad \frac{h_2}{+d_2} \quad \frac{0}{+\frac{w}{2}}$$

The calculation of the area will be made more convenient if now the opposite algebraic sign is placed on the opposite side of each denominator. The fractions then appear thus:

$$\frac{0}{-\frac{w}{2}+} \quad \frac{H_2}{-D_2+} \quad \frac{H_1}{-D_1+} \quad \frac{C}{0} \quad \frac{h_1}{+d_1-} \quad \frac{h_2}{+d_2-} \quad \frac{0}{+\frac{w}{2}-}$$

The area may now be computed by the following rule: *Multiply each numerator by the algebraic sum of the two adjacent denominators, using the signs facing the numerator. The algebraic sum of these products is double the area of the cross-section.*

Example: Below are the notes for an irregular cross-section; the width of the roadbed is 20 ft.; the cross-sectional area is to be calculated by the above rule. The fraction $\frac{0}{w/2}$ is recorded, and the double opposite algebraic signs are supplied as shown:

$$\frac{0}{-10+} \quad \frac{2.4}{-13.6+} \quad \frac{4.2}{-10.0+} \quad \frac{6.1}{0} \quad \frac{8.3}{+15.0-} \quad \frac{10.2}{+25.3-} \quad \frac{0}{+10-}$$

$$\begin{aligned}
2.4 \times (+10.0 - 10.0) &= 0.0 \text{ sq. ft.}\\
4.2 \times +13.6 &= +\ 57.1 \text{ sq. ft.}\\
6.1 \times (+10.0 + 15.0) &= +152.5 \text{ sq. ft.}\\
8.3 \times +25.3 &= +210.0 \text{ sq. ft.}\\
10.2 \times (-15.0 + 10.0) &= \underline{-\ 51.0} \text{ sq. ft.}\\
\text{Double area} &= +368.6 \text{ sq. ft.} \quad \text{Area} = 184.3 \text{ sq. ft.}
\end{aligned}$$

For side-hill sections, where the road is partly in fill and partly in cut, the cross-sectional areas may be conveniently calculated by dividing into partial areas consisting of trapezoids and triangles.

154a. When cross-sections are bounded by curved lines or are very irregular they are usually plotted as described in Art. 151, and the areas are determined either by subdividing into rectangles and triangles each of whose areas is calculated, or with greater facility by traversing the perimeter of each section with a polar planimeter.

155. The Polar Planimeter.—Figure 155 shows a polar planimeter with adjustable tracing arm. The planimeter is supported at three points, namely, the anchor point or pole *P*, the roller *R*, and the tracing point *T*. The arm carrying the anchor point is hinged or pivoted to the frame of the planimeter. The tracing arm *A* slides through a sleeve in the frame of the planimeter and is clamped in position by the screw at *B*. On the arm are graduations which, when set to the index *J*, give known relations between the reading of the planimeter and area. Fine settings of the arm are made by means of the tangent screw *C*. The circumference of the drum *D* of the roller R is graduated into 100 parts. At E is a vernier by means of which revolutions of the roller may be read to thousandths. By means of a worm, the revolutions of the roller turn the disc F in the ratio 10 to 1. The disc is graduated by ten radial lines marked from 0 to 9. Hence each space on the disc is the equivalent of one complete revolution of the roller. By means of an index mounted on the frame the whole number of revolutions of the roller is read on the disc; decimals of a revolution are given to hundredths by the drum reading opposite the index of the vernier, and are estimated to thousandths by reading the vernier. The anchor is a needle point which is pressed into the paper. The anchor arm is held down by the weight W.

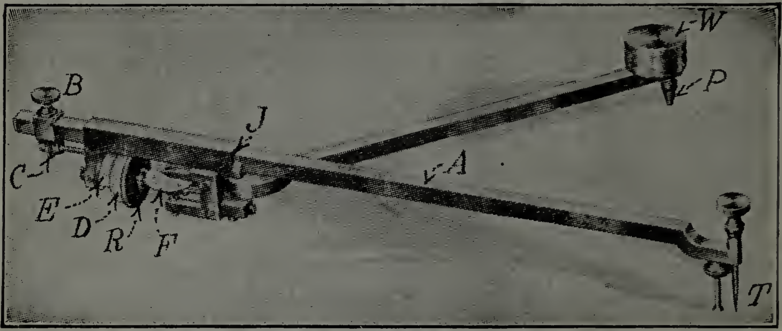


Fig. 155.—Polar planimeter.

When a plotted area is to be determined, the anchor point is fixed in the desired position on the paper, the tracing point is placed at some point on the perimeter of the figure, and the roller is set to read zero or preferably no attempt is made to set the index at zero, in which case an initial reading is taken. The perimeter is then completely traversed until the tracing point is brought to its original position. As the tracing arm is moved, the accompanying movement of the roller is partly sliding and partly rolling. By the calculus it can be shown that the amount the planimeter rolls is a direct function of the area traversed, provided the pole or anchor point is outside the figure; or in other words, the planimeter is a mechanical integrator.

It is evident that, when the direction of movement of the tracing point is parallel with the axis of the roller, the movement of the roller will be entirely by sliding. Hence, for a certain fixed distance between the anchor and the tracing point, the roller will not revolve while the point is being moved. Obviously, the path taken by the tracing point is the circumference of a circle of which the anchor is the center. The circle along which the tracing point may be moved without causing the roller to revolve is called the *zero circle*. When the anchor is placed inside any figure whose area is to be measured, the area is determined by *algebraically* adding the area indicated by the planimeter reading to the area of the zero circle. If the size of the figure is less than that of the zero circle and if the motion of the tracing point is clockwise, the planimeter reading will be negative; if greater than that of the zero circle, the planimeter reading will be positive. The area of the zero circle for various settings of the tracing arm is usually shown either on top of the arm or in the planimeter box. It can readily be determined by traversing a given figure first with the pole outside the figure and then with the pole inside. The algebraic difference between the two planimeter readings, when multiplied by the proper constant, determines the area of the zero circle.

Just to the right of each graduation on the tracing arm is indicated the number of units of area (for a given scale) per revolution of the roller when the graduation is set at index J. The accuracy of the planimeter for a given setting of the tracing arm is tested by traversing a figure of known area. Usually it is desirable that a simple relation exist between revolutions and square inches or square feet at the scale of the drawing. If the planimeter reading is too large to give a simple constant of multiplication, the length of the tracing arm is increased, and *vice versa*.

Many planimeters have a fixed length of tracing arm. For this type the common relation between revolutions of roller and square inches is 1 to 10.

When the horizontal and vertical scales are dissimilar, as for profiles, areas with the planimeter may be determined as readily as when the scale is the same in each direction. For example, if a profile were plotted to the scale of 1 in. = 400 ft. (horizontal) and 1 in. = 20 ft. (vertical), then each square inch on the paper represents 20 × 400 = 8,000 sq. ft.

156. Areas with the Planimeter.—The sheet on which the area is plotted is stretched flat and free from wrinkles. If the tracing arm is adjustable, it is set so that some convenient relation exists between area and revolutions of roller. For example, if a cross-section were plotted to the scale of 1 in. = 10 ft., the arm might be set at the 10 □″ graduation, indicating an area of 10 sq. in. or at the above scale, 1,000 sq. ft., per revolution of roller. Hence the least count of the vernier would be the equivalent of 1 sq. ft. of area. The accuracy of the tracing-arm setting is then tested by carefully tracing the boundary of a figure of known area, say a 2- by 5-in. rectangle, and if necessary the length of the arm is adjusted until by trial the desired relation is established. The anchor point is placed outside the test figure, and the boundary is tracted in a clockwise direction, thus increasing the score of the planimeter. The planimeter is read with tracing point at the point of beginning of the traverse and again after the figure has been traced. The difference between the two readings or scores gives the number of revolutions of the roller. Particular care should be exercised in returning the tracing point exactly to the point of beginning before taking the final reading.

An accessory furnished with some planimeters consists of a flat bar at one end of which is a needle point, and at the other end of which is a small hole into which the tracing point may be set. The distance between needle point and hole is equal to the radius of a circle whose area is 10 sq. in. The needle point is fastened to the paper, and the planimeter is quickly tested by tracing the circumference of the circle, the tracing point being held in the hole of the bar as it is revolved about the needle point.

After the planimeter has been tested, the figures whose areas are desired are traversed and the areas are computed by multiplying the difference between initial and final scores by the proper constant. Often the roller is set at zero with the tracing point at the point of beginning, in which case the final reading gives revolutions directly. In order to avoid consideration of the area of the zero circle, the anchor is generally placed outside the figure; to produce an increase in the score of the planimeter the tracing point is moved in a clockwise direction.

If the figure is larger than can be traced in one operation, it may be subdivided into smaller figures each of which may be traversed in the usual manner. When a considerable number of areas of large figures are to be determined, however, faster progress may be made by placing the anchor point inside the figure and algebraically adding the area of the zero circle to the area indicated by the planimeter. The area of the zero circle is determined as described in the preceding article. If the area of the figure is less than that of the zero circle the difference between initial and final readings of the planimeter is a negative quantity; if greater, the resulting score is a positive quantity. Some care must be exercised to avoid confusion between positive and negative results.

Attention is drawn to the fact that the constant by which revolutions of the roller must be multiplied to give area can readily be determined for any setting of the tracing arm. All that is necessary is to determine the planimeter score for a known area. Then by proportion, *any required area is to the corresponding score, as the figure of known area is to its score.*

157. Errors in Planimeter Measurements.—If the relation between revolutions and area is accurately established, the errors involved in planimeter measurements are accidental in character and are due principally to the inability of the observer to follow exactly the boundary of the figure with the tracing point. For the same care and skill on the part of the observer, the smaller the area, the larger the relative error of measurement. Hence it is desirable that the areas be plotted to a scale consistent with the relative accuracy with which it is desired to determine areas. Ordinarily, planimeter measurements of small areas may be expected to be correct within 1 per cent, and measurements of figures of considerable size may be correct within 0.1 or 0.2 per cent, or less. Few, if any, cases arise where the areas of plotted cross-sections for earthwork cannot be determined with a precision as great as is justified by the nature of the field measurements.

158. Volumes of Earthwork.—Volumes of earthwork are calculated by a variety of methods, depending upon the nature of the excavation and the data upon which computations are based. Where cross-sections have been taken along a route their areas are determined as described in preceding paragraphs, and the volumes of the prismoids between successive cross-sections are calculated either by the method of average end areas or by the prismoidal formula. The same procedure may be followed for borrow pits and other excavations of similar nature, or if observations of elevation after excavation has been completed are made at the same points as those obtained on the original ground, the volume may be calculated by dividing into vertical truncated prisms. Estimates for grading are frequently based on a topographic map showing the contours for the undisturbed ground and also those for the ground as it will appear when grading has been completed. The volumes are conveniently determined by dividing into prismoids with horizontal bases and sloping sides. Volumes are expressed in cubic yards.

159. Volumes by Average End Areas.—The most common method of determining volumes of excavation for railroads, highways, canals, and similar works is the *method of end areas*. It assumes that the volume between successive cross-sections is the average of their end areas multiplied by the distance between them, or expressed as a formula

$$V = \frac{l}{2}(A_1 + A_2) \tag{1}$$

in which V is the volume (cubic feet) of the prismoid between end bases or cross-sections having areas (square feet) A_1 and A_2, and l is the length (feet) of the prismoid, or in other words, the distance between cross-sections. If cross-sections are taken at full 100-ft. stations the volume in cubic yards between successive cross-sections A_1 and A_2 (square feet) is

$$V_y = 1.85(A_1 + A_2) \tag{2}$$

The above formulas are exact when $A_1 = A_2$, but are approximate for $A_1 \gtrless A_2$. As one of the areas approaches a point, as on running from cut to fill on side-hill work, a maximum error of $16\frac{2}{3}$ per cent would occur if the formulas were followed literally. In this case, however, the volume is usually calculated as a pyramid, *i.e.*, *volume* = *⅓ area of base times length.* Considering the fact that cross-sections are usually a considerable distance apart, and that minor inequalities in the surface of the earth between sections are not considered, the method of average end areas is sufficiently exact for ordinary earthwork.

160. Volumes by the Prismoidal Formula.—It can be shown that the volume of a prismoid is

$$V = \frac{l}{6}(A_1 + 4A_m + A_2) \tag{3}$$

in which l is the distance between end bases or sections, A_1 and A_2 are the areas of the end sections, and A_m is the middle area or area halfway between the end sections. A_m is determined by averaging the corresponding linear dimensions of the end sections and not by averaging the areas A_1 and A_2. The use of the formula is best illustrated by an example.

Example: Following are shown the three-level cross-section notes for two stations 100 ft. apart. The width of the roadbed is 20 ft. The volume of earthwork between the two stations is to be calculated by the prismoidal formula. Below the regular cross-section notes are shown those for the mid-section obtained by averaging the values given for sections at stations 115 and 116. In the column headed "Area, square feet," are cross-section areas computed by Formula (2), Art. 153. Then by the prismoidal formula given above

$$V = \frac{100}{6}(212.0 + 4 \times 154.0 + 103.0) = 15{,}520 \text{ cu. ft. or } 575 \text{ cu. yd.}$$

Station	Cross-section			Area, square feet	Volume, cubic yards
	L	*C*	*R*		
115	$\frac{c4.0}{16.0}$	$\frac{c6.0}{0}$	$\frac{c12.0}{28.0}$	212	
					575
116	$\frac{c2.0}{13.0}$	$\frac{c3.0}{0}$	$\frac{c8.0}{22.0}$	103	
Mid-section	$\frac{c3.0}{14.5}$	$\frac{c4.5}{0}$	$\frac{c10.0}{25.0}$	154	

While the prismoidal formula gives the true volume of a prismoid, the difference between results obtained through its application and values obtained by the method of average end areas is not large except where the change in cross-section is abrupt. For the above example, the volume calculated by average end areas is 583 cu. yd. and the difference between the results obtained by the two methods is 8 cu. yd. or about 1.4 per cent. It may be concluded that, so far as volumes of earthwork are concerned, the use of the prismoidal formula is justified only when cross-sections are taken at short intervals, when the observations are so conducted that small surface deviations will be measured, and when the areas of successive cross-sections differ widely.

161. Earthwork from Contours.—For preliminary estimates for grading areas, especially where the graded surface is itself more or less irregular in shape, the common practice is to utilize the topographic map directly as a basis for calculations of volume. On the map are shown the contours for the natural ground and also the contours for the proposed graded surface. One method consists of determining the volumes of earth to be moved between the horizontal planes marked by successive contours. A second method, sometimes more conveniently applied, consists of calculating volumes between upper and lower surfaces bounding certain increments of cut or fill. In either case, horizontal projections of areas are taken from the map, usually with the planimeter, and the volume between any two successive areas is determined by multiplying the average of the areas by the increment of depth between them.

The first of these two methods will now be described in detail.

161a. The full lines of Fig. 161 represent contours of the original ground, and the dash lines represent contours of the proposed graded surface. The heavy full lines are drawn through points of no cut or fill. Thus the line *abcdefa* bounds an area which is entirely in fill and the line *dghjked* bounds an area that is entirely in cut, as will be seen by studying the figure. The "no cut or fill" lines are seen to pass through the points of intersection between full contours and the corresponding dash contours (as at *a*, *b*, *d*, *e*, *h*, and *j*). The conditions surrounding the problem make it possible to estimate the position of the lines where the cut or fill runs out between contours (as the lines *bcd*, *efa*, and *jke*). The cross-hatched portions

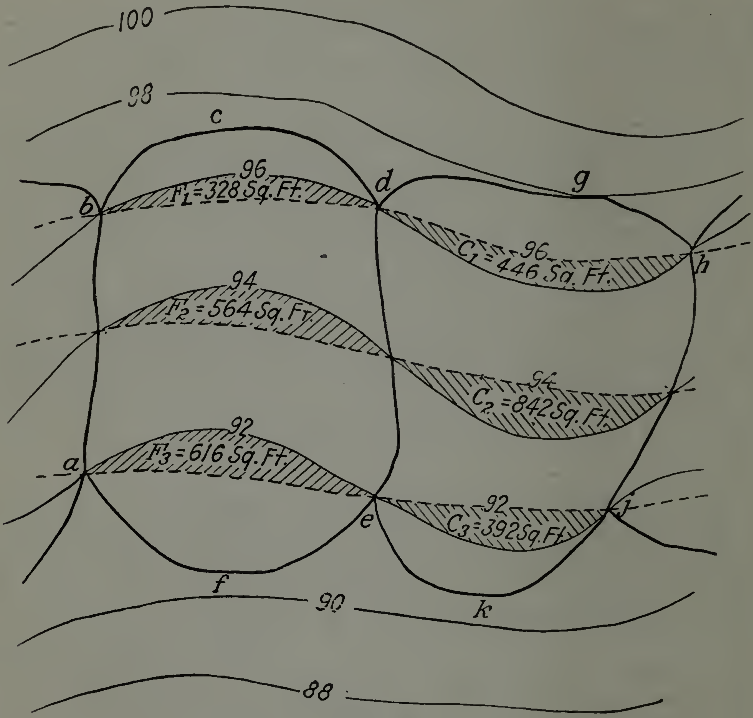


Fig. 161.

are the horizontal sections of earth cut or filled at the contour elevations. Thus F_1 represents the horizontal section of earth filled at elevation 96. It is evident that the volume of earthwork between two successive contours is a solid whose altitude is the contour interval and whose top and bottom bases are the horizontal projections of the cut or fill at the contour elevations (as between the 94- and 96-ft. contours, for fill *abcdefa*, the height is 2 ft. and the bases are F_1 and F_2). Where the cut or fill runs out between contours (as along line *bcd*) the height of the end volume will be less than the contour interval. This height may be estimated by assuming the slope of the ground to be uniform between contours (thus point *c* is estimated to be at elevation 97.2 and the volume above the 96-ft. contour is a solid whose base is F_1 and whose altitude is $97.2 - 96 = 1.2$ ft.). The end volumes may be considered as pyramids.

Example 1: It is desired to determine the volume of earthwork in fill bounded by the line *abcdefa* (Fig. 161). The intermediate volumes are to be calculated by the method of average end areas; the end volumes are to be considered as pyramids. The areas of fill at the contours are as shown in the figure. The point *c* is estimated to lie 1.2 ft. above the 96-ft. contour and the point *f* is estimated to be 1.6 ft. below the 92-ft. contour.

Elevation	Base area, square feet	Altitude, feet		Volume, cubic feet
c = 97.2	0			
		1.2	⅓ × 1.2 × 328	131
96	328			
		2.0	½ × 2.0 × 892	892
94	564			
		2.0	½ × 2.0 × 1,180	1,180
92	616			
		1.6	⅓ × 1.6 × 616	328
f = 90.4	0			

Total........2,531 cu. ft.
or 94 cu. yd.

162. Volumes from Road Profiles.—Preliminary estimates of earthwork for highways, railroads, and canals made prior to the location of the route are based on the preliminary profile. If the side slopes were vertical, the volume of any cut or fill would be a direct function of the area between grade line and ground line on the profile. Since the side slopes are inclined, the volume in cut or fill increases at a relatively greater rate than does the depth, hence, except for the purpose of rough estimates, the area representing cut or fill on the profile cannot be directly taken as a measure of volume, as would be the case for a trench.

Usually the cut or fill at each full station is scaled from the profile and the corresponding volume in cubic yards per 100 ft. is calculated for a level section whose depth is the scaled cut or fill. The level section at the station is assumed to exist over a length of 100 ft.

(*i.e.*, 50 ft. in advance and 50 ft. back of the station). The total volume for a given cut or fill is obtained by summing up the volumes per station obtained in the manner just described.

163. Numerical Problems.

1. From the notes of Fig. 138 calculate by the method of average end areas the volume in cut and in fill between station 605 and station 613. The roadbed is 20 ft. wide in cut and 16 ft. wide in fill and the side slopes are 1½:1. Tabulate data in the following form: "Sta." "Cross-section," "Area," "Volume."

2. From the data of problem 1, compute volumes by the prismoidal formula. Note the discrepancy in per cent between volumes as determined by the prismoidal and end-area methods.

3. What error in volume between station 605 and station 606 (Fig. 138) would be introduced if the recorded cuts at centers and slope stakes are 0.1 ft. too great? What is the error in terms of per cent of the volume by average end areas?

4. Suppose that between the above two stations a sag gradually takes place over a width of 24 ft. becoming a maximum of 1 ft. at 605 + 50. What error is introduced in the calculated volume?

✓5. Following are the notes for an irregular cross-section. The width of the roadbed is 24 ft. Calculate the cross-sectional area by the methods of Art. 154.

$$\frac{c4.2}{18.3} \quad \frac{c6.8}{12.0} \quad \frac{c11.2}{0} \quad \frac{c14.4}{10.0} \quad \frac{c16.8}{25.0} \quad \frac{c18.4}{39.6}$$

✓6. The tracing arm of a planimeter is set so that the roller reads 0.583 rev. for 10 sq. in. A figure for which the vertical scale is 1 in. = 8 ft. and the horizontal scale is 1 in. = 20 ft. is traversed with pole outside, and the planimeter reading is 1.932 rev. What is the actual area of the figure in square inches? And what area in square feet does it represent?

7. With the tracing arm of the planimeter set as in problem 6, the perimeter of an area is traversed first with pole outside and then with pole inside. The corresponding scores are 2.095 and −7.786. What is the area of the zero circle in square inches?

8. The perimeter of a figure is traversed with the pole inside and the tracing arm of the planimeter set as in problems 6 and 7. The score of the roller is −3.781. What is the area in square inches?

9. A given fill for a railroad is 1,350 ft. long. The profile is plotted at the horizontal scale of 1 in. = 400 ft. and at the vertical scale of 1 in. = 20 ft. The perimeter of the area between ground line and grade line is traversed with a planimeter set so that 1 revolution of the roller is equal to 10 sq. in. on the paper, and the score is 0.269. What is the average depth of the fill in feet? Estimate the volume of the fill in cubic yards, assuming a level section of the average depth, roadbed 18 ft. wide, and side slopes 1½ to 1.

CHAPTER XI

MEASUREMENT OF ANGLES AND DIRECTIONS

GENERAL METHODS

164. Remarks.—As has been stated, the purpose of a survey is to determine the relative positions of points on or near the surface of the earth. The location of a point is fixed if measurements are made of (1) its direction and distance from a known point, (2) its direction from two known points, (3) its distance from two known points, or (4) its direction from one known point and its distance from another. If the relative position of points as seen in horizontal projection is desired, the field operations involve the measurement of horizontal distances, as described in Chap. VI, and the determination of direction in the horizontal plane. In addition, if the relative elevations of points are required, they are determined by one or another of the methods of leveling described in Chaps. VII to IX.

For horizontal projection or plan, the direction of any line as defined by two points, is determined by horizontal angular measurements between the line and some reference line. For vertical projection, the direction of one point with respect to another is defined by the vertical angle between the plane of the horizon and the line joining the two points. In general, therefore, the angular measurements of surveying are either horizontal or vertical, or approximately so. When the angle between two points is spoken of, it is understood to mean the horizontal angle between lines passing through the two points and converging at a third. Thus at O the angle between B and C is the horizontal angle BOC. The vertical angle to a point is its angle of elevation or depression from the horizontal as measured from some point of reference; the angle is positive or negative according as the observed point is above or below the horizontal. Thus the vertical angle to B as measured from A is positive if B is higher than A. Measurement of vertical angles as applied to indirect leveling was briefly considered in Chap. VII, and will be described more in detail in Art. 193. The present chapter treats of angles and directions in the horizontal plane.

165. Angles and Directions.—The relative directions of lines connecting survey points may be obtained in a variety of ways.

Figure 165*a* shows lines about a point. The direction of any line (as OB) with respect to an adjacent line (as OA) is given by the horizontal angle between the two lines (as a) and the direction of rotation (as clockwise). The direction of any line (as OC) with respect to a line not adjacent (as OA) is not given by any of the measured angles but may be computed by adding the intervening angles (as $a + b$).

Figure 165*b* shows the same system of lines but with angles measured from a line of reference OM. The direction of any line (as OA) with respect to the line of reference (as OM) is given by the angle between the lines (as 1) and its direction of rotation (as clockwise). The angle between any two lines (as AOC) is not given directly, but may be computed by taking the difference between the direction angles of the two lines (as $\angle 3 - \angle 1 = \angle AOC$).

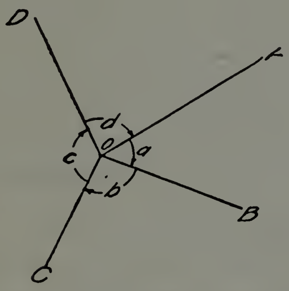


Fig. 165*a*.

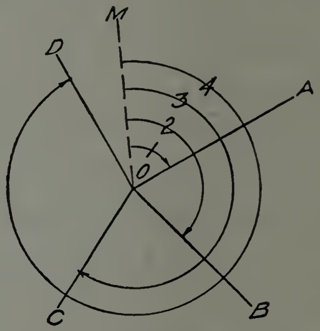


Fig. 165*b*.

In the first case it will be noted that a given angle denotes the direction of a line with respect to an adjacent line. In the second case a given angle denotes the direction of a line with respect to a fixed line of reference. In surveying, angular measurements may fall under either of the above general cases. The fixed line of reference may be any line in the survey or may be purely imaginary. It is termed a *meridian.* If it is arbitrarily chosen without special reference to the points of the compass, as is often the case, it is called an *assumed meridian;* if it is a true north and south line passing through the geographical poles of the earth it is called a *true meridian;* if it lies parallel with the magnetic lines of force of the earth as indicated by the direction of the magnetic needle it is called a *magnetic meridian.*

166. The True Meridian.—The true meridian is determined by astronomical observations to be described in a later chapter. For any given point on the earth its direction is always the same, and hence directions referred to the true meridian remain unchanged regardless of time. The lines of most extensive surveys, particularly those of higher precision, are referred to the true meridian, as are also, in general, the lines marking the boundaries of landed property.

167. The Magnetic Meridian.—The direction of the magnetic meridian is that taken by a freely suspended magnetic needle. Unlike the geographical poles, the magnetic poles are constantly varying in position. Hence the direction of the magnetic meridian is by no means constant nor is it in general parallel with the true meridian. The magnetic meridian is employed as a line of reference on the rougher surveys where a magnetic compass is used, or in connection with more precise surveys where angular measurements are approximately checked by means of the compass needle.

168. The Magnetic Needle.—Any slender symmetrical bar of magnetized iron when freely suspended at its center of gravity takes up a position parallel with the lines of magnetic force of the earth. In horizontal projection these lines define the magnetic meridian. In elevation, the lines are inclined downward toward the north in the Northern Hemisphere, and downward toward the south in the Southern Hemisphere. Since the bar takes a position parallel with the lines of force, it becomes inclined with the horizontal. This phenomenon is called the *magnetic dip*. The angle of dip varies from 0° at or near the equator to 90° at the magnetic poles. The needle of the magnetic compass rests on a pivot. To counteract the effect of dip, so that the needle will take a horizontal position when directions are observed, a counterweight is attached to one end (the south end in the Northern Hemisphere). So long as the needle is used in a given locality and it loses none of its magnetism, it will remain balanced. When for any reason it becomes unbalanced, it is adjusted to the horizontal by sliding the counterweight along the needle. At the mid-point of the needle is a jewel which forms a nearly frictionless bearing for the pivot.

169. The Magnetic Declination.—The angle between the true meridian and the magnetic meridian is called the magnetic declination. If the north end of the compass needle points to the east of the true meridian, or in other words, if the direction of rotation from the true meridian to the magnetic meridian is clockwise, the declination is said to be east; if the north end of the needle points to the west of the true meridian, the declination is said to be west.

170. The Isogonic Chart.—This chart (Fig. 170) shows lines of equal declination for the date Jan. 1, 1925, as based on observations made by the U. S. Coast and Geodetic Survey at stations widely scattered throughout the country. The agonic line, or line of zero declination, is the heavy, full, irregular line extending in a southeasterly direction from the Great Lakes. The isogonic lines or light, full lines east of the agonic line mark the paths where the declinations were on the above date, 1° west, 2° west, etc.; similarly, those west of the agonic line show the routes along which the declinations were 1° east, 2° east, etc.

171. Variations of Declination.—Changes in the direction of the magnetic meridian cause corresponding variations in the magnitude of the declination.

The *secular variation* is a gradual change extending over a long period of years. Although its causes are not well understood, the indications are that it is more or less periodic in character, and in some localities in the United States, it amounts to several degrees. Like a pendulum, the magnetic meridian swings in one direction for perhaps a century and a half, or more, until it gradually comes to rest, and then swings in the other direction. In Fig. 170 are shown by dash lines the annual rates of change in the secular variation for the year 1925. North and east of the line marking no change, the west declinations are increasing and east declinations are decreasing at the annual rates indicated on the chart; south and west of the line of no change the east declinations are increasing. When *variation* is spoken of without further qualification it is taken to mean the secular variation.

The solar-diurnal variation is a periodic swing of the magnetic needle occurring during each day. For points in the United States the north end of the needle reaches its extreme easterly swing at about 8 a.m., and its extreme westerly swing at about 1 p.m. The needle reaches its mean position between 10 and 11 a.m. and between 6 and 7 p.m. The average daily change for points in the United States is less than 07′, a quantity so small as to need no consideration for most of the work on which the compass needle is employed.

Other variations are the *annual variation*, a periodic annual swing, quite distinct from the secular variation, amounting to less than 01′ for most places in the United States, and *irregular variations* due to magnetic disturbances.

122. Local Attraction.—Objects of iron or steel, some kinds of iron ore, and electric wires alter the direction of the lines of magnetic force in their vicinity and hence are likely to cause the compass needle to deviate from the magnetic meridian. The deviation arising from such local sources is called *local attraction*. In certain localities, particularly in cities, its effect is so pronounced as to render the magnetic needle of no value for determining directions. Local attraction is not likely to be the same at one point as at another,

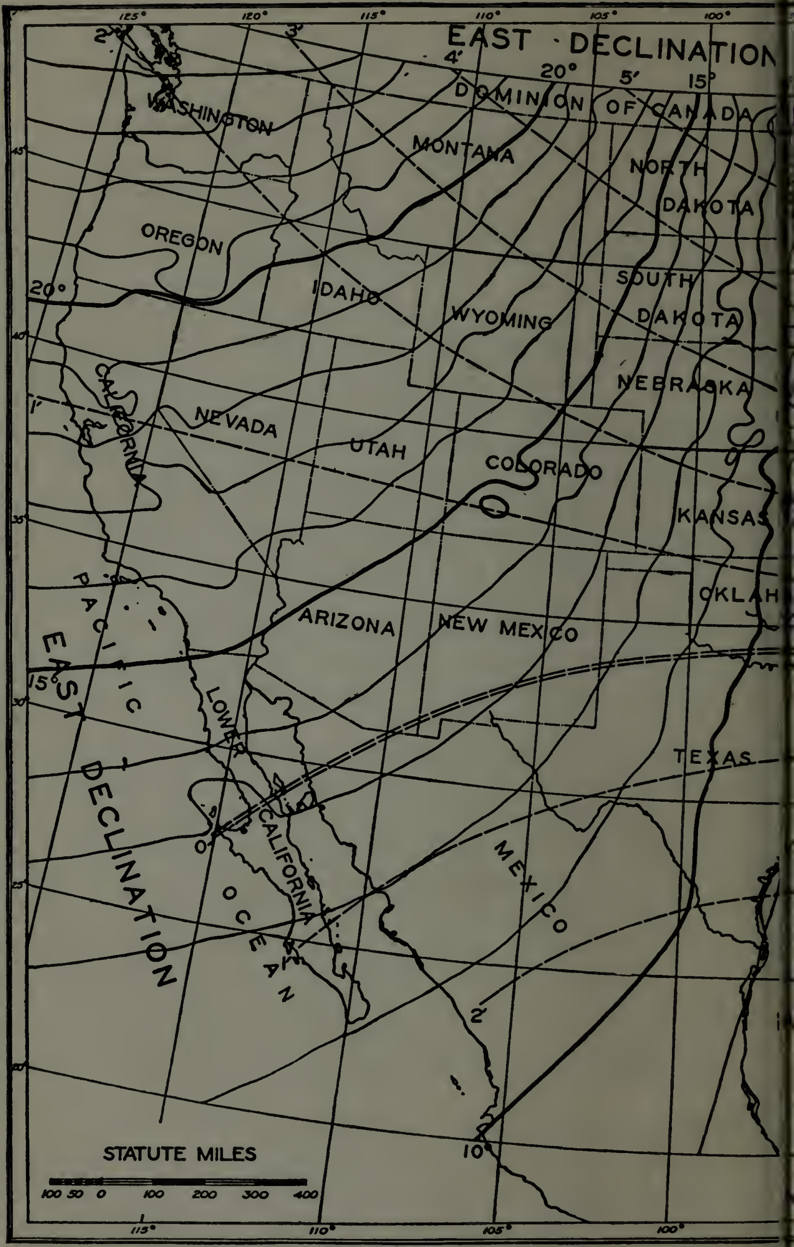
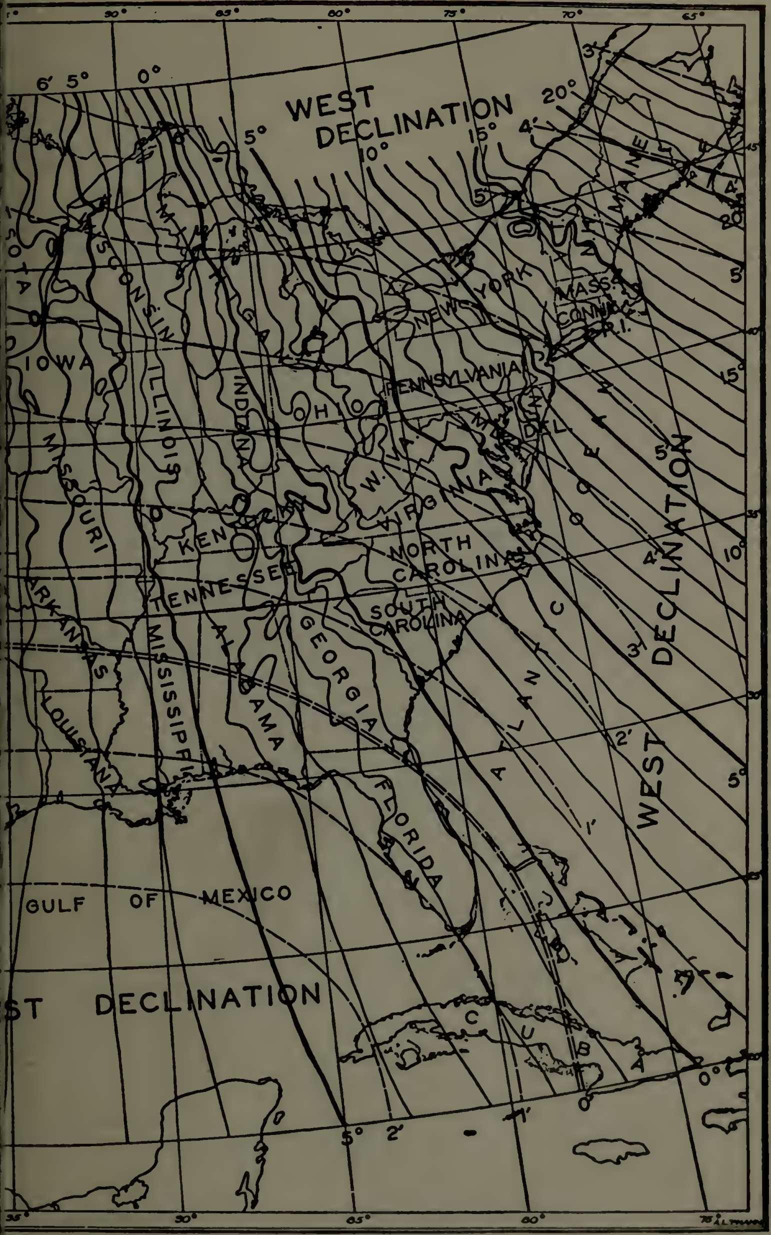

EAST DECLINATION
DOMINION OF CANADA
WASHINGTON
MONTANA
NORTH DAKOTA
OREGON
IDAHO
WYOMING
SOUTH DAKOTA
NEBRASKA
CALIFORNIA
NEVADA
UTAH
COLORADO
KANSAS
OKLAH
ARIZONA
NEW MEXICO
TEXAS
LOWER CALIFORNIA
MEXICO
PACIFIC OCEAN
EAST DECLINATION
125°
120°
115°
110°
105°
100°
2'
3'
4'
5'
1'
0
2'
20°
15°
10°
STATUTE MILES
100 50 0 100 200 300 400

Fig. 170.—Isogonic chart of the United States f

The lines of equal magnetic declination (full lines) apply to January 1, 1925. East
west of that line it points east of north.

The north end of the compass needle is moving to the westward for places north of t
cated by the lines of equal annual change (dash lines)


WEST DECLINATION
6′ 5° 0°
5° 10° 15° 20°
4′ 3′ 5′
IOWA
WISCONSIN
MICH
MINN
OHIO
INDIANA
ILLINOIS
MISSOURI
KENTUCKY
TENNESSEE
ARKANSAS
LOUISIANA
MISSISSIPPI
ALABAMA
GEORGIA
FLORIDA
SOUTH CAROLINA
NORTH CAROLINA
VIRGINIA
W. VA.
PENNSYLVANIA
NEW YORK
MASS.
CONN.
R.I.
N.J.
DEL.
MD.
MAINE
ATLANTIC OCEAN
WEST DECLINATION
2′ 1′ 0°
GULF OF MEXICO
ST DECLINATION
CUBA
5° 2′ 1′ 0
90° 85° 80° 75° 70° 65°
45° 40° 35° 30° 25°
95° 90° 85° 80° 75°

25. *(From the U. S. Coast and Geodetic Survey.)*

e line marked 0° (agonic line) the north end of the compass needle points west of north,

e of no change and to the eastward for places south of that line at an annual rate indi-

(*Facing page* 148.)

even though the points be but a short distance apart. By methods later to be described, the magnitude of local attraction may usually be determined and directions observed with the compass may be corrected accordingly.

173. Bearings.—The direction of any line with respect to a given meridian may be defined by the acute angle which the line makes with the meridian. In Fig. 173*a* let *SN* represent a meridian, either true, magnetic, or assumed, and let *OA*, *OB*, *OC*, and *OD* be lines whose directions with respect to the meridian are desired. The bearing of a line is indicated by the quadrant in which the line falls and the angle which the line makes with the meridian in that quadrant. Thus the line *OA* is in the northeast quadrant and makes an angle of 37° in that quadrant with the meridian. The bearing of *OA* is read as North 37° East and is written N37°E. The bearings of *OB*, *OC*, and *OD* are respectively S62°E, S50°W, and N20°W. In all cases values of bearing angles lie between 0° and 90°. If the direction of the line is parallel with the meridian and north, it is written as N0° or *Due North.* If perpendicular to the meridian and east, it is written as N90°E or *Due East.*

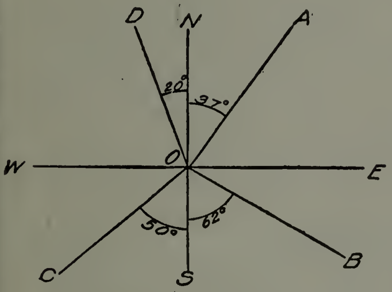


FIG. 173*a*.

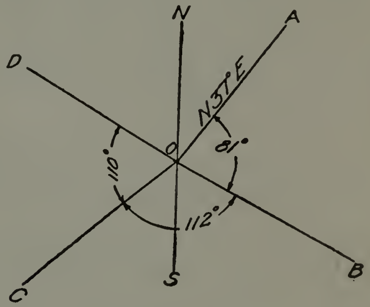


FIG. 173*b*.

When the bearing of a line is referred to the true meridian, it is called a *true bearing;* when referred to the magnetic meridian, it is called a *magnetic bearing;* and when referred to an assumed meridian, it is called an *assumed bearing.*

Observed bearings are those for which the actual bearing angles are directly obtained by field observations. *Calculated bearings* are those indirectly obtained by computation. Thus if in Fig. 173*b* the observed bearing of *OA* is N37°E and the angles *AOB* = 81°, *BOC* = 112°, and *COD* = 110°, then the calculated bearings are

Bearing *OB* = 180 − 37 − 81 = S62°E
Bearing *OC* = 112 − 62 = S50°W
Bearing *OD* = 180 − 50 − 110 = N20°W

174. Azimuths.—The azimuth of a line is its direction as given by the angle between the meridian (either true, magnetic, or assumed) and the line, always measured in a clockwise direction from the south point (or from the north point) of the meridian. In astronomical observations azimuths are always reckoned from the true south; in surveying, some surveyors reckon azimuths from the south point and some from the north point of whatever meridian is chosen as a reference, but on any given survey the direction of zero azimuth is either always south or always north. Azimuths are called *true azimuths*, *magnetic azimuths*, or *assumed azimuths* according as the meridian to which they are referred is true, magnetic, or assumed.

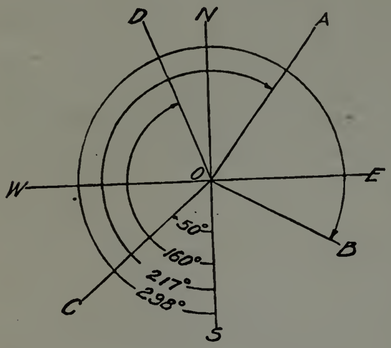


Fig. 174*a*.—Azimuths from south.

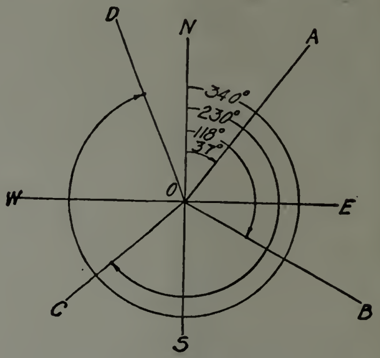


Fig. 174*b*.—Azimuths from north.

Azimuths may have values between 0° and 360°.

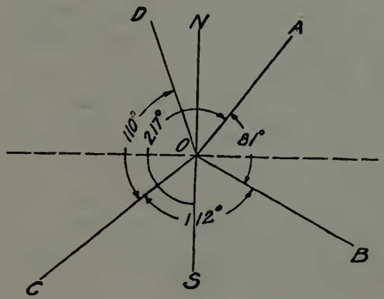


Fig. 174*c*.

An observed azimuth is one which is directly obtained by a field observation. Thus in Fig. 174*a* azimuths measured from the south point are Az. *OA* = 217°, Az. *OB* = 298°, Az. *OC* = 50°, and Az. *OD* = 160°; (or in Fig. 174*b* in which are shown the same lines with azimuths measured from the north point, Az. *OA* = 37°, Az. *OB* = 118°, Az. *OC* = 230°, Az. *OD* = 340°).

Calculated azimuths are those obtained by computation. Thus in Fig. 174*c* if the azimuth of *OA* as reckoned from the south is 217° and the angle *AOB* is 81°, the calculated azimuth of *OB* is 217 + 81 = 298°. Similarly the calculated azimuth of *OC* is 298 + 112 − 360 = 50°, and the calculated azimuth of *OD* is 50 + 110 = 160°.

Since both azimuths and bearings give directions with respect to a meridian, the operation of calculating azimuths from observed bearings, or *vice versa*, is a simple matter if a sketch is drawn to show the existing relations.

Example 1: Calculate azimuths (reckoned from south) corresponding to the following bearings: *OB*, N16°E; *OC*, S75°E; *OD*, S38°W; *OE*, N89°W. The azimuths are *OB*, $180 + 16 = 196°$; *OC*, $360 - 75 = 285°$; *OD*, 38°; *OE*, $180 - 89 = 91°$.

175. Deflection Angles.—The angle between a line and the prolongation of the preceding line is called a *deflection angle*. Thus in Fig. 175 if *AB* is the preceding line, then the deflection angle to the line *BC* is as indicated. Deflection angles are recorded as *right* or *left* according as the angle as measured from the preceding line is clockwise or counter-clockwise, or in other words, according as the line to which measurement is taken lies to the right or left of the prolongation of the preceding line. Thus in Fig. 175 the deflection angle at *B* is clockwise 22° and is recorded as 22°R; the deflection angle at *C* is counter-clockwise and is recorded as 33°L. Deflection angles may take any value between 0° and 180° but are not usually employed for angles greater than 90°. In any closed polygon the algebraic sum of the deflection angles (considering right deflections as of sign opposite to left deflections) is 360°.

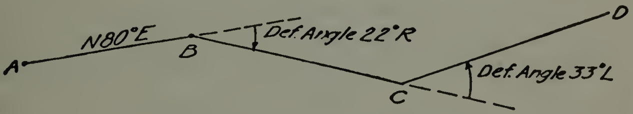


Fig. 175.—Deflection angles.

If the bearing of any line is known and the deflection angles are observed, the bearings of other lines may be calculated. Thus in the above figure the bearing of *AB* is N80°E, hence the bearing of *BC* is $180 - 80 - 22 =$ S78°E.

176. Interior Angles.—In a closed polygon the angles inside the figure between adjacent lines are called interior angles. If n is the number of sides in a closed polygon, then the sum of the interior angles is $(n - 2)180°$.

177. Traverses.—The broken line connecting a succession of established points along the route of a survey is called a *traverse* or a *traverse line*. Distances along the line between successive points are determined by direct measurement. At each point where the traverse changes direction an angular measurement is taken. If the traverse forms a closed figure, as for example, the boundary of a parcel of land, it is called a *closed traverse;* if it does not form a closed figure, as for example, the survey for a highway, it is called a *continuous traverse* or an open traverse. The points defining the traverse line are called *traverse stations* or traverse points.

Generally speaking, deflection angles are employed on continuous traverses where the change in direction of the line is less than 90° at each traverse station. For the location of railroads, highways, canals, etc., angular measurements are nearly always taken by deflections. Azimuths are widely used on topographic surveys and similar surveys where a large number of details are located by angular measurements from the traverse stations. Ordinarily the interior angles of traverses run to establish the boundaries of land are observed. Traverses by bearings are rarely run except on rough surveys where the magnetic compass is employed, though magnetic bearings are generally observed as a rough check on deflection angles, azimuths, or interior angles determined by more precise methods.

178. Triangulation.—Where the lines of a survey form triangular figures whose angles are measured and whose distances are determined by trigonometric computations, the operation of making the necessary field observations is called triangulation. The simplest case is that of a single triangle, one of whose sides is of known length. If any two angles of the triangle are measured, sufficient data are obtained for calculating the lengths of the other two sides. Further, if the third angle is measured, the angular measurements may be checked. When the survey is made up of a series of triangles so connected that having measured the angles of the triangles and the length of one line, the lengths of other lines may be calculated, it is called a *triangulation system.* The side of known length, upon which all calculated distances are based, is called a *base line.*

The subject of triangulation is discussed in Chap. XXV.

Triangulation is often necessary in connection with traversing where the direct measurement of one or more lines is impossible. Simple triangulation is also employed for the location of tunnel shafts and bridge piers. A simple chain of triangles affords a convenient means of locating points on opposite sides of a stream. A group of triangles is suitable for the survey of an area. The advantage of triangulation over traversing lies in the small number of linear measurements which are necessary; the disadvantage lies in the increased labor of computing.

179. Angles and Directions with the Transit.—On most surveys angles are measured with the engineers' transit, the use and adjustment of which will be described in the following chapter. The essential features of the transit here to be considered are (1) a horizontal circle, graduated in degrees, which may be rotated and may be clamped in any position, (2) a plate which may be rotated inside the graduated circle and which carries verniers for reading the graduated circle, and (3) a telescopic line of sight which may be rotated in altitude but which is attached to the vernier plate and rotates with it. By means of the verniers the graduated circle on most instruments may be read to the nearest minute of arc, but on some precise transits to the nearest 10 seconds.

If the angle *ABC* in Fig. 179 is to be measured, the transit is set at *B*. The index of the vernier is set at zero on the graduated circle and a sight is taken to *A*. The graduated circle is clamped in position and the line of sight is turned to *C*. Since the vernier plate is moved with the line of sight, it is rotated through the angle *ABC* and hence the vernier reads the angle. If azimuths are to be observed (and the circle is graduated from 0° to 360° in a clockwise direction) then the backsight to *A* is taken with the vernier set to read the azimuth of the line *BA*. When the line of sight is rotated to *C*, the vernier reading will be the azimuth of the line *BC*. If the deflection angle at *B* (Fig. 179) is to be observed, a backsight is taken on *A* with the vernier set at 0°, the line of sight is rotated first in altitude to point in the direction of *BD*, and then in azimuth until the point *C* is sighted; the vernier then reads the deflection angle *DBC*.

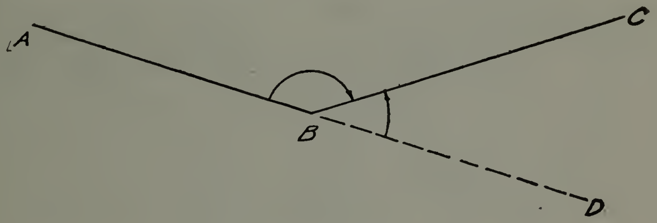


Fig. 179.

In running a traverse, angular measurements are made at successive stations. If an error occurs in any angle, then the observed or calculated directions of all succeeding lines in the traverse will be affected by the amount of the error.

180. Direction with the Magnetic Compass.—The types of compass used by the surveyor are: (1) those of the ordinary pocket variety designed for the traveler, sometimes useful on very rough exploratory surveys, and (2) those designed primarily for surveying.

The essential features of the latter are: (1) a compass box with circle graduated from 0° to 90° in either direction from the N and S points and having the E and W points interchanged as illustrated in

Fig. 180*a*; (2) a line of sight in the direction of the SN points of the compass box, and (3) a magnetic needle which may be raised from its pivot and clamped in position. When the line of sight is pointed in a given direction the compass needle (when brought to rest) gives the magnetic bearing. Thus in the figure the bearing of *AB* is N60°E. If the N point of the compass box is nearest the object sighted, the bearing is read by observing the north end of the needle. The varieties exhibiting the three features just mentioned are: (1) pocket compasses which are generally held in the hand when bearings are observed, and which are used on reconnaissance or other rough surveys, (2) the surveyors' compass which is mounted on a light tripod and which was formerly much used in all kinds of land surveying, but is now little employed except for forest surveys, and (3) the transit compass, a compass box mounted on the upper or vernier plate of the engineers' transit, a description of which is given in the following chapter.

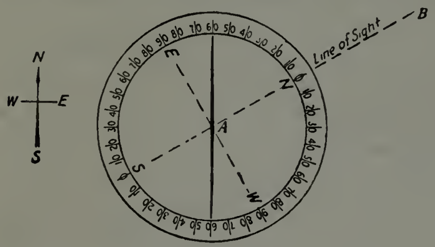


Fig. 180.

181. The Surveyors' Compass (Fig. 181*a*) consists of a compass box mounted on a plate *A* to the ends of which are fastened the vertical sight vanes *B*, *B*. The plate is rigidly connected to a vertical spindle beneath the center of the compass, which is free to revolve in a conical socket at *C*. At *D* is a leveling head consisting of a ball and socket joint and at *E* are level tubes by means of which the compass may be leveled. The upper portion of the socket at *D* is a thumb nut which is tightened until the ball is held securely by friction. *F* is a screw for lifting and clamping the needle, *G* is an outkeeper for registering chainage in land surveying, *H* is a screw for clamping the vertical spindle, and *J* is a spring catch which when pulled allows the removal of the spindle from the cone-shaped socket. The compass circle is usually graduated in half-degrees, and bearings may be read to 05′ or 10′. In order that *true* bearings may be read directly, some compasses, as the one shown in the illustration, are so designed that the compass circle may be rotated with respect to the plate on which it is mounted. When the circle is in its normal position, the line of sight as defined by the vertical slits in the sight vanes is in line with the N and S points of the compass circle, and the observed bearings are magnetic. If the circle is turned through an angle equal to the magnetic declination, the observed bearings will be true, as is evident

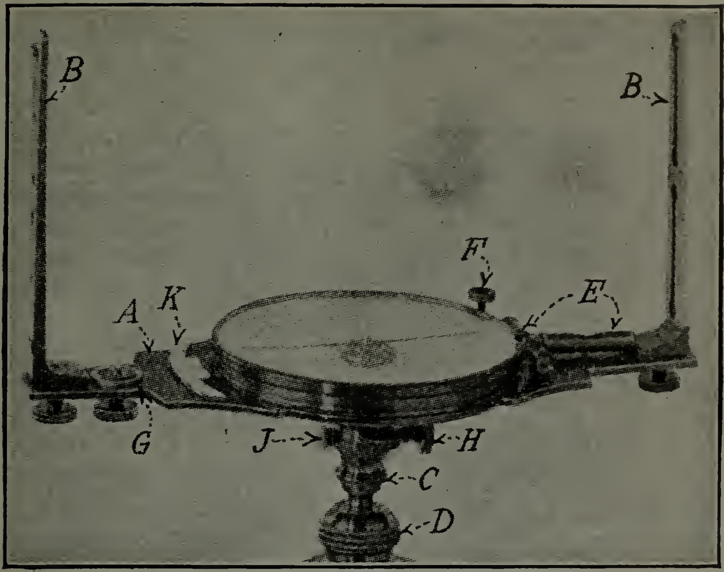

B
B
F
E
A
K
G
J
H
C
D

FIG. 181*a*.—Surveyors' vernier compass.

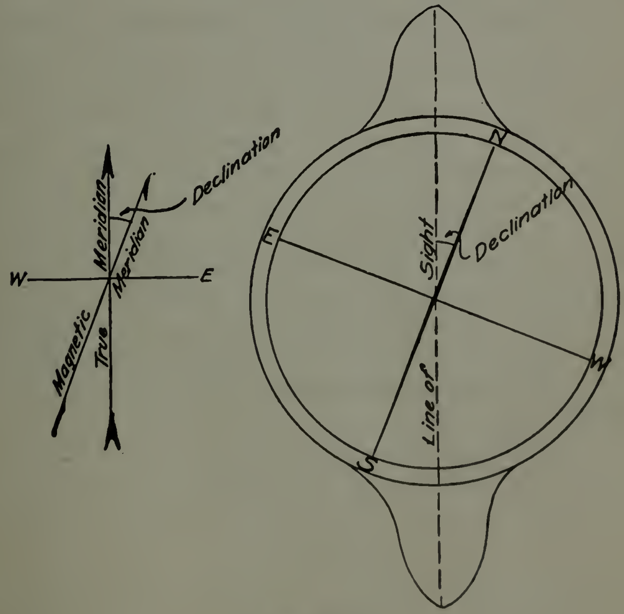

Declination
Meridian
Meridian
W
E
Magnetic
True
N
E
W
S
Declination
Line of Sight

FIG. 181*b*.

from Fig. 181*b*. At *K* (Fig. 181*a*) is a graduated arc and vernier by means of which the declination may be laid off to the nearest minute.

When the direction of a line is to be determined, the compass is set up on line and is leveled. The needle is released, and the compass is rotated about its vertical axis until a range pole or other object on line is viewed through the slits in the two sight vanes. When the needle comes to rest the bearing is read. Ordinarily the sight vane at the S end of the compass is held next to the eye, in which case the bearing is given by the north end of the needle.

When a traverse is run, only alternate stations need be occupied, but a check is secured and local attraction will be detected if both a backsight and a foresight are taken from each station. Unlike the transit traverse, where an error in any angle affects the observed or calculated directions of all following lines, an error in the observed bearing of one line in a compass traverse has no effect on the observed directions of any of the other lines.

182. Corrections for Local Attraction.—If local attraction exists at any station in a traverse both the back and forward bearings taken from that station will be affected by the same amount. Disregarding for the time being the accidental errors due to observing, it is probable that the terminal points of any line, as *AB*, are free from local attraction if the back bearing from *B* is the reverse of the forward bearing from *A*. Keeping in mind that the calculated angle between the forward and back lines from any station may be correctly determined from the observed bearings taken from that station regardless of whether or not the needle is locally affected, the direction of a line free from local attraction may be chosen as a basis, the traverse angles may be computed from observed bearings, and starting from the unaffected line, the correct bearings of successive lines may be calculated. Thus from the observed bearings tabulated below it is seen that points *A* and *B* are free from local attraction since the back and forward bearings of *AB* are opposite. Hence the correct bearing of *BC* is S60°E. The back bearing of *BC* shows an attraction of 2°. The angle at *C* as given by the bearings taken at that point is $180 - 62 - 31 = 87°$. The correct bearing of *CD* is therefore $180 - 60 - 87 =$ S33°W.

Station	Line	Observed bearing	Calculated angle	Corrected bearing	Correction
A					
	AB	N45°E		N45°E	0°
	BA	S45°W		S45°W	0°
B			75°		
	BC	S60°E		S60°E	0°
	CB	N62°W		N60°W	2°
C			87°		
	CD	S31°W		S33°W	2°
	DC	N30°E		N33°E	3°
D			100°		
	DE	N70°W		N67°W	3°
	ED	S67°E		S67°E	0°
E					

Owing to errors of observation there are likely to be discrepancies between the observed back and forward bearings of lines, even though no local attraction exist. If the discrepancies are small and are apparently not of a systematic character, it is fair to assume that the errors are due to other causes than local attraction.

When the compass traverse forms a closed figure, the interior angle at each station may be calculated from the observed bearings. The sum of the interior angles should equal $(n - 2)\ 180°$ in which n is the number of sides in the traverse. Since the error of observing a bearing is accidental in character, the error of closure of the traverse (as indicated by the sum of the calculated interior angles) is assumed to be distributed equally and the interior angles are corrected accordingly. The bearings are then adjusted by starting from some line whose observed bearing is assumed to be correct and calculating the bearings of successive lines by means of the corrected interior angles, after the manner described in the first paragraph in this article.

183. Methods of Determining the Meridian.—The true meridian is established by astronomical observations, a description of which is given in Chapter XVI. Any of the heavenly bodies whose astronomical position is known may be observed, but those commonly observed by surveyors are the sun and Polaris (the north star). For surveys of ordinary precision the instrument utilized is the transit.

A magnetic meridian may be established by setting up the compass over any convenient point marking one end of the proposed line, and then setting a series of points on a stake or other object marking the other end of the meridian. The mean of the points thus established is assumed to be on the magnetic meridian, provided the observations are taken at a time of day when the declination is about at its mean value.

184. Numerical Problems.

4 1. The magnetic bearing of a line is S47°30′W and the magnetic declination is 12°10′W. What is the true bearing of the line?

5 2. The true bearing of a line is N18°17′W and the magnetic declination is 7°12′E. What is the magnetic bearing of the line?

3. In 1856 the magnetic bearing of a line was N35°15′E and the declination was 2°10′W. At the present time the declination is 3°15′E. What is the true bearing and its present magnetic bearing?

4. Following are the observed magnetic bearings of a compass traverse: *AB*, N37°45′E; BC, N84°30′E; *CD*, S66°40′E; *DE*, S79°0′E; *EF*, N55°15′E. Calculate the deflection angles.

6 5. Following are deflection angles of traverse *A* to *F*: *B*, 37°21′L; *C*, 12°39′L; *D*, 63°31′R; *E*, 14°07′L. The true bearings of *AB* is *S*37°56′ E. Calculate the bearings of the remaining lines.

6. For the traverse of problem 4, the declination is 7°15′E. Compute the true azimuths reckoned from the north point.

7. For the traverse of problem 5 compute the true azimuths reckoned from the south point.

8. The interior angles of a five-sided closed traverse are as follows: *A*, 117°36′; *B*, 96°32′; *C*, 142°54′; *D*, 132°18′. The angle at *E* is not measured. Compute the angle at *E*, assuming the above values to be correct.

9. What are the deflection angles of the traverse of problem 8? What are the calculated bearings if the bearing of *AB* is due north?

10. The following azimuths are reckoned from the north: *AB*, 187°12′; *BC*, 273°47′; *CD*, 318°48′; *DE*, 0°48′; *EF*, 73°00′. What are the corresponding bearings? What are the deflection angles?

11. Following are the deflection angles of a closed traverse: *A*, 85°16′L; *B*, 10°11′R; *C*, 83°32′L; *D*, 63°27′L; *E*, 34°18′L; *F*, 72°56′L; *G*, 30°45′L. Calculate the error of closure. Adjust the angular values on the assumption that the probable error is the same for one angle as for any other.

12. In triangulating across a river a base line *AB* of the triangle *ABC* has a measured length of 536.27 ft. and the angles at *A* and *B* are respectively 87°32′ and 68°48′. Calculate the distance *AC*.

7 13. Below are bearings taken for a continuous compass traverse. Correct for local attraction.

Line	Forward bearing	Back bearing
AB	N37°15′E	S36°30′W
BC	S65°15′E	N65°30′W
CD	S31°00′E	N31°00′W
DE	S89°45′W	N89°45′E
EF	N46°30′W	S46°45′E
FG	N15°00′W	S14°45′E

14. The following are bearings taken on a closed compass traverse. Compute the interior angles and correct them for observational errors. Assuming the observed bearing of the line *AB* as being correct, adjust the bearings of the remaining sides.

Line	Forward bearing	Back bearing
AB	S37°30′E	N37°30′W
BC	S43°15′W	N44°15′E
CD	N73°00′W	S72°15′E
DE	N12°45′E	S13°15′W
EA	N60°00′E	S59°00′W

CHAPTER XII

THE USE AND ADJUSTMENT OF THE ENGINEERS' TRANSIT

THE ENGINEERS' TRANSIT

185. Description.—The engineers' transit is sometimes called the universal surveying instrument by reason of the wide variety of uses for which it is adapted. Figure 185*a* is a photograph of the type of transit in most common use and Fig. 185*b* is a vertical section of the same instrument. It is seen to consist of an upper or vernier plate to which are attached A-shaped standards supporting the telescope, and a lower plate to which is fixed a graduated circle. The upper and lower plates are fastened respectively to the inner and outer spindles, the axes of the spindles being concentric with the graduated circle and with the vernier plate. The outer spindle is seated in the cone-shaped socket of the leveling head. Near the lower extremity of the leveling head is a ball-and-socket joint, which secures the instrument to the foot plate yet permits rotation of the instrument about the joint as a center. When the lower plate is rotated the outer spindle revolves in its socket in the leveling head. By means of the lower clamp screw, the outer spindle, and hence the horizontal graduated circle, may be clamped in any position. After the lower clamp has been tightened, small movements of the horizontal circle may be secured by turning the lower tangent screw. Similarly, when the upper plate is rotated, if the lower plate is clamped, the inner spindle revolves within the outer spindle. By means of the upper clamp the inner spindle may be secured to the outer spindle, and any small movement of the verniers with respect to the graduated circle may be obtained by turning the upper tangent screw. The axis about which the spindles revolve is called the vertical axis.

Plate levels, one of which is fastened to the upper plate and the other of which is secured to one of the standards, are provided for leveling the instrument so that the plane of the horizontal circle will be truly horizontal when observations are made. The four leveling screws are threaded into the leveling head and bear against the foot plate; when the screws are turned, the instrument is rotated about the ball-and-socket joint. When all four screws are loosened, the transit may be shifted laterally with respect to the foot plate. To the end of the spindle a chain with hook is suspended, from which the plumb-line may be hung. The instrument is mounted on a tripod by screwing the foot plate onto the tripod head.

The telescope is fixed to the horizontal axis which rests in bearings at the upper extremities of the standards. The telescope may be rotated about the horizontal axis and may be fixed in any position in a vertical plane by means of the telescope clamp screw. Small movements about the horizontal axis may be secured by turning the telescope tangent screw. Fixed to the horizontal axis is the vertical circle, and attached to one of the standards is the vertical vernier. Beneath the telescope is the telescope level tube.

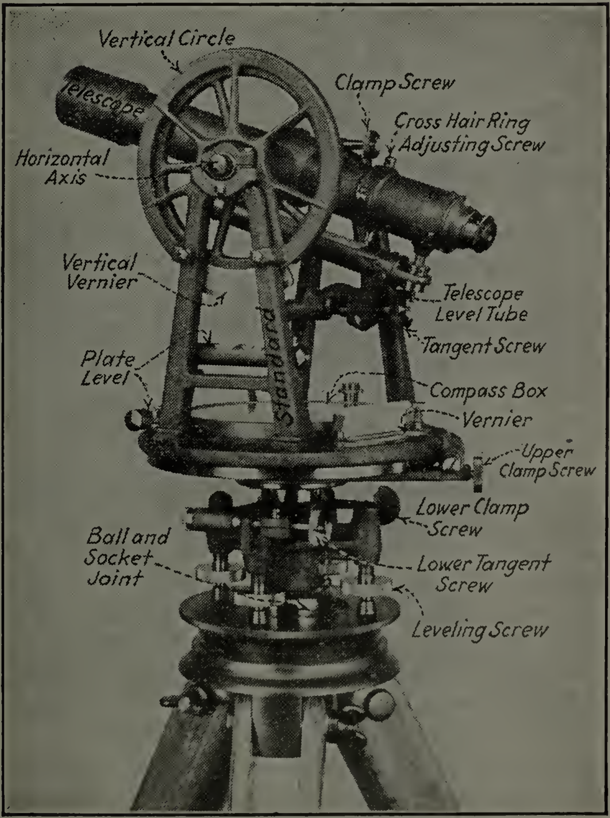

Vertical Circle
Telescope
Clamp Screw
Cross Hair Ring Adjusting Screw
Horizontal Axis
Vertical Vernier
Standard
Telescope Level Tube
Tangent Screw
Plate Level
Compass Box
Vernier
Upper Clamp Screw
Lower Clamp Screw
Ball and Socket Joint
Lower Tangent Screw
Leveling Screw

Fig. 185*a*.—Engineers' complete transit.

On the upper plate is placed the compass box, the details of which are the same as for the surveyors' compass described in Art. 181.

At the side of the compass box is a screw by means of which the magnetic needle may be lifted from its pivot and clamped.

Summing up the several features: (1) the center of the transit may be brought over a given point by loosening the leveling screws and shifting the transit laterally; (2) the instrument may be leveled by means of the plate levels and the leveling screws; (3) the telescope may be rotated about either the horizontal or the vertical axis; (4) when the upper clamp screw is tightened and the telescope is rotated about the vertical axis, there is no relative movement between the verniers and the horizontal circle; (5) when the lower clamp screw

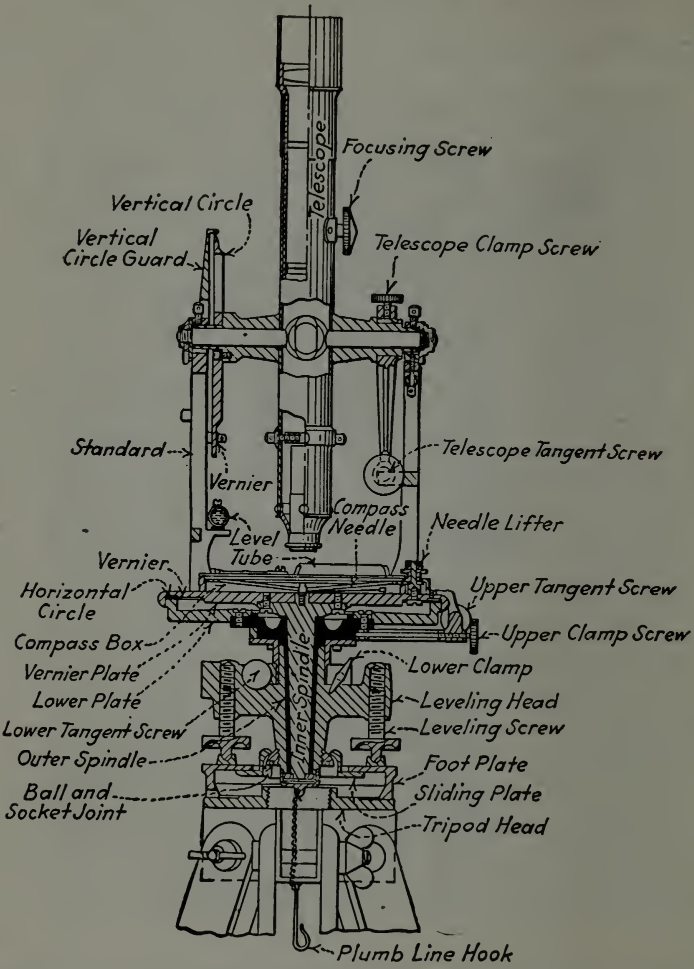

Telescope
Focusing Screw
Vertical Circle
Vertical Circle Guard
Telescope Clamp Screw
Standard
Telescope Tangent Screw
Vernier
Compass Needle
Needle Lifter
Level Tube
Vernier
Horizontal Circle
Upper Tangent Screw
Upper Clamp Screw
Compass Box
Vernier Plate
Lower Clamp
Lower Plate
Leveling Head
Lower Tangent Screw
Leveling Screw
Inner Spindle
Outer Spindle
Foot Plate
Ball and Socket Joint
Sliding Plate
Tripod Head
Plumb Line Hook

Fig. 185*b*.—Section of transit.

is tightened and the upper one is loose, a rotation of the telescope about the vertical axis causes the vernier plate to revolve but the horizontal circle remains fixed in position; (6) when both upper and lower clamps are tightened the telescope cannot be rotated about the vertical axis; (7) the telescope may be rotated about the horizontal axis and may be fixed in any direction in a vertical plane by means of the telescope clamp and tangent screws; (8) the telescope may be leveled by means of the telescope level tube, and hence the transit may be employed as an instrument for direct leveling; (9) by means of the vertical circle and vernier, vertical angles may be determined, and hence the transit is suitable for trigonometric leveling; (10) by use of the compass, magnetic bearings may be determined; and (11) by means of the horizontal circle and vernier, horizontal angles may be measured.

185a. There are several modifications of the instrument just described. A transit without vertical circle and telescope level tube is called a *plain transit.* One without compass and having U-shaped, one-piece standards, but otherwise the same as that illustrated in Fig. 185*a*, is often called a *city transit.*

Other types employ three leveling screws, a vertical vernier which is movable, a striding level, a telescope tangent screw with gradienter, and a vertical vernier level. When this level is centered, the vertical vernier readings give correct vertical angles, regardless of whether or not the plates are leveled. The movable vertical vernier with control bubble, as just described, is a feature of considerable value in topographic surveying where numerous vertical angles are taken from a single set-up.

186. Telescope.—The telescope of the transit is similar to that of the engineers' level (Arts. 87 to 94). When the transit is used as an instrument for direct or trigonometric leveling, the line of sight is defined by the optical center of the objective lens and a point on the horizontal cross-hair; when the transit is used for establishing lines, measuring angles, or taking bearings, the line of sight is defined by the optical center of the objective lens and a point on the vertical cross-hairs. Most instruments of recent manufacture are equipped with stadia hairs, which are usually mounted in the same plane with the cross-hairs. The magnifying power ranges from 18 for small instruments to 30 for larger ones designed for precise work. For the transit, as for the level, the erecting eyepiece is generally employed, but the superior optical properties of the inverting eyepiece make it the favorite of many surveyors, and it is the type used on instruments of precision. For the relative merits of the inverting and erecting eyepieces, see Art. 94.

187. The Graduated Circles.—The vertical circle has two opposite zero points and is graduated usually in ½°, the numbers increasing to 90° in either direction from the zero points. When the telescope is level, the index of the vernier is at 0°.

The horizontal circle is likewise usually graduated in ½°. It may be numbered from 0° to 360° clockwise and 0° to 90° in quadrants, or 0° to 360° in each direction.

The horizontal circles of transits designed for work of moderately high precision are graduated to 20′ or to 15′.

188. Verniers.—The verniers employed for reading the horizontal and vertical circles are identical in principle with those for the target rod described in Art. 103. Practically all transit verniers of recent manufacture are of the direct type.

Figure 188*a* shows the usual type of double direct vernier. The vernier on the left of the index is for reading clockwise angles and that on the right is for reading counterclockwise angles. Thirty spaces on the vernier are equal to 29 spaces on the circle and a single space on the circle is equal to 30′. The differences between the length of a space on the vernier and one on the circle is therefore $\frac{1}{30} \times 30 = 01'$, which is the least reading or least count of the vernier. Considering clockwise angles, the vernier index is seen to lie between 162°30′ and 163°00′. The number of minutes greater than 162°30′, determined by observing the graduation on the left vernier that coincides with one on the circle, is seen to be 05. Hence the angle is 162°30′ + 05′ = 162°35′. Similarly reading the right vernier, the counterclockwise angle is 17°00′ + 25′ = 17°25′.

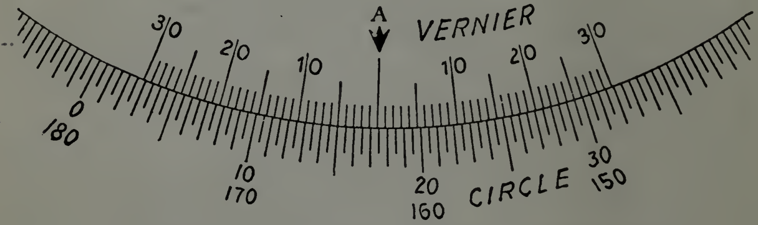


FIG. 188*a*.—Double direct vernier reading to minutes.

Figure 188*b* illustrates a double direct vernier for which 40 spaces are equal to 39 on the circle and one space on the circle is 20′. The least count is therefore $\frac{1}{40} \times 20' = 30''$. Reading clockwise, the angle is 49°40′ + 10′30″ = 49°50′30″. Reading counterclockwise the angle is 130°00′ + 09′30″ = 130°09′30″.

Figure 188*c* represents a folded vernier reading to 20″. The full length of the vernier is employed for reading angles in either direction.

Sixty spaces on the vernier are equal to 59 on the circle, and the circle is graduated to 20′. The vernier is read from the index towards either of the extreme divisions and then from the other extreme division in the same direction to the center. The index of the vernier and its 20′ mark are the same. Reading clockwise, the angle is 30°00′ + 02′40″ = 30°02′40″. Reading counterclockwise, the angle is 149°40′ + 17′20″ = 149°57′20″. The folded vernier is employed where the length of the corresponding double vernier would be so great as to be impracticable.

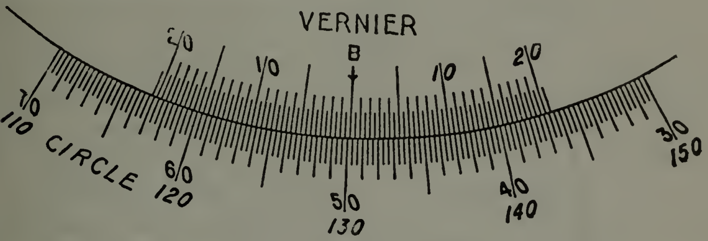


Fig. 188*b*.—Double direct vernier reading to 30 seconds.

188a. All transits have two verniers for reading the horizontal circle, their indices being 180° apart. The one nearest the upper clamp and tangent screw is known as the A vernier, and the opposite one is known as the B vernier. Failure of the verniers to register identical readings, *i.e.*, not exactly 180° apart, may be due to either or both of two causes: (1) the verniers may have become displaced so that a line joining their indices does not pass through the center of rotation of the vernier plate, or (2) the spindles may have become worn or otherwise damaged so that the center of rotation of the upper plate does not coincide with the center of rotation of the horizontal circle. The former condition is known as "eccentricity of verniers" and the latter as "eccentricity of centers."

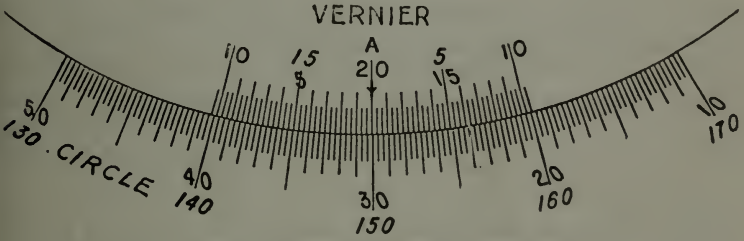


Fig. 188*c*.—Folded vernier reading to 20 sec.

To correct either of these defects requires the services of an instrument maker, but neither defect limits the precision with which angles may be determined. If the verniers only are eccentric, it is clear that they will lack the same amount of being 180° apart for all parts of the graduated circle. If the centers only are eccentric there will be one setting on the graduated circle for which the indices are 180° apart (first position, Fig. 188*d*), and 90° therefrom another setting for which the verniers fail to register 180° apart by a maximum amount (second position, Fig. 188*d*). Obviously the mean of the two vernier readings eliminates errors due to eccentricity whether it be due to verniers, to centers, or to both. Furthermore if the verniers only are eccentric no error is introduced in an angle so long as the same vernier is used for making the initial setting as for making the final reading.

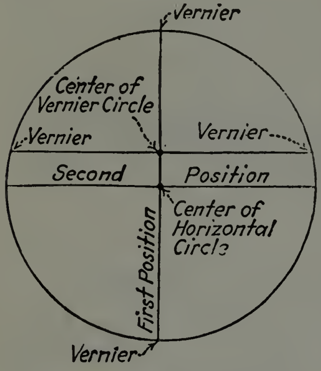


Fig. 188*d*.—Eccentricity.

USE OF THE TRANSIT

189. Remarks.—The succeeding articles describe the elementary processes employed in running lines and in measuring horizontal and vertical angles by means of the graduated circles and verniers of the transit. Transit surveys are considered in detail in the following chapter. The process of taking magnetic bearings with the transit is the same as with the surveyors' compass. The transit may be employed for running direct levels in the same manner as when using the engineers' level, the telescope bubble being centered each time a rod reading is taken.

The operation of rotating the telescope about the horizontal axis is called "plunging the telescope." When the telescope level tube is below, the telescope is said to be in the *normal* or *direct* position; when the level tube is above, the telescope is said to be in the *reversed* position.

190. Setting Up the Transit.—Ordinarily the transit is set over a definite point, such as a tack. For centering the transit, a plumb-line is suspended from the hook and chain beneath the instrument. To set up the transit, place it approximately over the point and adjust the legs until the tripod head is nearly level. Pick up the instrument bodily without disturbing the position of the legs, set it carefully over the point, and press each shoe firmly into the ground, at the same time adjusting each leg until the plumb-bob falls within ½ in. or less of the tack and the instrument is somewhere near level. Loosen two adjacent leveling screws and shift the instrument laterally until the plumb-bob is over the tack. Turn the screws to a firm, but not tight, bearing. Level the instrument by means of the plate bubbles, first bringing each level tube parallel with a pair of leveling screws.

It is quite useless to center one bubble carefully until the other has been brought approximately to the level position. In the absence of any device for adjusting the height of the plumb-bob, the string should be tied in a sliding bowknot. The plumb-bob should be adjusted until it just clears the tack.

191. Measuring a Horizontal Angle.—If any angle, as AOB, is to be measured, the transit is set up over O as described in the preceding article. The upper motion is clamped and by means of the upper tangent screw one of the horizontal verniers is set at 0°. The telescope is sighted to A, the lower motion is clamped, and by means of the lower tangent screw the line of sight is set on a range pole or other object marking the point. The upper clamp is loosened and the telescope is turned until the line of sight cuts B. The upper clamp is tightened, and the line of sight is set exactly on B by turning the upper tangent screw. The reading of the vernier which was initially set at 0° gives the value of the angle.

Following is a list of suggestions bearing on transit work in general:

1. Make settings reasonably close by hand so that the tangent screws will not have to be turned through more than one or two revolutions.

2. Make the last direction of movement of the tangent screw in a clockwise direction.

3. When reading the vernier, have the eye directly over the coinciding graduations. It is also helpful to observe that the graduations on either side of those coinciding fail to concur by the same amount.

4. The plate bubbles should be centered before measuring an angle, but between initial and final settings of the line of sight the leveling screws should not be disturbed.

5. Test the telescope for parallax (Art. 88) by moving the head slightly from side to side, meanwhile observing the magnified image. The eyepiece slide should be focused until the cross-hairs are sharp in outline, and the objective should be moved until there is no apparent movement of the cross-hairs over the image.

6. Do not allow the hand to rest on the instrument nor the clothing to brush against it while a sight is being taken. Do not step near the feet of the tripod.

192. Common Mistakes.—In measuring horizontal angles mistakes often made are:

1. Turning wrong tangent screw.
2. Failing to tighten clamp.

3. Confusion of numbers on the horizontal scale, as reading from the outer row when the angle turned is indicated by numbers on the inner row.
4. Reading angles in the wrong direction.
5. Dropping 30′ or 20′ by failure to take the full scale reading before reading the vernier, as for example, with a circle graduated to 30′ calling the angle 21°14′ when it is actually 21°44′, the vernier reading being 14′.
6. Reading the vernier in the wrong direction.
7. Reading the wrong vernier.

193. Measuring a Vertical Angle.—The transit is set up and leveled as when measuring horizontal angles. If the vertical vernier is fixed, considerable care should be taken in centering the plate bubbles. If the vernier is movable and has a control level, the bubble should be centered when the telescope is sighted in the general direction of the point to which a vertical angle is to be taken.

In measuring the vertical angle the telescope is sighted approximately at the point and the horizontal axis is clamped. The horizontal cross-hair is set exactly on the point by means of the telescope tangent screw, and the angle is read by means of the vertical vernier.

If the plane of the horizontal circle is not truly horizontal no error will be introduced in vertical angles measured with a transit having the movable vernier and control bubble, so long as the instrument is in adjustment. For angles measured with a transit having a fixed vertical vernier, any inclination of the upper plate introduces an error in the measured vertical angle equal in amount to the angle through which the vernier is displaced about the horizontal axis. This error, called the "*index error,*" is determined by leveling the telescope and reading the vertical circle. If the error is due solely to poor adjustment of the vernier, it will be a constant regardless of the direction in which the telescope is pointed; if it is due to nonadjustment of the plate levels, it will vary with the direction in which the telescope is pointed. The true vertical angle is determined by applying the *index correction* to the measured angle. Thus, if the observed angle is +12°14′ and the index error is +02′, the true value of the vertical angle is +12°14′ − 02′ = +12°12′.

In ordinary trigonometric leveling, vertical angles are usually taken by sighting at a leveling rod, the horizontal cross-hair being set at a rod reading equal to the height of the horizontal axis above the station over which the transit is set up. In precise trigonometric leveling the distance between stations is usually great and vertical angles are measured with a theodolite by sighting at points defined by signals erected at the distant stations.

194. Prolonging a Straight Line.—If any line as AB (Fig. 194a) is to be prolonged to P (not already defined upon the ground), which is beyond the limit of sighting distance or is invisible from A and B, the line is extended by establishing a succession of stations, C, D, etc., each of which stations is occupied by the transit. Any of the following three methods may be employed, but the second method is usually the most convenient.

Method 1.—The transit is set up at A, a sight is taken to B, and the point C is established on line. The transit is moved to B, a sight is taken to C, and point D is set on line. So the process is continued until point P is set.

Method 2.—The transit is set up at B and a backsight is taken to A. With both upper and lower motions clamped, the telescope is plunged and C is set on line. If the line of sight is perpendicular to the horizontal axis, as it will be if in perfect adjustment, it will generate a vertical plane as the telescope is revolved, and the point C will lie on the prolongation of AB. The transit is then moved to C, a backsight is taken to B, the telescope is plunged and D is established; and so the process is repeated until P is set.

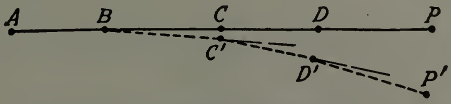


Fig. 194a.—Prolonging a straight line.

If the line of sight is not perpendicular to the horizontal axis, as the telescope is plunged from the reversed to the normal position, the line of sight will generate a portion of a cone whose vertex is at the center of the instrument and two of whose elements are AB and, let us say, BC' (the angle CBC' representing double the angle which the line of sight makes with the normal to the horizontal axis), and C' will not lie on the prolongation of AB. If the instrument is set up at C', a backsight taken to B with the telescope in the reversed position as before, and it is plunged to normal, a second and similar cone is generated and D' will not lie on the prolongation of BC'. Thus, if the line is extended by the method outlined and all backsights are taken with the telescope in one position (normal or reversed), the points established will lie along a curve instead of a straight line and each segment of the line will be deflected in the same direction (to the right or to the left) by double the angular error of adjustment of the line of sight. When the instrument is in good adjustment and the number of set ups is not large, this method is sufficiently exact for most work.

Method 3.—This is known as the method of "double-sighting," and is employed when the instrument is in poor adjustment or when it is desired to establish the line with high precision. If the line AB (Fig. 194*b*) is to be prolonged to some point P, the transit is set up at B and a backsight is taken to A with the telescope in its *direct* position. The telescope is plunged and a point C' is set on line. The transit is then revolved *about its vertical axis* and a second backsight is taken to A with the telescope in its *reversed* position. The telescope is plunged and a point C'' is established on line beside C'. It is evident that C' will be as far on one side of the true prolongation of AB as C'' is on the other. Hence C, a point on the prolongation of AB, is set midway between the points C' and C''. In a similar manner D is established by setting up at C, double-sighting to B and setting points at D' and D''. Thus the process is repeated until the desired distance is traversed.

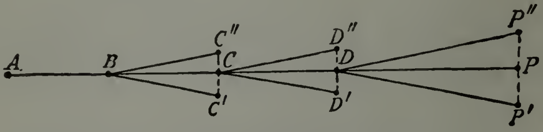


Fig. 194*b*.—Double sighting.

195. Laying Off a Horizontal Angle.—If an angle AOB is to be laid off from the line OA, the transit is set up at O, the vernier set at 0° and the line of sight is set on A. The upper clamp is loosened, and the vernier plate is turned until the index of the vernier is approximately at the required angle. The upper clamp is tightened, and the vernier is set at the given angle by means of the upper tangent screw. The point B is established on the line of sight.

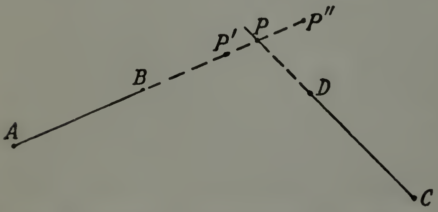


Fig. 196.—Intersection of lines.

196. Intersection of Lines. The point of intersection of two lines as AB and CD (Fig. 196) is established as follows: By methods described in Art. 194 prolong one of the lines, AB, and establish points P' and P'', a short distance on either side of the estimated position of the prolongation of CD. Stretch a string between P' and P''. Prolong the line CD until it intersects the string at P. A point set at P marks the intersection of the two lines.

197. Prolonging Line Past Obstacle.—Figure 197 illustrates one method that may be employed to prolong a line past an obstacle when the offset space is limited, the line *GH* representing the prolongation of *AB*. The transit is set up at *A*, a right angle is turned, and point *C* is established at a convenient distance from *A*. Similarly the point *D* is established, the distance *BD* being made equal to *AC*. The line *CD* is now parallel to *AB*. By the methods of Art. 194 *CD* is prolonged and points *E* and *F* are established in convenient positions beyond the obstacle. From points *E* and *F* right-angle offsets are made and *G* and *H* are set as were *C* and *D*. It is evident, if errors be neglected, that *GH* marks the prolongation of the line *AB*, and also that the distance *AH* is directly determined by measuring the lengths of the lines *AB*, *DE*, and *GH*. If the chainage is to be carried forward with precision it is necessary to erect the perpendiculars *AC*, *BD*, etc., with some care, and if the line is to be prolonged accurately it is essential not only that the offset distances be measured carefully but also that *AB* and *EF*, the distances between offsets, be relatively large.

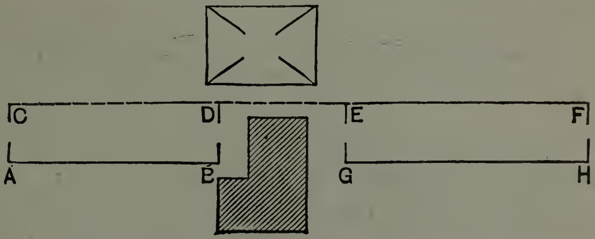


Fig. 197.

198. Determination of an Inaccessible Distance.—This involves triangulation (see Art. 178). One simple method is described below:

AB (Fig. 198) represents a line whose length cannot be measured directly, *B* being visible from *A* and from other points in the vicinity. The transit is set up at *A*, a sight is taken to *B*, an angle of 90° is laid off, and *C* is set at any convenient point from which *B* is visible. The length of the line *AC* is measured. The transit is set up at *C*, and the angle *ACB* is observed. Then $AB = AC \tan ACB$.

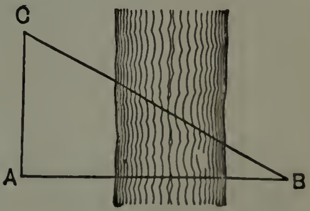


Fig. 198.

199. Running a Straight Line between Two Points.—If the terminal points of a line are fixed and it is desired to establish intervening points on the straight line joining the terminals, the method to be employed depends upon the length of the line and the character of the terrain. Three common cases are considered below:

Case 1. *Terminals Intervisible.*—If the terminals are *A* and *B*, the transit is set up at *A*, a sight is taken to *B*, and intervening points are established on line.

Case 2. *Terminals Not Intervisible, but Visible from an Intervening Point.*—By trial the location of the line at the intervening point is determined. In Fig. 199*a*, *A* and *B* represent the terminals both of which can be seen from the vicinity of *C*. The transit is set up on the estimated position of the line, a backsight is taken to *A*, the telescope is plunged, and the position of the line of sight at *B* is noted. The amount that the transit must be shifted laterally is estimated, and the process is repeated until when the telescope is plunged, the line of sight falls on the point at *B*, as in Fig. 199*a*. The position of the instrument so found should be tested by the method of double-sighting to eliminate possible effects of inaccurate adjustment of line of sight or horizontal axis.

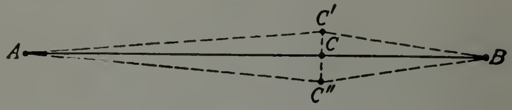


Fig. 199*a*.

Other intermediate points may then be established by setting up the transit at *C* and proceeding as in Case 1.

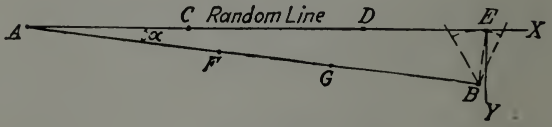


Fig. 199*b*.

Case 3. *Terminals Not Visible from Any Intermediate Point.*—*AB* (Fig. 199*b*) represents a straight line along which it is desired to establish intermediate points, the character of the ground being such that it is not possible to find any one intermediate position from which both terminals are visible. By one or more of the methods previously explained, a straight line, called a *random line,* is run from *A* in the estimated direction of *B*. Let *AX* be such a random line, and *C* and *D* be stations established during the process of extending it. The transit is set up at *D* and sighted towards *X*. The tape is swung about *B* as a center and the offset distance *BE* from the line *DX* to the point *B* is determined by sighting at the tape through the transit and taking the least reading.

The length AE is measured, and the angle which the random line makes with AB is calculated by the expression $\tan \alpha = \frac{BE}{AE}$. The transit is again set up at A, a sight is taken to C, the calculated value of α is laid off, and the line is run towards B, intermediate stations as F and G, being established at desired points. With the instrument at G, a backsight is taken to F, the telescope is plunged, and the linear offset error at B is noted. If the error is sufficiently large to be of importance, the points at F and G are corrected by linear measurements so as to place them on the true line, the correction being made proportional to the distance from the point to the terminal A. That is, the offset correction at F is to FA as the error at B is to BA. In many cases, points on the random line, as C and D, can be transferred to the true line by the same method.

If the offset distance is short, compared with the length of the line AB, the position of E along the line AX may be estimated with sufficient precision. If the offset distance BE is large, the tape may be swung through a considerable arc in the vicinity of the perpendicular without materially changing the offset reading, and hence the position of E needs to be determined by a more exact method than simply by estimation. Usually the transit is set up at the estimated position of E, a perpendicular EY is laid off from the line EA, and the distance from this perpendicular EY to B is measured by a swing offset. This offset gives the distance that E must be moved along the line AX to place it on the perpendicular through B.

200. Measuring an Angle by Repetition.—One of the advantages of the transit not possessed by other instruments is that a horizontal angle may be mechanically multiplied and the product can be read with the same degree of precision as the single value. Thus, with the ordinary transit having verniers reading to single minutes, an angle for which the true value is between the limits $30°00'30''$ and $30°01'30''$ will be read as $30°01'$, and the limits of possible error will be $\pm 30''$. If the true angle is multiplied, let us say, six times on the horizontal circle, the product, likewise read to the nearest minute, might be $180°04'$, its true value being within the limits $180°03'30''$ and $180°04'30''$, and the limits of possible error, so far as reading the vernier is concerned, being likewise $\pm 30''$. Dividing the observed product $180°04'$ by six, the single value becomes $30°00'40''$ for which the limits of possible "reading" error are $\pm 30'' \div 6 = \pm 05''$. This method of determining an angle is called "*measurement by repetition.*" While it might appear from what has been said that the accuracy with which an angle can be measured by this method varies directly with the number of times the angle is multiplied or repeated, on account of lost motion in the instrument, errors due to setting the line of sight, and to other causes, usually the precision is not appreciably increased by more than six or eight repetitions.

To repeat an angle, as *AOB*, the transit is set up at *O*, the vernier is set at 0°, a sight is taken to *A*, and the lower motion is clamped. The upper clamp is loosened, a sight is taken to *B*, the upper clamp is tightened, and the single value of the angle is read. The vernier setting is left unaltered, the instrument is turned on its lower motion, and a second sight is taken to *A*. The upper clamp is loosened, and the telescope is again sighted to *B*. The angle has now been doubled. In this way the process is continued until the angle has been multiplied the required number of times. The vernier is read, and the value of the angle is determined by dividing this reading by the number of times the angle was turned. To obviate mistakes, this value is compared with the angle observed at the completion of the first turn.

201. Laying Off an Angle by Repetition.—If it is desired to establish an angle with a precision greater than that possible by a single observation, the methods of the preceding article may be employed in the following manner: In Fig. 201, *OA* represents a fixed line and *AOB* the angle which is to be laid off to establish the line *OB*. The transit is set up at *O*, the vernier is set at 0°, and a sight is taken to *A*. The vernier is set as closely as possible to the given angle and a point *B'* is established with the line of sight in its new position. *AOB'* is then repeated a sufficient number of times to fix the error within prescribed limits, and a measured value of the angle is determined by dividing the total angle turned by the number of repetitions. The length of *OB'* is measured.

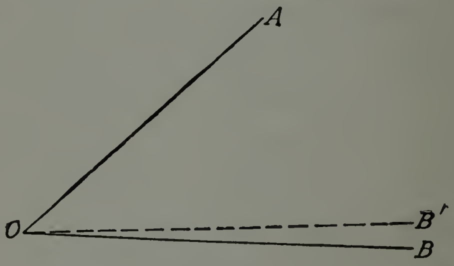


Fig. 201.

Let *AOB* represent the required angle and *AOB'* the measured value of the angle which was laid off. Then $B'OB = AOB - AOB'$ represents the correction which must be applied to the measured angle. The correction, while too small to be laid off by angular measurement, is applied by offsetting the distance $B'B = OB' \tan B'OB$, thus establishing the point B. Finally, as a check on the work, AOB is measured by repetition. It is convenient to remember that the tangent or sine of $01' = 0.0003$ (very nearly).

ADJUSTMENTS OF THE TRANSIT

202. Remarks on Adjustments.—Most of what was said in Art. 110 concerning adjustments of the level applies equally well to those of the transit.

For a transit in perfect adjustment the following relations should exist:

1. The axes of the plate levels should lie in a plane perpendicular to the vertical axis, so that when the instrument is leveled the vertical axis will be truly vertical and horizontal angles may be measured in a horizontal plane.

2. The optical axis, the axis of the objective slide, and the line of sight should coincide for any position of the objective, and each should be perpendicular to the horizontal axis at its mid-point. In this way when the telescope is rotated about the horizontal axis the line of sight will generate a plane regardless of whether the objective is focused for a near sight or for a far sight, and the plane will pass through the center of the instrument.

3. The vertical cross-hair should lie in a plane perpendicular to the horizontal axis so that any point on the hair may be employed when measuring horizontal angles or running lines.

4. The horizontal axis should be perpendicular to the vertical axis so that when the telescope is plunged the line of sight will generate a vertical plane.

5. The axis of the telescope level should be parallel to the line of sight so that the transit may be employed in direct leveling.

6. If the transit has a fixed vernier for the vertical circle, the vernier should read zero when the plate bubbles and telescope bubble are centered, in order that vertical angles may be measured; or if the vernier is movable and has a control level, the axis of the control level should be parallel to that of the telescope level when the vernier reads zero.

7. If the transit has a striding level, its axis should be parallel to the horizontal axis so that when the telescope is plunged the line of sight will generate a vertical plane.

203. Adjustments.—In the description of the following adjustments it is assumed that the objective slide does not admit of adjustment, but is permanently fixed in the telescope tube so far as lateral motion is concerned; and that the maker has done his part by so constructing the instrument that the optical axis and the axis of the objective slide coincide and are perpendicular to the horizontal axis. This ideal construction is never exactly attained; but in most modern instruments the departure is so slight that it need not be considered in ordinary transit work, and in precise surveying the resulting errors are eliminated by methods of procedure.

For transits with adjustable objective slides, the adjustments are the same as for those with permanently fixed objective slides, except as noted below.

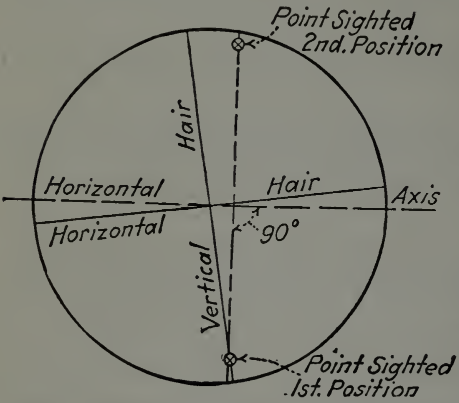

Point Sighted 2nd. Position
Hair
Horizontal
Hair
Axis
Horizontal
90°
Vertical
Point Sighted 1st. Position

Fig. 203*a*.—Adjustment of vertical cross-hair.

1. *To Make the Vertical Cross-hair Lie in a Plane Perpendicular to the Horizontal Axis.*

Test.—Sight at a well-defined point about 200 ft. from the instrument. Swing the telescope through a small vertical angle, so that the point traverses the length of the vertical cross-hair. If the point appears to move continually on the hair, the cross-hair lies in a plane perpendicular to the horizontal axis (see Fig. 203*a*).

Correction.—If the point does not appear to move always on the cross-hair, loosen two adjacent screws and turn the cross-hair ring in the telescope tube until the point traverses the entire length of the hair. Tighten the *same* two screws.

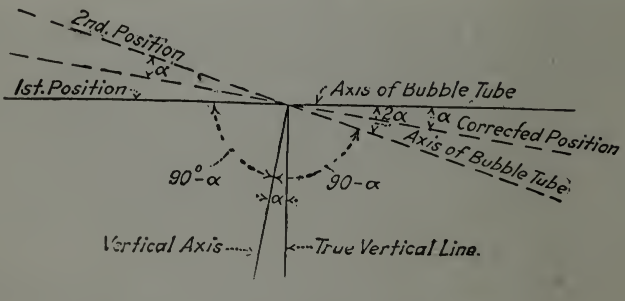

2nd. Position
α
1st. Position
Axis of Bubble Tube
2α
α
Corrected Position
Axis of Bubble Tube
90°-α
90-α
α
Vertical Axis
True Vertical Line.

Fig. 203*b*.—Adjustment of plate levels.

2. *To Make the Axes of the Plate Levels Lie in a Perpendicular to the Vertical Axis.*

Test.—Rotate the transit about the vertical axis until each level tube is parallel to a pair of foot screws. Bring the bubbles to the centers of the tubes by means of the foot screws. Rotate the transit 180° in azimuth. If the bubbles remain centered, the axes of the level tubes are in a plane perpendicular to the vertical axis (see Fig. 203*b*).

Correction.—If the bubbles become displaced, bring them *halfway* back by means of the adjusting screws. Level the instrument again and repeat the test to verify the results.

3. *To Make the Line of Sight Perpendicular to the Horizontal Axis.*

Test.—Set up and level the instrument. Sight on a point *A* (see Fig. 203*c*) about 500 ft. away, plunge the telescope, and set another point *B* on the line of sight and about the same distance away on the opposite side of the transit. Turn the instrument in azimuth, and again sight at *A* (with telescope in reversed position). Plunge the telescope; if *B* is on the line of sight, the desired relation exists.

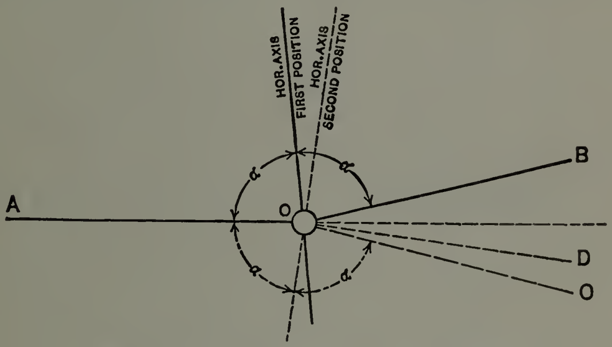


Fig. 203*c*.—Adjustment of line of sight.

Correction.—If *B* is not on line, set a point *C* beside it on the line of sight. Mark a point *D*, *one-quarter* of the distance from *C* to *B*, and adjust the cross-hair ring by means of the two opposite horizontal screws, until the line of sight passes through *D*.

4. *To Make the Horizontal Axis Perpendicular to the Vertical Axis.*

Test.—Set the transit near a building, flag staff, or other object upon which is some well-defined point at a considerable angle of elevation. Level the instrument very carefully, thus making the vertical axis truly vertical. Sight at the high point *A* (see Fig. 203*d*); and with the horizontal motions clamped depress the telescope and set a point *B* on the ground. If the horizontal axis is truly horizontal, *A* and *B* will be in the same vertical plane, since the line of sight has been made perpendicular to the horizontal axis. Plunge the telescope, rotate in azimuth, and again sight on A. Depress the telescope as before; if the line of sight falls on B, the horizontal axis is perpendicular to the vertical axis.

Correction.—If the desired relation does not exist, set C on the line of sight beside B. A point D *one-half* the distance from B to C will lie in the same vertical plane with the high point. Loosen the screws of the bearing cap, and raise or lower the adjustable end of the horizontal axis until the line of sight passes through both A and D when the telescope is swung in a vertical arc. *The high end of the axis is always on the same side of the vertical plane through the high point as the point last set.* In readjusting the bearing cap, care should be taken not to bind the horizontal axis, but it should not be left so loose as to allow the objective end of the telescope to drop of its own weight when not clamped.

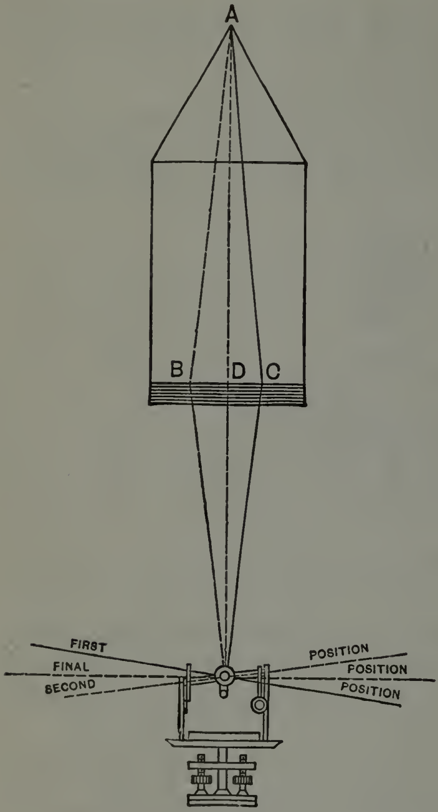


Fig. 203*d*.—Adjustment of horizontal axis.

5. *To Make the Axis of the Telescope Level Parallel to the Line of Sight.*

Test and Correction.—Same as the peg adjustment of the dumpy level (Art. 111) except that the correction is made by moving the level tube until the axis is horizontal after having set the line of sight on the rod reading for a level line.

6. *To Make the Vertical Circle or Arc Read Zero When the Telescope Bubble Is in the Center of the Tube.*[1]

Test.—With the plate bubbles centered, level the telescope bubble and read the vernier of the vertical circle.

Correction.—Loosen the vernier and move it until the desired relation is established.

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[1] This adjustment is not applicable to a transit having a level attached to the vertical vernier.

6a. *To Adjust the Auxiliary Level on the Vernier of the Vertical Circle so That Its Axis Is Parallel to the Axis of the Telescope Level When the Vernier Reads Zero.*[1]

Test.—Bring the telescope bubble to the center of its tube, and move the vernier by means of the tangent screw until it reads zero.

Correction.—By means of the capstan-headed screws, move the bubble attached to the vertical vernier until it is in the center of its tube.

204. Suggestions.—The adjustments of the transit are more or less dependent one upon another. For this reason if the instrument is badly out of adjustment, time will be saved by first making corrections roughly for those adjustments which are related until all the tests have been tried, and then repeating the tests and corrections (beginning at the first). The plate levels will not be disturbed by other adjustments, and should be exactly corrected before other adjustments are attempted. Any movement of the screws controlling the cross-hair ring is likely to produce both lateral displacement and rotation of the ring; hence any considerable adjustment of the line of sight is likely to disturb the vertical hair so that it will no longer remain on a point when the telescope is rotated about the horizontal axis. The adjustment of the telescope level depends upon the unaltered position of the horizontal cross-hair and hence should not be tested until the line of sight and horizontal axis have been corrected.

Care should be taken not to strain unduly the adjusting screws, but they should be brought to a firm bearing. The threads of the reticule screws are in the ring and not in the barrel of the telescope; hence the cross-hair ring is drawn towards the screw which is turned clockwise.

In making adjustment 4, the end of the horizontal axis is raised or lowered by trial until when the telescope is rotated, the line of sight will pass through both the high point and the mid-point on the ground.

ERRORS

205. Remarks.—The following articles are calculated to point out the sources of error in transit work and to show the effect of the various errors on the precision of angular measurements.

Except in field astronomy, a measured angle is always closely related to a measured distance, and as has been previously stated (Arts. 11 and 12) there ought, in a general way, to be a consistent relation between the precision of these two classes of measurements. It is important, both from the standpoint of accuracy and from the standpoint of expediting the work, that the surveyor be able to visualize the effect of errors both in terms of angle and in terms of distance, that he appreciate what degree of care must be exercised to keep certain errors within specified limits, and that he know under what conditions various instrumental errors may be eliminated.

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[1] This adjustment is not applicable to a transit having a fixed vernier.

206. Instrumental Errors.—These errors are caused by imperfections in the instrument itself. The adjustments previously described, even though carefully made, are never exact. Likewise the graduations are not perfect, and the centers are not absolutely true.

1. *Errors in Horizontal Angles Caused by Nonadjustment of Plate Levels.*—When bubbles in nonadjustment are centered, the circle is inclined and hence measured angles are not truly horizontal angles. Also the horizontal axis is inclined to a varying degree depending upon the direction the telescope is sighted. There will be one vertical plane which will include the vertical axis in its inclined position. When the line of sight is in this plane the horizontal axis is truly horizontal and the line of sight will generate a vertical plane when the telescope is plunged; hence no error in direction is introduced, regardless of the angle of elevation to the point sighted. As the transit is rotated about the vertical axis from this position the horizontal axis becomes inclined, making a maximum angle with the horizontal when the angle of rotation reaches 90°. With the horizontal axis in this latter position, the line of sight generates a plane making a maximum angle with the vertical equal to the error in the position of the vertical axis, and with the line of sight inclined at a given angle, the maximum error in determining the direction of a line is introduced. The larger the vertical angle, the greater the error.

2. *Errors in Vertical Angles Due to Nonadjustment of Plate Levels.* These errors obviously vary with the direction in which the instrument is pointed. With the fixed vertical vernier they are eliminated by observing the index error of the corresponding observed vertical angle. With the movable vernier having a control level, errors are eliminated by keeping its bubble centered as described in Art. 193.

3. *The Effect of the Line of Sight's Not Being Perpendicular to the Horizontal Axis.*—If the telescope is not reversed between backsight and foresight and the sights are of the same length, so that it is not required to refocus the objective, also if both points sighted are in the horizontal plane through the telescope, no error is introduced in the measurement of horizontal angles even though the adjustment be badly out. If the instrument is plunged between backsight and foresight, the resultant error in the observed angle is double the error of adjustment. If there is a considerable movement of the objective between sights, an appreciable error may be introduced due to the line of sight's not making a constant angle with the horizontal axis

for both sights. The error may be eliminated by taking the mean of two angular observations, one with the telescope in the direct position and the other with the telescope reversed. In prolonging a line, errors are avoided by the method of double sighting described in Art. 194. The effect of the error depends upon the inclination (from the horizontal) of the line of sight towards the distant signal. For all ordinary cases this error may be taken as

$$E = e \sec h$$

in which e is the angle between line of sight and normal to horizontal axis, and h is the vertical angle of the pointing.

4. *Errors Due to Horizontal Axis Not Being Perpendicular to the Vertical Axis.*—No error is introduced in horizontal angles so long as the points sighted are in the same horizontal plane. It can be shown that the angular error in the observed direction of any line is with sufficient precision $\theta = E \tan a$ in which E is the angle which the horizontal axis makes with the horizontal and a is the angle of inclination of the line of sight with the horizontal. The error in observed horizontal angle may become large, but if any angle is measured with the telescope first in the direct and then in the reversed position one value will be too great by the amount of the error and the other will be correspondingly too small; thus the error is eliminated by taking the mean of the two values.

5. *Effect of Lack of Coincidence between Line of Sight and Optical Axis.*—Under these conditions if the line of sight is perpendicular to the horizontal axis for one position of the objective, it will not be perpendicular for other positions. If an angle is measured without disturbing the position of the objective no error is introduced. The error from this source is not likely to be sufficiently large to be of consequence in ordinary transit work. It is eliminated by taking the mean of two angles, one observed with the telescope direct and the other with it reversed.

6. *Errors Due to Eccentricity.*—With the modern transit in good condition, these errors are of no consequence in the ordinary measurement of angles. In any case, such errors are eliminated by taking the mean of readings indicated by opposite verniers.

7. *Imperfect Graduations.*—Errors from this source are of consequence only in work of high precision. They are reduced to a negligible amount by taking the mean of several observations for which the readings are distributed over the circle and over the vernier.

Summing up, it is seen that:

(*a*) Errors in horizontal angles due to nonadjustment of plate levels or of horizontal axis become large as the inclination of the sights increases.

(*b*) Nonadjustment of the line of sight becomes of consequence only when the telescope is plunged.

(*c*) Errors due to instrumental imperfections or nonadjustment are all systematic in character, and without exception they may be eliminated, or at least reduced to a negligible quantity, if proper methods of procedure are employed. In general, this consists in obtaining the mean of two values—observed, one before and one after a reversal of the horizontal plate (by plunging the telescope and rotating about the vertical axis)—one of these values being as much too large as the other is too small. An exception is the error in either horizontal or vertical angle due to inclination of the vertical axis, which cannot be so eliminated but which is eliminated, so far as its systematic character is concerned, by releveling the plate bubbles in addition to the reversal of the plate above mentioned.

207. Personal Errors.—Personal errors arise from the limitations of the human eye in setting up and leveling the transit and making observations.

1. *The Effect of Not Setting Up Exactly over the Point.*—This produces an error in all angles measured at a given station, the magnitude of the error varying inversely with the length of sight. It is convenient to remember that 1 in. is the arc whose angle is 01′ when the radius is approximately 300 ft. In general, the error may be kept within negligible limits by reasonable care, but many instrumentmen waste time by exercising needless care in setting up when the sights to be taken are long.

2. *The Effect of Not Exactly Centering the Plate Bubble.*—This produces an error in horizontal angles after the manner described in the preceding article when the plate levels are out of adjustment. The probable error from this source is small when the sights are nearly level, but may be large for steeply inclined sights. The average transitman does not appreciate the importance of careful leveling for steeply inclined sights and often uses more care in leveling than is necessary when sights are nearly horizontal.

3. *Errors in Setting and Reading the Vernier.*—These are functions of the least count of the vernier and legibility of scale and vernier lines. For the usual 01′ transit the probable error is less than 30″; for the 30″ transit the probable error is about 15″. The use of a reading glass enables closer reading, particularly for finely graduated circles. Also in reading the vernier it is helpful to observe the position of the graduations on either side of the ones that appear to

coincide, and to note that they appear to lack coincidence by the same amount.

4. *Not Sighting on Exact Point.*—This is likely to be a source of rather large error on ordinary surveys where sights are taken on the range pole and often only the upper portion is visible from the transit. The effect upon a direction is, of course, the same as the effect of not setting up exactly over the point. For short sights greater care should be takenth an for long ones and usually the plumb-line is employed in place of the range pole.

5. *Imperfect Focusing or Parallax.*—The error due to imperfect focusing is always present to a greater or less degree, but with reasonable care it may be reduced to a negligible quantity. The manner of detecting parallax was described in Art. 88.

All of the personal errors are accidental in character and hence cannot be eliminated. They form a large part of the resultant error in transit work. Of the personal errors, those due to inaccuracies in reading and setting the vernier and to not sighting on the exact point are likely to be the ones of larger magnitude.

208. Natural Errors.—Sources of natural errors are (1) settlement of the tripod, (2) unequal atmospheric refraction, (3) unequal expansion of parts of the telescope due to temperature changes, and (4) wind producing vibration of the transit or making it difficult to plumb accurately.

In general, the resulting angular errors are not of sufficient magnitude appreciably to affect measurements of ordinary precision. Large errors, however, are likely to arise from the first source when the transit is set up on boggy or thawing ground. Attention is directed to the fact that settlement is usually accompanied by an angular movement about the vertical axis as well as linear movements both vertically and horizontally. Errors due to adverse atmospheric conditions may nearly always be rendered negligible by choosing appropriate times for observing.

209. The Precision of Angular Measurements.—The angular precision to be expected in transit work depends upon so many factors that it would be absurd to attempt to lay down an exact procedure to be followed to ensure a required accuracy. However the following values represent, in a general way, the maximum errors likely to occur under the average conditions of practice, instruments being in fair condition and in fair adjustment except as otherwise stated.

Case 1.—Short sights, point indicated by range pole obscured near ground. Range pole plumbed by eye. Single observation of angles. Error 02′ to 04′.

Case 2.—Long sights, but otherwise as stated for case 1. Error 01′ to 02′.

Case 3.—Unobscured but steeply inclined sights; no special attention given to making horizontal axis truly horizontal; single measurement of angle. Error 01′ to 02′.

Case 4.—Unobscured sights on well-defined points; sights not steeply inclined. Single observation of angle, vernier reading to minutes. Error 30″ to 01′.

Case 5.—As for case 4, but transit in excellent condition and in good adjustment. Angles estimated to ½ min. Error 20″ to 30″.

Case 6.—As for case 4 but angle doubled, the telescope being reversed between sights. Error 15″ to 30″.

Case 7.—Unobscured sights on well-defined points. Sights not steeply inclined. Verniers reading to 30″. Single observation of angle represented by mean of readings of both verniers. Transit in excellent condition and in good adjustment. Error 15″ to 30″.

Case 8.—As for case 7, but verniers reading to 10″. Also instrument set up with great care. Error 10″ to 15″.

Case 9.—Unobstructed sights on well-defined points. Instrument set up with great care. Sights not steeply inclined. Transit in good condition. Vernier reading to 30″. Angles repeated six times with telescope direct and six times with it inverted. Error 02″ to 04″.

Case 10.—As above, but transit reading to 10″. Observations taken at favorable times. Error 01″ to 02″.

The average angular errors will of course be materially less than the values given for the preceding cases. Also, since the errors are largely accidental in character, the resultant error in the sum of a series of measured angles may be expected to vary as the square root of the number of angles involved.

Generally speaking, the measurement of angles is closely related to the measurement of distances, and as was pointed out in Arts. 12 and 13, there ought to be a consistent relation between these two classes of measurements. On surveys of ordinary precision it usually requires much more care to keep linear errors within prescribed limits than to maintain a corresponding degree of angular precision. The general tendency among surveyors is to pay undue attention to securing accuracy in angular measurements while at the same time overlooking large and important errors in the measurement of distances.

210. Numerical Problems.

1. Thirty spaces on a transit vernier are equal to 29 spaces on the graduated circle and one space on the circle is 15′. What is the least count of the vernier?

2. Sixty spaces on a transit vernier are equal to 59 on the circle and the circle is divided into 15′ spaces. What is the least count of the vernier?

3. Referring to Fig. 198, suppose that the distance AC is 317.2 ft. and the angle ACB is 67°13′. What is the distance AB?

4. Referring to Fig. 198*b*, a straight line AX is run at random from A in the general direction of AB, point B not being visible from A. A swing offset is measured from B to the line AX and found to be 63.4 ft. The transit is set up at E and EY (perpendicular to AX) is erected. The swing offset from B to EY is 6.1 ft. Also the distance AE is 2638.9 ft. Calculate the angle α which must be laid off from the random line in order to establish points on the straight line AB, and determine the length AB. What must be the precision of α in order that the line established shall fall within 0.1 ft. of the point B?

5. What error would be introduced in the computed value of the angle α of problem 4 if the swing offset distance from B to EY had been neglected and AE had been assumed to be the base of a triangle of which AB is the hypotenuse?

6. Having given the data of problem 4, it is proposed to establish points on the line AB by perpendicular offsets from C and D. What must these offsets be if AC = 937.6 ft. and AD = 1932.0 ft.?

7. A transit for which the graduated circle is numbered from 0° to 360° in each direction is used to measure an angle by repetition. The angle is multiplied five times in both clockwise and counterclockwise directions and the following readings are observed:

	Measured clockwise	Measured counterclockwise
Initial	63°17′	318°26′
One repetition	189°49′	84°58′
Five repetitions	335°58′	231°09′

Calculate the most probable value of the angle.

8. Before beginning the construction of a 300- by 500-ft. armory, lines marking its sides were laid off on the ground. This was accomplished by first establishing one of the 500-ft. sides and then erecting 300-ft. perpendiculars at either end. The 90° angles were laid off as closely as possible with a 01′ transit. The angles laid off were then repeated six times with the telescope normal and six times with it reversed. The actual angles laid off were found to be 89°59′50″ and 90°00′25″. What are the values of the offsets from the ends of the 300-ft. trial perpendiculars to establish the true corners?

9. Two points A and B, 5,280 ft. apart, are to be connected by a straight line. A random line run from A in the general direction of B is found by calculation to deviate 03′18″ from the true line. On the random line at a distance 1,250.6 ft. from A an intermediate point C was established. What must be the offset from C to locate a corresponding point D on the true line?

10. What error would be introduced in the measurement of a horizontal angle if, through nonadjustment, the horizontal axis were inclined 03′ with the horizontal and sights were taken to points at the same elevation as the transit? If the horizontal axis were inclined 3° and sights were taken to points at the same elevation as the transit?

11. Same as problem 10 but sights inclined at positive angles of 30°.

12. Same as problem 10 but one sight inclined at +30° and the other at −30°.

13. In prolonging a straight line the transit is set at *B*, a backsight is taken to *A*, and the telescope is plunged to set *C* 1,000 ft. in advance of *B*. *A* and *B* are at the same elevation but the vertical angle from *B* to *C* is +15°00′. If the vertical axis were inclined 01′ with the true vertical in a vertical plane making 90° with the direction of the line, what would be the linear offset error in the located position of *C*?

14. Same as problem 13, but *A*, *B*, and *C* all in the same plane.

15. Same as problem 14, but vertical angles of +15°00′ for both backsight and foresight.

16. A line *AB* is prolonged by setting up the transit at succeeding points *B*, *C*, *D*, etc., backsighting on preceding points, *A*, *B*, *C*, etc., and plunging the telescope. If the line of sight made an angle of 15″ with the normal to the horizontal axis and the procedure were such that each backsight was taken with the telescope in the normal position, what would be the angular error in the segment *FG*? What would be the offset error in the position of the point *G* if the segments *AB*, *BC*, etc., were uniformly 500 ft. long?

17. In measuring a horizontal angle the error of setting up the transit was 0.03 ft., the direction of displacement being such as to produce a maximum angular error. What error is introduced in a 60° angle if the length of sights is 50 ft.? If the length of sights is 1,000 ft.?

18. Same as problem 17, but 120° angle.

19. If the ratios of linear precision to be maintained on the various parts of a survey were $\frac{1}{1{,}000}$, $\frac{1}{5{,}000}$, $\frac{1}{20{,}000}$, and $\frac{1}{40{,}000}$, about how closely should the corresponding horizontal angles be observed in order that a consistent relation should exist between precision of angles and precision of distances?

20. It is desired to determine by calculation the length of a side of a right triangle, the angle opposite and the hypotenuse being measured. If the angle is 20°, with what precision should it be measured in order that the ratio of precision in the calculated length be $\frac{1}{20{,}000}$?

21. Same as problem 20, but a measured angle of 70°.

22. It is desired to determine by calculation the length of a side of a right triangle, the angle opposite and the side adjacent thereto being measured. If the angle is 20°, with what precision should it be measured in order that the ratio of precision be $\frac{1}{20{,}000}$? If the angle is 70°?

CHAPTER XIII

SURVEYS WITH TRANSIT AND TAPE

211. Remarks.—This chapter treats of the general methods utilized on the large variety of surveys of ordinary precision for which the transit is employed for the measurement of horizontal angles and the tape is used for the measurement of distances. The practices here described are, with minor modifications, common to land, city, topographic, hydrographic, and location surveys for highways, railroads, and the like. Probably 90 per cent of all surveying work, not including leveling, is with modifications, carried on as herein described.

212. Transit Party.—The transit party is usually composed of three men, the transitman, the head chainman, and the rear chainman. The transitman directs the work of the party, operates and cares for the transit, and keeps all notes in the field notebook. The head chainman performs the duties of that position as described in Arts. 64 and 65, gives line as directed by the transitman, and is responsible for the accuracy and speed of the chaining operations. Where stakes are set he attends to their proper marking. The rear chainman carries out the duties of that office as described in Arts. 64 and 65, gives backsights as directed by the transitman, often carries and drives stakes, and assists in removing obstructions to the vision of the transitman. In wooded country axemen (up to five or six in number) are employed in clearing the transit line of trees and brush. The head chainman keeps in close communication with the axemen and assists in their direction. Where sights are long, a rear flagman is often stationed at the transit point preceding that at which the transit is set, his duty being to give backsights to the transitman. Where a large number of observations are to be taken, the notes are kept by a recorder, who may also act as the chief of party.

213. Equipment of the Transit Party.—The equipment of the transit party usually consists of a transit, 100-ft. steel tape, two range poles, stake bag, stakes, tacks, axe or hammer, two or three plumb-bobs, field notebook, chaining pins, and marking crayon. A wool or silk hood is provided as a protection to the transit against rain. Often large nails are conveniently used as markers, and not infrequently a cold chisel is useful in marking points on stone or other hard objects.

214. Transit Stations.—Any point of reference over which the transit is set up in the process of making a survey is called a transit station. The object marking the point may be either temporary or permanent in character. On most surveys the transit station is a stake driven flush with the ground with a tack in its top marking the exact point to which both angular and linear measurements are referred. Such a stake is generally termed a hub. On city streets the transit station may be a driven nail or a cross cut in the pavement or curb. In land surveying the stations are often iron pipes, stones, or other more or less permanent monuments set at the corners.

The transit stations of a survey are usually identified either by number or by letter, the identification mark being placed either upon the station marker or upon some other object near by, and also of course being appropriately indicated in the field notes. Where the station is a hub, its location is usually indicated by a stake extending some distance above the ground and driven sloping so that its top is over the hub. This stake, called *a witness stake* or *guard stake*, carries the number of the transit station over which it stands. Usually the number is marked with lumber crayon and reads down the stake. The hubs are often made square, say 2 by 2 in., and the witness stakes are usually flat, perhaps ¾ by 3 in.

215. Transit Lines.—Lines connecting transit stations are termed transit lines. If a system of lines run with the transit forms a traverse (as described in Art. 177) it is called a transit traverse, to distinguish from traverses run with other instruments, though the term *traverse* without qualification would usually be understood to be one for which the angles were measured with the transit and distances were measured with the tape. The transit lines forming a triangulation system are called lines of triangulation, and the points at which the transit is set up are called triangulation stations. Both in the field notes and as a part of the identification mark left in the field, the station number, if for a traverse, is preceded by the symbol ⊙, and if for a triangulation station, by the symbol ⚠.

In most cases the transit stations are identified by consecutive numbers as the survey progresses. Sometimes triangulation stations are given names suggested by the locality where each is established; this is particularly true for the more precise triangulation systems covering large areas.

For many continuous traverses where lengths are measured with the tape, distances are referred to the point of beginning of the survey and stakes marked with the distance from the initial point are commonly set every 100 ft. These 100-ft. points are called *full* stations. Points falling between full stations are called *plus* stations. The distance to any plus station is indicated as the number of *hundreds of feet* from the initial point to the preceding full station plus the distance in feet from the preceding full station to the point in question. Thus, a full station 1,200 ft. from the point of beginning would be numbered 12 + 00, or simply 12, and a plus station at say 1,927.2 ft. from the point of beginning would be numbered 19 + 27.2. The stations intermediate between transit points are usually marked by flat stakes driven vertically with the number on the side towards the zero end of the traverse, the number reading down the stake. Where conditions will not permit the driving of stakes, there may be employed nails, chisel marks, painted marks, etc. Transit points are witnessed as described above, the number for a given point being its chainage from the zero point, as for any other station on the traverse. Thus the witness stake for a transit point 1,216.3 ft. from the initial station would be marked as ⊙ 12 + 16.3.

216. Transit Surveys.—The field work involved in making a survey with the transit may ordinarily be divided into two parts: first, the establishment of transit stations and transit lines by angular and linear measurements; and second, the location of objects and points with respect to the transit lines. The transit lines may be said to form the skeleton of the survey and are sometimes spoken of as *control* or *horizontal control;* the measurements made to determine the location of objects or to establish the position of points furnish the details with which the transit lines are clothed. For some surveys the amount of detail secured from the transit lines is little, as in establishing the boundaries of land. For other surveys the location of features away from the transit lines forms the greater portion of the work, as on certain topographic surveys. On some surveys the collection of details may take place as the work of laying out the transit lines proceeds; on others the system of transit lines is first established and after it has been checked, the details are obtained. The latter procedure is most likely to be employed where the survey covers a considerable territory and where the methods and instruments used in running the transit lines are not those used in the collection of details.

Surveys may be said to be of two general classes: first, those that have for their object the location of certain features of the landscape; and second, those that have for their primary purpose the establishment of points and lines of predetermined length and direction which are to be employed as a guide to the future enterprises of man. Often the two classes overlap and a single survey may be useful in each of the above capacities. For a given character of work the methods employed are fundamentally the same whether the survey is of the first or of the second class.

The simplest survey to be made with the transit and tape consists of a single transit station over which the instrument is set. Angles and distances to surrounding points are observed, and thus objects within the range of vision of the transit are located. This is often called the method of radiation.

One nearly as simple consists of two transit stations connected by a single transit line, called the base line. From each station, angles with respect to the transit line are observed to objects which it is desired to locate. Thus any point is defined by the two angles taken from the transit stations and by the length of the base line. This is generally called the method of intersection, and is a variety of triangulation.

On surveys of any considerable extent the method of traversing is generally employed to establish most of the transit lines. The two preceding methods, together with some others later to be described, are used in locating details. On surveys of ordinary precision, transit traverses in one form or another probably make up more than 95 per cent of the systems of control.

On surveys of ordinary precision, the triangulation system as a means of providing control is not generally used except in hydrographic surveying and in topographic surveying in rough, open country. Where conditions are favorable, it is a very economical method insofar as time spent in the field is concerned. Generally, additional control stations are established by traverses run between triangulation points. The method of triangulation is employed extensively on surveys of precision covering wide areas, particularly in connection with topographic and hydrographic work. The opportunities where the triangulation system may be employed to good advantage over traversing are more numerous than is generally realized. The subject of triangulation is reserved for a later chapter.

217. Method of Radiation.—This method alone is applicable only to surveys covering small areas. The transit is set up at any convenient station from which can be seen all points which it is desired to locate. The distance from the transit station to each of the points is measured, and the horizontal angle is observed. The angles between successive points may be measured, or the true, magnetic, or assumed bearing or azimuth of each of the lines joining the points with the transit station may be observed.

Where it is necessary to locate only the points, as, for example, where the survey is being made for a map, the method is excellent.

But trigonometric computations are necessary if the lengths and directions of land lines are to be determined.

The method is one quite generally employed for the location of details on more extensive surveys, the stations occupied being traverse or triangulation points.

218. Method of Intersection.—In Fig. 218 let the points A, B, C, etc., represent objects which it is desired to locate, and let OP be some convenient line from either end of which the unknown points A, B, C, etc., are visible. The length of the base line OP is measured with the tape. The transit is set up at O, and angles to the unknown points are observed. These may be expressed as azimuths, as bearings, or as angles between successive points. A similar series of observations is made with the transit at P. In this manner each of the unknown points, A, B, C, etc., becomes the vertex in a triangle of which the base line OP is the side of measured length, and the angles adjacent thereto are observed values; the positions of the unknown points are thus defined.

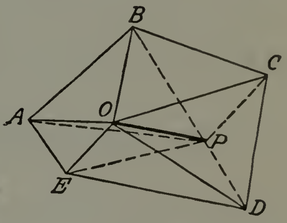


Fig. 218.

The method is not usually employed alone but is used in conjunction with other methods, particularly traversing. Where well-defined details at a considerable distance are visible from two or more transit stations, they may often be located by this method much more expeditiously than by angle and distance. Also when some landmark is visible from the number of stations of a traverse, angles to the mark taken from the several stations provide a means of checking the accuracy of the angular and linear measurements of the traverse.

In hydrographic surveying it is a very useful method of locating soundings, observations upon a boat signal being made simultaneously by two transits located at triangulation or traverse stations on shore.

219. Traversing.—The process of running a transit traverse is much the same for one variety of traverse as for another. If the traverse is for the purpose of locating features already existing in the field, the positions of transit stations are chosen so as to facilitate the work of locating these features. For example, in running a traverse for the control of a hydrographic survey, the transit stations would be located at points from which a good view of the areas to be sounded could be obtained, and the traverse for the preliminary survey for a proposed railroad would follow the probable general route of the railroad. If the traverse is for the purpose of establishing points and lines in accordance with predetermined measurements, it becomes simply a matter of laying off these measurements in the field. For example, the location survey for a railroad would consist of a traverse along the center line of track.

Following is a general description of the work of running a closed traverse, the transit stations being established in advantageous locations as the survey progresses, and distances being measured between successive transit stations: In Fig. 219 let A and B be selected locations for transit stations marking the first line of a traverse. Hubs defining the points are driven and properly identified by witness stakes. The transit is set up at B, the horizontal vernier is set to a given angular value, a backsight is taken on a range pole at A, and the lower motion is clamped. The line AB is then chained as described in Chap. VI, except that usually the head chainman is lined in by the transitman rather than by the rear chainman. The distance AB is recorded. The position of C is selected, and the transit point is established. The transit is turned on the upper motion until a foresight is secured on C. The upper motion is clamped, and the angular value is read and recorded. The distance BC is chained and recorded in a manner similar to that for AB. The transit is moved forward to C, a backsight is taken to B, the point D is chosen, a foresight is taken to D, and the angle is read. The line CD is chained. And so the process is repeated for points E, F, etc., until the traverse is finally brought to a closure on the initial point A. Usually, as a means of checking the angular measurements against large errors, *magnetic* bearings are observed on both backsight and foresight from each station and are compared with *calculated* bearings determined from the measured angles as described in Chap. XI. Also as a check, after the traverse is brought to a closure, the initial station is occupied by the transit and the angle between the first and last lines of the traverse is measured; in this way the angular error of closure is determined. Where the traverse is not long, a sketch of the traverse, together with any other desirable features of the survey, is usually shown on the right-hand page of the notebook, and the measurements are tabulated on the left-hand page, the numerical notes being kept down the page in order of the observations. Even though measurements to details are not included, it is usually

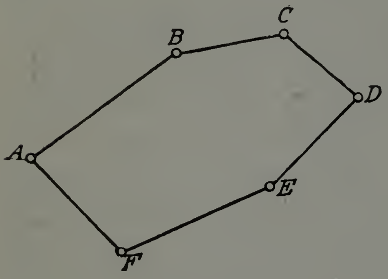


FIG. 219.

customary to show on the sketch the approximate location of important objects.

A continuous or open traverse may be run in exactly the same manner, but the angular error of closure can not, of course, be determined as described above. Not infrequently the continuous traverse may begin and close on previously established lines, the directions and positions of which are known. Where stakes are set every 100 ft. the chainage is referred to the point of beginning; and if the traverse is of considerable length, the notes are usually taken up the page, and for sketches the traverse line is considered as being on the center line of the right-hand page. Also in the notebook a line is given to each station and plus. This manner of keeping notes facilitates sketching, since objects represented will, when the transitman has his back to the zero end of the line, appear in the same relative position with respect to the transit line in the notebook as they appear with respect to the transit line on the ground.

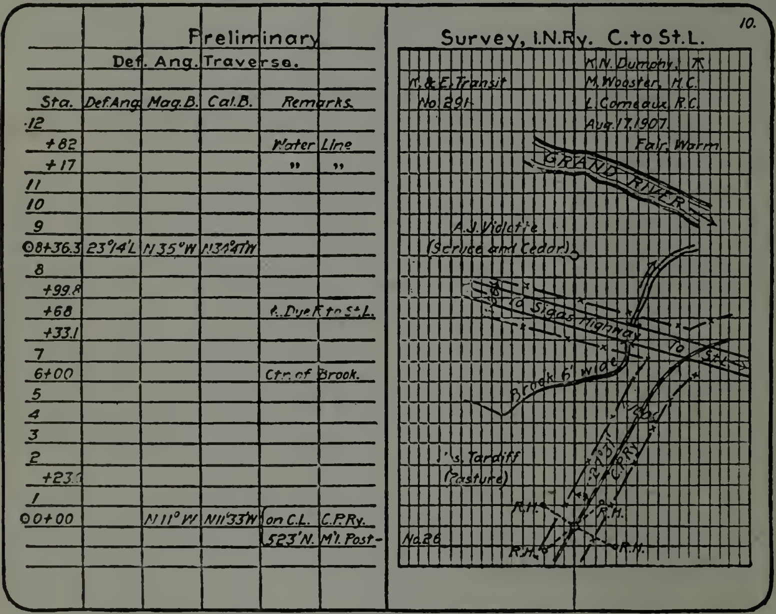
The common methods of running transit traverses are: (1) by deflection angles, (2) by azimuths, and (3) by interior angles. The magnetic compass with which the transit is equipped may be employed for running magnetic traverses, and is in this respect more useful than is generally appreciated, particularly for rough preliminary or reconnaissance surveys. Formerly it was common practice to run bearing traverses by means of the horizontal circle graduated in quadrants, but the azimuth method has such marked advantages over the bearing method that the latter is not often used except with the compass.

220. Deflection Angle Traverse.—This method of running traverses is probably more commonly employed than any other, especially on continuous traverses where only a few details are located as the traverse is run. It is used, practically to the exclusion of other methods, for the location surveys for roads, railroads, canals, pipe lines, etc. It is employed to a less extent in land surveying and in establishing control traverses for topographic and hydrographic surveys.

Successive transit stations are occupied and at each station a backsight is taken with the *A* vernier set at zero and the telescope in the reversed position. The telescope is then plunged to the direct position, the foresight is taken by turning the instrument on its upper motion, and the deflection angle is observed. The angle is recorded at right (*R*) or left (*L*), according to whether the upper motion is turned clockwise or counterclockwise.

Figure 220 shows the notes for a portion of a continuous or open deflection-angle traverse where stakes are set every 100 ft. For the sketches the center line of the right-hand page represents the traverse. The notes are kept up the page, and observations are checked against large errors by the close agreement between calculated bearings and observed magnetic bearings. The notes are typical of the form used on railroad, highway, and other route surveys.

The precision of measurements is increased somewhat, and at the same time each observation is checked, by doubling the angle, a common practice on important surveys. Both the single and the doubled values are usually recorded. The deflection angle is considered as being one-half of the doubled value. To eliminate instrumental errors, the first backsight from a given station is taken with the telescope direct and the second backsight is taken with the telescope reversed. The telescope is usually plunged between backsight and corresponding foresight. The procedure just outlined checks the angles within the lease count of the vernier and hence furnishes a much closer verification than does the method of checking by magnetic bearings.



Preliminary
Def. Ang. Traverse.

Sta.	Def. Ang.	Mag. B.	Cal. B.	Remarks.
12				
+82				Water Line
+17				" "
11				
10				
9				
⊙8+36.3	23°14′L	N 35° W	N 34°47′ W	
8				
+99.8				
+68				A. Dyek to St. L.
+33.1				
7				
6+00				Ctr. of Brook.
5				
4				
3				
2				
+23.3				
1				
⊙0+00		N 11° W	N 11°33′ W	on C.L. C.P.Ry.
				523′N. M'l. Post-

10.

Survey, I.N.Ry. C. to St. L.

K. & E. Transit
No. 291

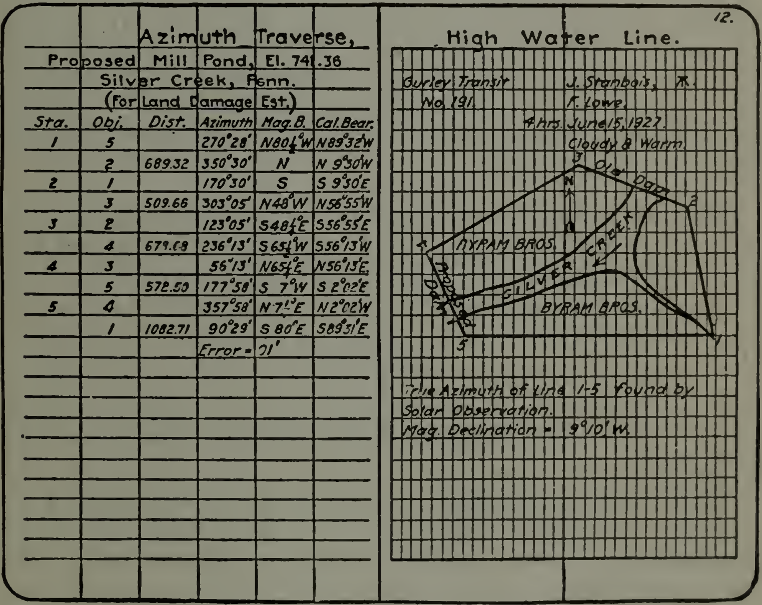
K. N. Dumphy, ⊼
M. Wooster, H.C.
L. Comeaux, R.C.
Aug. 17, 1907
Fair, Warm.

GRAND RIVER
A. J. Violette
(Spruce and Cedar)
Road to Siggs Highway
To St. L.
Brook 6′ wide
Road
J's. Tardiff
(Pasture)
To St. J. C.P.Ry.
R.H.
R.H.
R.H.
R.H.
No. 26

FIG. 220.—Notes for continuous deflection angle traverse.

221. Azimuth Traverse.—The azimuth method possesses an advantage over the other common methods of traversing with the transit in that the simple statement of an angular value fixes the direction of the line to which it is referred. The method is extensively used on topographic and other surveys where a large number of details are located by angular and linear measurements from transit points. The simple relation existing between azimuths and bearings makes it possible to calculate one from the other at a glance. Also the angular error of closure of a traverse is at once evident by the difference between the initial and final observations taken along the first line.

The azimuth of the initial line of the traverse may be referred either to a true or an assumed meridian. Successive stations are occupied beginning with the line of known azimuth. At each station the transit is *oriented* by setting the *A* vernier to read the back azimuth of the preceding line and then backsighting to the preceding transit point. The instrument is then turned on the upper motion, and a foresight on the following transit point is secured. The reading indicated by the *A* vernier is the azimuth of the forward line.



Azimuth Traverse,
Proposed Mill Pond, El. 741.36
Silver Creek, Penn.
(For Land Damage Est.)

Sta.	Obj.	Dist.	Azimuth	Mag. B.	Cal. Bear.
1	5		270°28'	N80¼°W	N89°32'W
	2	689.32	350°30'	N	N 9°30'W
2	1		170°30'	S	S 9°30'E
	3	509.66	303°05'	N48°W	N56°55'W
3	2		123°05'	S48¼°E	S56°55'E
	4	679.68	236°13'	S65¼°W	S56°13'W
4	3		56°13'	N65¼°E	N56°13'E.
	5	572.50	177°58'	S 7°W	S 2°02'E
5	4		357°58'	N 7½°E	N2°02'W
	1	1082.71	90°29'	S 80°E	S89°31'E
			Error =	01'	

12.

High Water Line.

Gurley Transit No. 191.

J. Stanbois, ⊼.
F. Lowe.
4 hrs. June 15, 1927.
Cloudy & Warm.

True Azimuth of Line 1-5 Found by Solar Observation.
Mag. Declination = 9°10' W.

FIG. 221.—Notes for azimuth traverse.

Figure 221 illustrates the notes for a closed traverse for which the azimuths were observed as just described. Azimuths set off on the circle and used as backsight readings are called *back azimuths*, and those observed when the foresights have been taken are called *forward azimuths*. As indicated in the notes, each line has both a forward and a back azimuth whose values differ by 180°. The traverse is started from the line 1-5 whose azimuth is 270°28′ reckoned from true north. The forward azimuth of line 1-2 is found to be 350°30′. When the transit is set up at Sta. 2, the back azimuth is calculated by subtracting 180° from the forward azimuth (350°30′- 180° = 170°30′), and this value is set on the vernier before the backsight to Sta. 1 is taken. Magnetic bearings are observed, and a check against large errors is secured by noting that the calculated bearings vary from corresponding magnetic bearings by about 9°10′, the amount of the magnetic declination.

Often it is not convenient or necessary to use the true meridian as the initial or reference meridian from which the azimuths of the lines of a traverse are measured. In this case, the first line of the traverse may arbitrarily be taken as the reference meridian, sometimes called the line of zero azimuth. In other cases, a direction for the initial meridian is chosen from the first transit station to some definite object in the general direction of North.

222. Interior Angle Traverse.—This method of traversing is used principally in land surveying. So far as the field operations are concerned it is not materially different from the deflection method. At any station the vernier is set at zero and a backsight to the preceding transit point is taken. The instrument is then turned on the upper motion until the advance station is sighted, and the interior angle is read. The notes may be kept in the form of a sketch on which the observed angles and distances are shown, or the numerical values may be tabulated.

223. Methods of Checking Traverses.—At the expense of some repetition, the common methods of checking traverses will here be discussed. The errors involved in traversing are of two kinds, angular and linear.

For a closed traverse there are certain fixed geometrical relations that exist so that it is possible to determine readily the angular error of closure, *i.e.*, the amount that the measured sum of angles differs from the true sum. Thus, for the deflection angle traverse the algebraic sum of the deflections should be 360°; for the interior angle traverse the sum of the interior angles should equal $(n\text{-}2)180°$; and for the azimuth traverse the azimuth of the first line as observed at closure should equal the known or assumed azimuth of the same line employed at the beginning of the survey.

The actual total error in measured distances can not be determined. But the coordinates of the first point as calculated by successive angles and distances around the traverse should equal the coordinates of this same point as used at the beginning of computations, and this condition renders it possible to calculate the *linear error of closure*, due to errors both in angles and in distances, or the difference between the initial and final computed positions of the initial point of the traverse. The process of computing the linear error of closure is given in the following chapter.

For an open or continuous traverse no such means of checking the measurements as a whole are available. But while linear errors can not be detected, angular errors may be closely determined by astronomical observations taken at intervals as the traverse progresses. Where conditions allow, cut-off lines run between certain intermediate points are sometimes established, these lines making it possible to check certain parts of the traverse. Thus in Fig. 223 the line *ABC . . . N* represents a continuous traverse. If the direction of the cut-off line *AG* is observed at both *A* and *G*, the angular measurements of the traverse from *A* to *G* may be checked, and further if the distance *AG* is measured the linear error of closure may be computed. Similar measurements for the cut-off line *DK* would enable the checking of the traverse up to *K*.

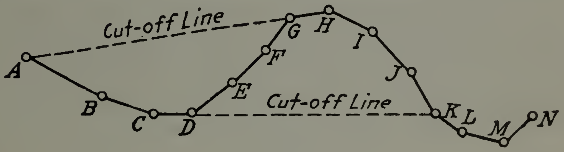


Fig. 223.

By observing the angle to some distant land-mark from each of several stations several determinations of the rectangular coordinates of the mark may be obtained. If these values agree closely, it is good evidence that all angular and linear measurements which are involved are free from serious errors. If there is disagreement between computed coordinates, there is no way of always telling with certainty just where the error lies but usually its location may be fixed approximately.

While the methods of checking continuous traverses already described are employed where opportunity offers, ordinarily it is impossible thus to verify more than a small portion of the total number of observations taken, and the errors in the traverse as a whole can not be determined.

Where conditions are such as to render the magnetic needle a dependable device, magnetic bearings offer an excellent means of checking observed angles against large errors or mistakes, and on traverses of ordinary precision this check is usually applied by reading the compass needle on the transit and then comparing the observed bearing with the bearing calculated from observed transit angles.

On important traverses, particularly those of extensive surveys or surveys that do not contain closed figures, the angular values are usually verified by doubling the angles, and often the linear measurements are checked by chaining forward and back over each line.

224. The precision of the transit traverse is affected by angular as well as by linear errors of measurement. The precision of linear measurements with the tape was discussed in Art. 74; the precision of angular measurements with the transit was discussed in Art. 209. It will be remembered that under ordinary conditions the angular errors are largely accidental in character, and hence, in general, the probable error in the direction of any line in a traverse with respect to any other may be expected to vary as the square root of the number of set-ups between the two lines. On the other hand, the important linear errors on traverses of ordinary precision are likely to be systematic in character. The precision of the position of any transit point therefore is generally influenced much more by the systematic linear errors than by the accidental errors, and except on surveys of high accuracy, the precision is usually found to vary approximately as the length of the traverse lines. In stating limits of error in transit work the ratio of linear precision $\left(\text{as } \frac{1}{5{,}000}\right)$ is used.

Limitations as regards the skill of surveying personnel, quality of instruments, and field conditions are so numerous as to make any statement of the accuracy to be attained in traversing with the transit of only very general value. The following specifications have been prepared with this in mind, and the values therein given are not by any means to be considered as fixed. The specifications give maximum values, and if executed by well-trained men, instruments in good adjustment, and under average field conditions, the error of closure should, in general, be not more than half the specified amount.

225. Specifications for Traversing. *Class* 1.—Total error of closure not to exceed $\frac{1}{1{,}000}\cdot$ Transit angles to be read to the nearest minute. Sights taken on a range pole plumbed by eye. Allowable angular error of closure, $1'30''\sqrt{n}$. Distances measured with a 100-ft. steel tape. Disregard slopes under 3 per cent. On slopes over 3 per cent, distances to be measured, either on the slope and a correction applied, or the tape held level using an estimated standard pull. Pins or stakes to be set within 0.1 ft. of the end of the tape. Precision sufficient for many preliminary surveys, horizontal control for topographic surveys plotted to intermediate scale, and for land surveys where the value of the land is low.

Class 2.—Total error of closure not to exceed $\frac{1}{3,000}$. Transit angles read carefully to the nearest minute. Sights taken on range pole carefully plumbed. Allowable angular error of closure to be $01'\sqrt{n}$. A correction to be applied to the linear measurements if the temperature differs more than 15°F. from standard. Disregard slopes under 2 per cent. Distances on slopes over 2 per cent to be measured either on the slope and corrections applied, or the tape to be held level and the standard pull carefully estimated. Pins or stakes to be set within 0.05 ft. of the end of the tape. Precision sufficient for most land surveys, for the location of railroads, highways, etc. By far the greater number of transit traverses fall in this class.

Class 3.—Total error not to exceed $\frac{1}{5,000}$. Transit angles read twice with the instrument reversed between readings. Sights taken on a plumb-line or a range pole very carefully plumbed. Allowable angular error of closure, $30''\sqrt{n}$. Temperature to be determined within 10°F. and corrections applied to the linear measurements. All tape measurements on slopes to be corrected. If the tape is held level, the pull should vary not more than 5 lb. from the standard, and corrections for sag should be applied. Pins to be set within 0.05 ft. of the end of the tape. Precision sufficient for much of the work of city surveying, for surveys of important boundaries, and those forming the control for extensive topographic surveys.

Class 4.—Total error of closure not to exceed $\frac{1}{10,000}$. Transit angles read twice with the instrument reversed between readings, each reading being taken as the mean of both A and B vernier readings. Verniers reading to 30″. Instrument in excellent adjustment. Allowable angular error of closure $15''\sqrt{n}$. Temperature of tape to be determined within 5°F. and corrections applied. Slopes to be determined within 2 per cent and corrections applied. When the tape is held level, the tension should not vary more than 3 lb. from standard and corrections for sag should be applied. Pins to be set within 0.05 ft. of the end of the tape. Precision sufficient for accurate city surveying and for other specially important surveys.

226. Referencing Transit Stations.—In many instances transit stations are in locations where they are likely to be disturbed or to be destroyed before the usefulness of the transit lines as lines of reference has expired. Where the need for relocation is likely to extend over a short period only, the hubs marking the transit stations are usually tied by angular or linear measurements to temporary wooden hubs or other objects that are not likely to be disturbed. A transit station is said to be *referenced* when it is so tied to nearby objects that it can be readily replaced. Hubs employed as reference marks are called *reference hubs*, and guard stakes placed beside the reference hubs indicate the character of the hub and the transit station to which it is referred. Thus a stake marked *RHA* ⊙ 172 + 32.2 would designate the hub beside which it was placed as reference hub *A* for traverse station 172 + 32.2. The manner of referencing a transit station is indicated in the notes by an appropriate sketch showing not only the angular and linear reference measurements, but also the character of the reference marks. On many surveys, particularly land surveys, the corners are tied to nearby objects which can be readily found, are not likely to be moved or obliterated, and are of a more or less permanent character.

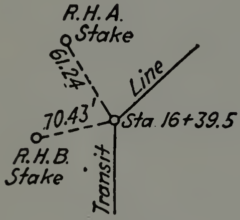


FIG. 226*a*.

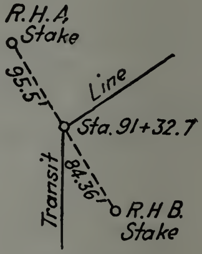


FIG. 226*b*.

Figures 226*a* to 226*d* illustrate several methods of referencing a transit station. The station shown in Fig. 226*a* is tied by linear measurements to two reference hubs. While the original measurements can be made very expeditiously the field work connected with replacement of the station is not so simple, and if either of the reference hubs is disturbed the station can not be relocated.

Figure 226*b* shows a station referenced by setting the two reference hubs on a line passing through the station and by taking linear measurements from these hubs to the station. Replacement is accomplished by setting up the transit at *RHA* and sighting to *RHB* (or *vice versa*) and then laying off along the line thus established the given distance from reference hub to station. If either of the reference hubs is destroyed the station can not be relocated except by rerunning the transit line from a station the position of which is known.

Figure 226*c* represents a third method and one that is quite often employed when there is likelihood of any of the reference hubs being accidentally lost. The station is at the intersection of the line *AB* and the line *CD*. With the measurements as shown, any two reference hubs may be destroyed and still the station may be relocated.

Figure 226*d* is typical of the manner in which corners are referenced in land surveying.

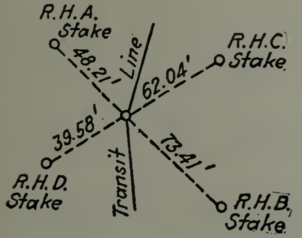


Fig. 226*c*.

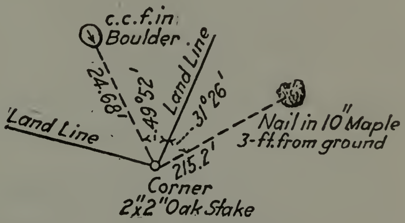


Fig. 226*d*.

227. Details from Transit Lines.—On nearly all transit surveys, a part of the work consists of the location of certain details with respect to the transit lines. The nature and number of details to which measurements are taken depends upon the purpose of the survey and the character of the country through which the survey runs. On the one hand, a survey for the purpose of establishing or relocating boundaries of land would include the location of only a few important objects close to the transit lines. On the other hand, a complete topographic survey might include the location of all natural and artificial features of the terrain. The accuracy with which details are located likewise depends upon the purpose of the survey. In retracing property lines, not infrequently the actual lines are obstructed by hedges or buildings, so that it is necessary to run transit lines approximately parallel with the property lines, and to locate the corners by measurements from transit lines. Such measurements would need to be taken with a precision as great as that for the transit line. The survey for a map ought to be so conducted that all objects may be correctly shown within the scale of the map. Thus if the scale of the map were 1 in. = 1,000 ft., there would be no particular advantage in taking measurements from a transit line to, say, a building closer than the nearest 10 ft., but if the scale were 1 in. = 10 ft. measurements ought to be taken to tenths of feet. Generally, if details are located solely for map-making purposes, the required precision of measurements to details is less than the required precision of measurements for the transit lines; this is particularly true for extensive surveys.

Angular measurements are usually made with the transit. In most cases angles are read to minutes, but where angles are to be used only in mapping operations and their correct numerical value is of no particular consequence, usually nothing is to be gained by reading closer than 05′. For the ordinary transit, angles may be estimated to the nearest 05′ without the aid of the vernier, and where a number of observations are to be made, estimating the angles in this way results in a material saving in time.

Linear measurements to details are made with the 100-ft. steel tape, with the metallic tape, with the stadia, and sometimes by pacing. Where distances are long and a high precision is required, the steel tape is employed. Where the survey is for the purpose of securing data for a map, distances to details are very often obtained by stadia (see Chap. XVII).

On most surveys details are located as the work of establishing the transit lines progresses; thus as a traverse is being run, at any transit station after the foresight to the following station has been observed, the transit is left in position until angles to details have been read. These observations with the transit are called *side shots*, to distinguish them from the traverse angles.

228. Following are descriptions of the common methods of locating details with the transit and tape:

1. *By Angle and Distance from Transit Station.*—As illustrated by Fig. 228*a*, any given point on the object is located by an angle and a distance. At a given transit station distances are measured to such details as may be conveniently located from that station, and usually at the same time the transit is set up at the station and angles are observed between the transit line and the given details. The method is very widely used, particularly when the country through which the survey passes is open and the details are not remote from transit stations. Generally the notes show by sketches the character and general position of the objects thus located. Where details are not numerous the angles and distances are shown in the proper location on the sketch, but if a considerable number of observations are to be made from a single transit station, each observed point is given a number on the sketch, and angles and distances are tabulated

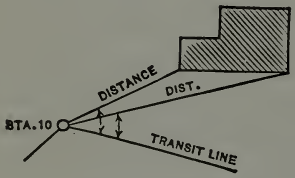


FIG. 228*a*.

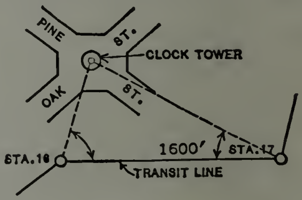


FIG. 228*b*.

opposite corresponding numbers on the left-hand page of the notebook. Where details are numerous, azimuth angles, rather than deflection angles, are observed for the reason that the chances of later confusion are much less.

2. *By Angles from Two Transit Stations.*—This method is particularly useful in locating distant or inaccessible objects which can be seen from two or more transit stations. As shown by Fig. 228*b*, the position of any point is located with respect to the transit line if angles to the point are taken from at least two stations. The particular advantage of the method is that no linear measurements, other than those made in running the transit lines, are required; hence the field work is reduced. The disadvantage is that the distance from transit station to point sighted can be determined only by computation (except that a rough value can be scaled from a map).

3. *By Distances from Two Stations.*—On transit traverses for which stakes are set every 100 ft. this method of locating details sometimes expedites the work, particularly when the details are close to the traverse line, yet distant from the nearest transit station. A given object is located by linear ties from two traverse stations. Thus, in Fig. 228*c* the stone bound at the fence corner is located by distances from stations 4 + 00 and 5 + 00 neither of which is a transit station. The method is used to advantage where the angle and distance method either would require additional transit stations or would make necessary the measurement of long distances.

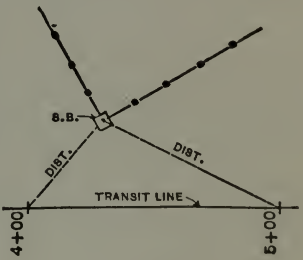


Fig. 228*c*.

4. *By Perpendicular Offsets from the Transit Line.*—This method is adapted to the location of irregular or curved boundaries, streams,

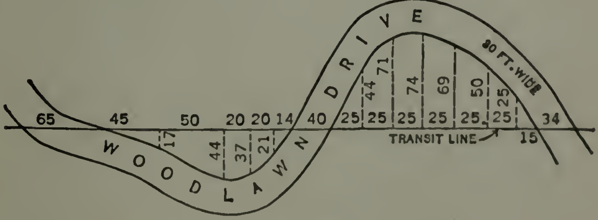


Fig. 228*d*.

drives, and other features of character similar to those mentioned which closely parallel transit lines. As indicated by Fig. 228*d*, a point is located by measuring the distance along the transit line to the foot of the perpendicular offset through the point and then measuring the length of the perpendicular offset. Features such as those mentioned are located sometimes by offsets at regular intervals, but more often at critical points which, in general, will make the offsets come at irregular intervals. Where stakes are placed every 100 ft. along the transit line, the station and plus of the foot of each perpendicular is secured, rather than the distance between offsets as shown.

5. *By Angle from One Station and Distance from Another.*—This method is not employed as frequently as the others just described, but occasionally will be found useful. Angles are measured with the transit set up at a transit station; distances are measured from stations intermediate between transit stations. Figure 228*e* illustrates a situation where the method would prove advantageous.

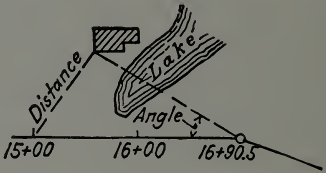


Fig. 228*e*.

6. *By Range Ties and Swing Offsets.*—In addition to the methods previously described, if the features to be located are buildings, the work of location may be facilitated by ranging in (by eye) the sides of the building and finding the points of intersection of lines thus defined with the transit line. The station and plus of a point of intersection together with the distance along the range line from the point of intersection to a corner of the building is called a range tie.

A swing offset is the perpendicular distance from a transit line to a point and is determined by swinging the tape about the point as a center and taping the least distance from the point to the transit line.

CHAPTER XIV

METHODS OF PLOTTING

229. General Remarks.—The methods of plotting described in this chapter are those employed in mapping areas of limited extent where the earth's surface is assumed to be plane and all meridians are assumed to be parallel. These methods are applicable to surveys for railroads, highways, irrigation and drainage systems, and the like, as well as to most topographic and hydrographic surveys, except those of wide extent, and to rural and urban land surveys.

230. Process of Making a Map.—As a usual thing, regardless of their purpose or kind, maps are so plotted that features are shown in the same relative position as they exist on the ground and are represented at a given scale. Hence the data of a survey furnish the information that is necessary to plot the map, and the operations of plotting are in a sense the reverse of the operations of surveying. The value of most maps depends upon the accuracy with which details are shown, and the general aim is to plot the more definite features so that they will be located within proper limits of error.

Just as the first step in the field work for a survey of any considerable extent which has as its object the location of points and lines in the horizontal plane consists of establishing points of horizontal control so that their relative positions will be known with a required degree of precision, so the first step in mapping is to plot these points of horizontal control so that their position one with respect to another will be correctly shown, irrespective of the distance which intervenes between them. And just as the lines joining points of horizontal control on the ground form the skeleton of the survey, so do the corresponding lines when represented upon paper form the skeleton upon which the details of the map are hung. In general, therefore, the process of mapping involves: (1) the plotting, by more precise methods, of points of horizontal control which are generally, though not always, transit stations and which may be either traverse or triangulation points, or both; and (2) the plotting, by less precise methods, of features which go to make up the map, generally called the map details, measurements to these details being given in the form of angles and distances from the lines and points in the horizontal control system.

231. Methods of Plotting Horizontal Control.—Horizontal control may be plotted: (1) by use of the protractor, (2) by the chord method, (3) by the tangent method, or (4) by the coordinate method. In any case, distances are measured with the engineers' scale, and points are pricked with a needle. Traverse and triangulation stations are indicated by appropriate symbols and control lines are carefully drawn with a hard pencil having a fine point. Points are numbered or lettered to conform to the system employed in the field work.

232. The Protractor Method.—Where the control system is not extensive and the map is small, a good-sized protractor provides a sufficiently accurate means of plotting angular values. Obviously the precision with which angles may be laid off is likely to vary directly with the diameter of the protractor. The largest protractor commonly manufactured has a diameter of 14 in. It can be shown that even for this maximum size and with most careful plotting the probable error for a single angle is not less than 05′ and for the 6 in. protractor (a size in quite common use) the accidental error of plotting may amount to 15′. It is apparent that the angular precision likely to be obtained with any protractor is low.

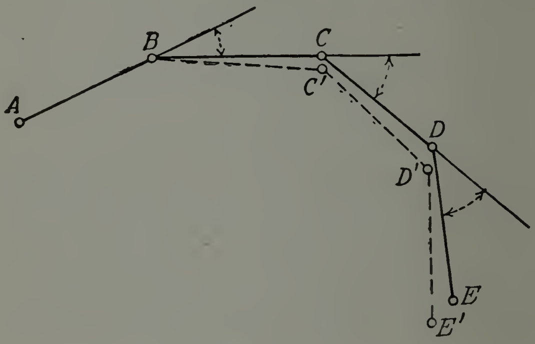


Fig. 232.

Where a traverse is to be plotted, the common practice is to fix by estimation the position of the first line (as AB, Fig. 232) and then by placing the protractor at the forward point (as B) to lay off the deflection angle to the succeeding line, and having determined its direction by a light line of indefinite length, to lay off along this line the given distance (as BC) to the succeeding point (as to C); and thus to repeat the process for each traverse point (as for C, D, etc.). An objection to this procedure lies in the fact that any error in the direction of one line affects to a like degree the directions of all succeeding lines, and the linear error in the position of succeeding points increases with the distance. Thus in Fig. 232 if *CBC′* represents the error made in laying off the angle at *B*, then the corresponding linear errors at the succeeding points are *CC′*, *DD′*, and *EE′*, and the magnitude of these errors varies directly with the distance from the point at which the angular error occurs. For example, if in Fig. 232 the angular error at *B* were 10′, and the distances *B* to *C* and *B* to *E* were respectively 10 in. and 30 in., then the linear error in the position of *C* due to the given angular error alone would be 0.03 in. and that in the position of *E* would be 0.1 in. When it is recalled that distances may be scaled within 0.02 in., it becomes obvious why this method will not yield results sufficiently accurate for extensive maps with many points of control.

233. The instances where the protractor may properly be employed for plotting triangulation stations are not numerous, but it is a sufficiently accurate method for rough surveys over small areas, and is often useful in checking continuous traverses when angular observations have been taken from several stations to a given landmark. The position of a station is determined by the intersection of lines laid off from two other stations marking the ends of a line of known length.

234. Plotting by Tangent Offsets.—This method is employed principally for plotting transit traverses for the control of maps of moderate size where the number of stations is small. In principle, it is similar to the protractor method described in Art. 232, with this difference: An angle is laid off by linear measurement which is a constant times the natural tangent of the angle. Usually deflection angles are plotted, but there is an advantage in plotting azimuths or bearings from a centrally located meridian. In Fig. 234, *AB* represents the plotted position of the initial line of a traverse and α is the deflection angle at *B* which it is desired to lay off in order to determine the direction of *BC*. By the tangent method the line last established, in this case *AB*, is prolonged some convenient distance, usually 10 in., to form a base line *Bb*, at the end of which a perpendicular *bb′* is erected and the distance *bb′* laid off equal to the length of the base line *Bb* multiplied by the natural tangent of α. A line drawn from *B* through *b′* defines the direction of *BC*. Then the point *C* is plotted by laying off to scale the given distance *BC*. Thus the process is repeated for succeeding lines, except that generally when the deflection angle is greater than 45°, instead of making the base line a prolongation of the preceding line, it is established as a perpendicular to the preceding line at the point last plotted. Thus in the figure, β is assumed to be such an angle and *Cc* is the base line perpendicular to BC. The perpendicular offset cc' is in this case the length of the base line Cc times the natural cotangent of the angle β. The line Cc' fixes the direction of CD, and the point D is then plotted by linear measurement from C. Ordinarily the length of base line is 10 in.

When a traverse is to be plotted by this method, deflection angles and distances are tabulated and the tangents of angles less than 45° and cotangents of angles greater than 45° are recorded. The traverse is roughly plotted to small scale using the protractor for laying off angles, and the shape of the traverse is noted. By means of the small-scale sketch a suitable position for the first line of the traverse is estimated, and the line is plotted. The work of plotting is then continued as described in the preceding paragraph. If base lines are made 10 in. long the numerical values of offsets are not recorded but are determined mentally from the tabulated values of tangents. Where the base line has a length other than 10 in. offsets are computed and tabulated.

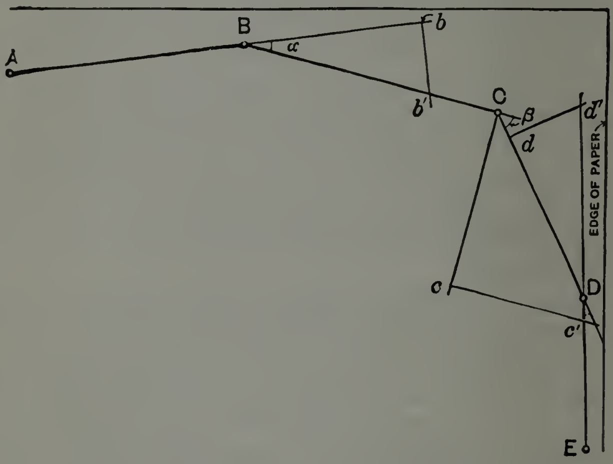


Fig. 234.—Plotting by tanget offsets.

The precision with which angles may be laid off by this method depends upon the care used in plotting and the length of base line employed. For accurate results all points should be pricked with a needle and all lines should be drawn with a hard, sharp pencil. When this careful procedure is followed the error in laying off an angle using a 10-in. base line need not exceed 03′, which is about the error that might be expected in using a protractor having a diameter of 20 in., if such a size were available.

235. Plotting by Chords.—This method is much like that of plotting by tangents described in Art. 234, except that instead of erecting a perpendicular at the end of a 10-in. base line, an arc of 10-in. radius is struck. The chord distance for the given angle is then scaled from the point of intersection between arc and base line to a point on the arc.

236. Checking.—Checking the correctness of the plotted horizontal control is quite as essential as is the verifying of field measurements. The methods of checking that may be employed are essentially the same for any one of the three methods of plotting just described, as for the others. Hence what follows may be said to apply to points plotted either by the protractor method, by the tangent method, or by the chord method. The process of checking the plotted location of points of horizontal control may proceed as the work of plotting progresses, but in any case it is completed before the work of plotting the details is begun. Not infrequently the process of checking serves to verify not only the work of plotting but to verify the field measurements as well.

When the horizontal control is in the form of a traverse which has been brought to closure in the field, it should likewise close on paper, and if it does, a check is secured on the correctness of both the field work and the drafting. If for the purpose of this discussion the field work be regarded as correct, then any error of paper closure will be due to inaccuracies of laying off angles and distances. If a traverse is of considerable length it will rarely close on paper, no matter how careful the work of plotting; if the error of closure is small, it is usually assumed to have gradually accumulated and the traverse is made to close by a gradual change in the position of the plotted lines.

237. When the horizontal control is a continuous or open traverse there is no absolute check on the precision of plotting, as is the case with the closed figure; a satisfactory way of checking distances is as follows: Cut a strip of paper somewhat longer than the combined length of the several lines in the traverse. Let *a*, *b*, *c*, *d*, etc., be points to be marked on this strip corresponding to traverse points *A*, *B*, *C*, *D*, etc., on the drawing. Near one end and close to the edge, with a needle mark point *a*. Place this mark at *A* of the traverse and with the edge of the strip along *AB*, mark point *b* on its edge opposite *B* of the traverse. Move the strip to *BC* with *b* opposite the corresponding point of the traverse, and mark point *c*. Continue in this way around the entire traverse. Scale the total length marked off on the strip. This scaled distance should agree very closely with the sum of the lengths according to the data used in plotting.

238. Cut-off Lines.—When during the process of running a continuous traverse, cut-off lines are established, as described in Art. 223, both the field work and the plotting may be checked. Each cut-off line may be considered as forming the closing side of an independent traverse. Where the length of the cut-off line is measured, both angles and distances within the length intercepted by the cut-off line are checked if it closes on paper. If it fails to close, the procedure of checking is the same as for any other traverse, and when large errors appear to have been eliminated the portion intercepted by the cut-off line may be adjusted until the cut-off line actually closes the plotted circuit.

239. Intersecting Lines.—When angles to a given station not on the traverse are observed from a number of stations of the traverse, both the field work and the plotting may be regarded as correct if the plotted positions of the lines to the given station intersect at a common point.

240. Plotting by Rectangular Coordinates.—This method, also known as the method of total latitudes and departures, is the only practical one for plotting extensive systems of horizontal control. It is always employed for plotting triangulation figures except they be of the simplest character, and it is considerably more accurate than any of the methods so far described for plotting traverses.

The coordinate axes are a reference meridian (either true, magnetic, or assumed) and a line at right angles thereto, called a reference parallel. The intersection of these lines, marking the origin, may be any point in the survey or may be a point entirely outside the survey. The azimuths or bearings of all lines are either given or are calculated from observed angles. With the direction of each line determined and its length known, the lengths of its orthographic projection upon the meridian and upon the parallel are computed. The projection upon the meridian is termed the *latitude* of the line; the projection upon the parallel is called the *departure* of the line. The origin having been chosen, the coordinates for the several control points are computed by using the latitudes and departures. The coordinate of a point measured normal to the parallel is called the *total latitude* of the point; the coordinate measured normal to the meridian is called the *total departure.* With the coordinate axes established on paper, a point is plotted by laying off its total latitude and its total departure to the required scale. The succeeding articles are devoted to a more detailed discussion of the processes involved.

241. Latitudes and Departures.—In Fig. 241*a*, *AB* represents any line the latitude and departure of which it is desired to determine, and the lines NS and EW represent any meridian and any parallel. The line AB makes the angle α with the meridian.

Since the latitude of a line is the orthographic projection of the line upon a meridian, the latitude of AB is $A'B' = Ab = AB \cos \alpha$.

And since the departure of a line is its orthographic projection upon a parallel, the departure of AB is $A''B'' = Aa = AB \sin \alpha$.

Stated in the form of a general rule applicable to any line:

$$\text{Latitude} = \text{Length} \times \text{cosine bearing angle}$$
$$\text{Departure} = \text{Length} \times \text{sine bearing angle.}$$

Latitudes are designated as *North* or *positive* for all lines having a northerly bearing, and *South* or *negative* for all lines having a southerly bearing.

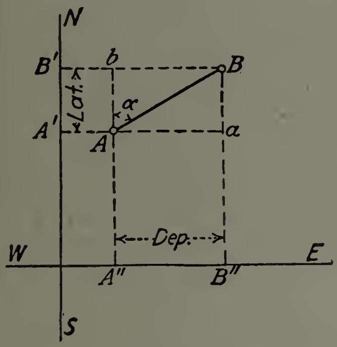


FIG. 241*a*.

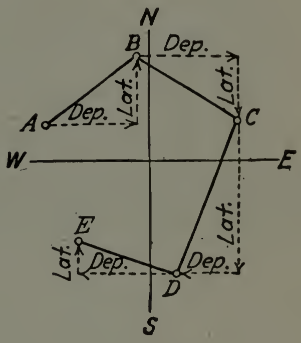


FIG. 241*b*.

Departures are designated as *East* or *positive* for lines having an easterly bearing, and *West* or *negative* for lines having a westerly bearing.

Thus, referring to Fig. 241*b*, for the line AB the latitude is North or + and the departure is East or +. For BC the latitude is South or −, and the departure is East or +. For CD the latitude is South or −, and the departure is West or −; and for DE the latitude is North or + and the departure is West or −.

If the latitudes and departures are determined solely for purposes of map construction, the number of places to be used in computations should be such that points may be correctly plotted within the scale of the map. Thus for a scale of 1 in. = 800 ft., distances can hardly be plotted closer than to the nearest 10 ft., but to insure all coordinates being correct to the nearest 10 ft. the latitudes and departures would probably be determined to the nearest foot, and to insure the latitudes and departures being correct to the nearest foot, the intermediate computations might be carried out to tenths of feet.

Example: For a given course in a traverse the calculated bearing is N34°21′W and the observed length is 1,215.3 ft. The traverse is to be plotted to the scale of 1 in. = 100 ft. and it is desired that the relative position of all points be correct within the scale of the map. It is desired to compute the latitude and departure with a degree of precision consistent with the purpose for which they are to be used.

At the given scale distances may be plotted within 2 ft. and therefore latitudes and departures should be correct to perhaps the nearest quarter foot if the traverse has many sides. Five place logarithms will be used. The computations are:

Latitude	1,003.3 ft.
Log. lat	3.00145
Log. cos	9.91677
Log. dist	3.08468
Log. sin	9.75147
Log. dep	2.83615
Departure	685.7 ft.

242. Error of Closure.—In any closed traverse it is obvious that the sum of the north latitudes should equal the sum of the south latitudes, and that the sum of the east departures should equal the sum of the west departures (since making a complete circuit of any kind involves going as far in one direction as in the opposite). In other words, for any closed traverse the algebraic sum of the latitudes (ΣL) should be equal to zero and the algebraic sum of the departures (ΣD) should be equal to zero. Due to errors in field measurements of both angles and distances, the chances of these conditions being realized are exceedingly remote, hence in general the unadjusted traverse would not close on paper even though the plotting were without error. The conditions just stated above make it possible to determine by means of the computed latitudes and departures, the magnitude of the error of closure. Figure 242 shows a traverse which does not close, the line $A'A = e$ being the side of error. If the sum of the latitudes and the sum of the departures of the measured sides $AB, BC \ldots EA'$

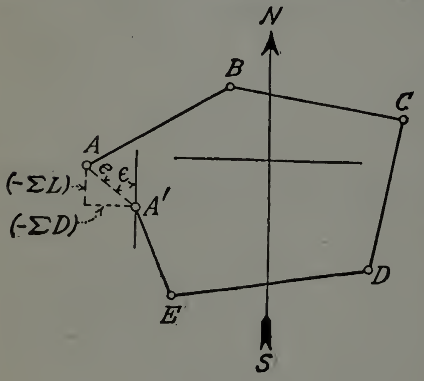


FIG. 242.

respectively be designated by the symbols ΣL and ΣD, then in order that the sum of the latitudes and the sum of the departures shall each be equal to zero, the latitude of the side of error must be $(-\Sigma L)$ and its departure must be $(-\Sigma D)$. These two quantities form the base and altitude of the right-angle triangle of which the side of error is the hypotenuse, hence the error of closure

$$e = \sqrt{(\Sigma L)^2 + (\Sigma D)^2}$$

The direction of the side of error is given by the relation

$$\tan \epsilon = \frac{-\Sigma D}{-\Sigma L}$$

which is satisfied by bearings differing by 180°. The data will make it apparent in which quadrant the bearing lies.

Example: In a given closed traverse the sum of the north latitudes exceeds the sum of the south latitudes by 9 ft. and the sum of the west departures exceeds the sum of the east departures by 12 ft. What is the linear error of closure and the bearing of the side of error?

The linear error of closure is $e = \sqrt{(9)^2 + (12)^2} = 15$ ft.

Since for the given sides the north latitudes exceed the south latitudes, the latitude of the side of error must be south, and by parallel reasoning the departure of the side of error must be east; hence the bearing angle is in the southeast quadrant.

$$\tan \epsilon = \frac{-\Sigma D}{-\Sigma L} = \frac{+12}{-9} = -1.33$$

$$\epsilon = -53°08'$$

$$\text{Bear.} = \text{S}53°08'\text{E}$$

243. Balancing the Survey.—(This problem deals with the adjustment of field observations which has been discussed at some length in Chap. V.)

When the error of closure has been determined as described in the preceding article, corrections are usually made so that the traverse will form a mathematically closed figure, and the corrections are applied to the latitudes and departures in such manner as to make their algebraic sum equal zero. This operation is called *balancing the survey* or *balancing the traverse*. It is an operation which is performed not only prior to the calculation of coordinates for plotting, but also before computing areas. There is, of course, no possible means of determining the true magnitude of the errors in angle and distance which occur throughout the traverse, but if conditions surrounding the field measurements have been uniform, it is fair to assume that errors have gradually accumulated and corrections should be made accordingly.

There are several rules for distribution of errors each of which will produce a mathematically closed figure, and each of which is assumed to be adapted to certain conditions as regards measurements. Many surveyors, however, rely upon their own judgment, in a large measure disregarding any established rule, and arbitrarily distribute the error in accordance with their estimation of the difficulties met in the field.

244. Adjustment of the Angular Error.—The values of the measured angles of a traverse are checked by the known geometrical conditions. The total angular error thus determined is distributed among the angles or calculated bearings of the traverse before computations of the latitudes and departures are made.

As has been stated in the previous article, this adjustment is often arbitrary, based on a knowledge of the field conditions; but, if all angles have been measured under like conditions, the error is distributed equally to each angle in the traverse.

The total angular error is the result of both accidental and systematic errors which have affected the work.

The adjustment does not yield true values, for each measured angle is subject to accidental errors whose sign and magnitude are unknown. All that can be said about the adjustment as regards the accidental errors, is that it meets the known geometrical conditions, and to that extent the adjusted values are the most probable values that can be assigned.

245. Rules for Balancing a Survey.—Two rules are commonly used to balance a traverse, namely the "Compass Rule" and the "Transit Rule." The Compass Rule states that the correction to be applied to the $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right\}$ of any course is to the total error in $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right\}$ as the length of the course is to the length of the traverse.

The Transit Rule states that the correction to be applied to the $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right\}$ of any course is to the total error in $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right\}$ as the $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right\}$ of that course is to the arithmetical sum of all the $\left\{\begin{matrix}\text{latitudes}\\ \text{departures}\end{matrix}\right\}$ in the traverse.

The Compass Rule is based on the assumptions: (1) that the errors in traversing are accidental in nature and therefore vary with the square root of the lengths of the sides, thus making the corrections to each side proportional to its length (Art. 51); and (2) that the effects of the errors in angular measurements are consistent with the effects of errors in chaining.

The Transit Rule is based on the assumptions: (1) that the errors in traversing are accidental in nature; and (2) that the angular measurements are more precise than those of chaining.

As an example of the effects of the two rules the computations are given below for a particular case.

Example: Below are tabulated the lengths, bearings, latitudes, and departures for a closed traverse of six sides. The survey is to be balanced by the Compass Rule and the Transit Rule.

Line	Length, feet	Bearing	Latitudes		Departures	
			N	S	E	W
AB	500.0	N	500.0	....	0.0	
BC	848.6	N45°00′E	600.0	....	600.0	
CD	854.4	S69°27′E	....	300.0	800.0	
DE	1,019.8	S11°19′E	....	1,000.0	200.0	
EF	1,118.0	S79°42′W	....	200.0	....	1,100.0
FA	656.8	N54°06′W	385.0	....	....	532.0
Sum	4,997.6	....	1,485.0	1,500.0	1,600.0	1,632.0

The error of closure in latitude is 15.0 ft. and in departure is 32.0 ft. The sum of the lengths of the sides is 4,997.9 ft. or practically 5,000 ft.

By the Compass Rule the correction in latitude for the course *AB* is $\frac{15}{5{,}000} \times 500 = 1.5$ ft. The north latitudes are too small and hence the correction is to be added, *i.e.*, the correction is +1.5 ft. The correction to the departure of *AB* is $\frac{-32}{5{,}000} \times 500 = -3.2$ ft. By the Compass Rule the correction to the latitude of *CD* is $\frac{-15}{5{,}000} \times 854.4 = -2.6$ ft., and the correction to the departure of *CD* is $\frac{+32}{5{,}000} \times 854.4 = +5.4$ ft.

Using the Transit Rule, the arithmetical sum of the latitudes is 2,985.0 ft. and the arithmetical sum of the departures is 3,232.0 ft. The correction to the latitude of *CD* is $\frac{-15}{2{,}985} \times 300 = -1.5$ ft., and the correction in departure is $\frac{+32}{3{,}232} \times 800 = +8.0$ ft.

Below are tabulated the corrections in feet as determined by the two rules, for the several courses in the traverse. In the last line are shown the sums of the values in the several columns, which should equal the error in latitude or departure as the case may be. This is a check on the correctness of the computations, and when the corrections have been applied, the algebraic sum of the latitudes and departures must be zero, and hence the survey is balanced.

Line	Correction in latitude		Correction in departure	
	Compass	Transit	Compass	Transit
AB	+ 1.5	+ 2.5	− 3.2	0
BC	+ 2.6	+ 3.0	+ 5.4	+ 6.0
CD	− 2.6	− 1.5	+ 5.4	+ 8.0
DE	− 3.1	− 5.0	+ 6.6	+ 2.0
EF	− 3.3	− 1.0	− 7.2	−10.8
FA	+ 1.9	+ 2.0	− 4.2	− 5.2
Sum	15.0	15.0	32.0	32.0

246. Omitted Measurements.—When for any reason it is impossible or impracticable to determine by field observations the length and bearing of every side of a closed traverse, the missing data may generally be computed provided not more than two quantities are omitted. When the missing quantities have been supplied, the coordinates may be computed and the traverse may be plotted (or the area may be calculated) as though all field measurements had been taken. In the process of calculating the unknown quantities it must be assumed that the observed values are without error, and hence all errors of measurement are thrown into the computed lengths or bearings. Measurements which may be supplied in this manner are:

1. Length and bearing of one side.
2. Length of one side and bearing of another.
3. Lengths of two sides for which the bearings have been observed.
4. Bearings of two sides for which the lengths have been observed.

247. Length and Bearing of One Side Unknown.—The problem of determining the length and bearing of one side of a closed traverse is exactly the same as that of computing the length and bearing of the side of error in any closed traverse for which field measurements are complete. The latitudes and departures of the known sides are computed. For reasons explained in Art. 242, if the sum of the latitudes and the sum of the departures of the known sides are respectively designated by ΣL and ΣD, then the length of the unknown side is

$$S = \sqrt{(\Sigma L)^2 + (\Sigma D)^2}$$

and the tangent of the bearing angle is

$$\tan \alpha = \frac{(-\Sigma D)}{(-\Sigma L)}$$

248. Length of One Side and Bearing of Another Side Unknown. Figure 248a represents a closed traverse for which the direction of the line $DE = d$ and the length of the line $EA = e$ are not determined by field measurements, but for which lengths and directions are measured for the lines $AB = a$, $BC = b$, and $CD = c$. Let an imaginary line extend from D to A, cutting off the unknown sides from the remainder of the traverse. Then $ABCDA$ forms a closed traverse for which the side $DA = f$ is unknown in both direction and length. By the method of the preceding article

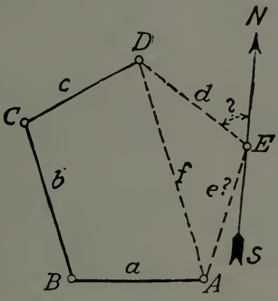


FIG. 248a.

$$\text{length of } f = \sqrt{(\text{Lat. } a + b + c)^2 + (\text{Dep. } a + b + c)^2}$$

and

$$\text{tan bearing angle of } f = \frac{\text{Dep. } f}{\text{Lat. } f}$$

The direction of f having been computed, and the direction of e being known, the angle between the lines e and f in $\triangle DEA$ is

$$\angle DAE = \text{azimuth of } AE - \text{azimuth of } AD$$

In the triangle ADE the length of the two sides d and f and one angle DAE are known. By the relation that sines of angles are proportional to sides opposite

$$\sin DEA = \sin DAE \cdot \frac{f}{d}$$

With angle DEA known, angle ADE can, of course, be computed, and the remaining unknown length is given by the expression

$$e = f\frac{\sin ADE}{\sin DEA} = d\frac{\sin ADE}{\sin DAE}$$

Also the azimuth of DE = azimuth of $DA - \angle ADE$.

The unknown length EA and the unknown bearing of DE having thus been determined, the latitudes and departures of these lines can be computed, and the numerical calculations involved in the solution of the triangle ADE will be verified if the algebraic sum of the latitudes and departures of the sides DE and EA are equal to the algebraic sum of the latitudes and departures of the lines AB, BC, and CD.

249. In general, for any closed traverse when two sides which are partially unknown are not adjacent one to the other, they are treated as though they were adjacent. By means of the latitudes and departures of the known sides, the length and direction of the side necessary to close the figure formed by the known sides is computed, after which the solution is identical with that described in the preceding article. To simplify the problem the data are usually plotted roughly to small scale. Following is the solution of a typical problem.

Example: Below are tabulated the measured lengths and bearings for the courses of a closed traverse *a* to *f*, together with the latitudes and departures of the known sides. The length of *b* and the bearing of *e* are not observed. The general direction of *e* is southwest. It is desired to compute the unknown length and direction. Quantities in parentheses are derived from following calculations.

Line	Length, feet	Bearing	Latitudes		Departures	
			N	S	E	W
a	500.0	N	500.0	....	0.0	
b	(889.9) unknown	N45°00′E	(629.3)	....	(629.3)	
c	854.4	S69°27′E	......	300.0	800.0	
d	1,019.8	S11°19′E	......	1,000.0	200.0	
e	1,118.0	(S78°57′W) unknown	......	(214.3)	......	(1,097.3)
f	656.8	N54°06′W	385.0	....	......	532.0
g	(625.5)	(N48°26′W)	(415.0)	....	......	(468.0)

In Fig. 249 the lines *a*, *c*, *d*, and *f* are the courses for which the length and bearing are known. The line *g* is the closing side of the figure formed by the known courses. From the tabulated quantities, the sum of the known latitudes is 1,300S + 885N = 415S, hence the latitude of *g* is 415N. Similarly the departure of *g* is 1,000 − 532 = 468W.

$$\tan \text{bearing } g = \frac{468.0}{415.0} = 1.127$$

and the bearing of *g* = N48°26′W.

The length of $g = \sqrt{(415)^2 + (468)^2} = 625.5$ ft.

Since the bearing of *b* is N45°00′E,

$$\angle \text{E} = 45°00' + 48°26' = 93°26'$$

$$\sin G = \frac{g}{e} \sin E = \frac{625.5}{1{,}118} \times 0.9982 = 0.5585$$

$$\angle G = 33°57', \ \angle B = 180°00' - 93°26' - 33°57' = 52°37'$$

$$\text{length } b = \text{length } e \cdot \frac{\sin B}{\sin E} = 1{,}118 \times \frac{0.7946}{0.9982} = 889.9 \text{ ft.}$$

$$\text{Bearing of } e = 180° - 48°26' - 52°37' = \text{S}78°57'\text{W.}$$

As a check on the calculations the latitudes and departures of the lines *b* and *e* are computed; the values are shown in parentheses. The sum of the latitudes and the sum of the departures for courses *a*, *b*, *c*, *d*, *e*, and *f* are found to be zero, and hence the computations for determining the unknown length and bearing are correct.

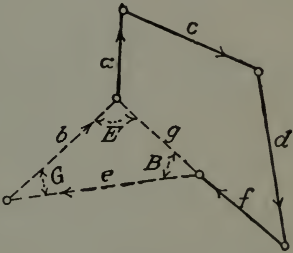


Fig. 249.

250. When the length of one side and the bearing of another are unknown, the solution described in the preceding article will in general render two values of each of the unknowns, and it is often impossible to tell which are the correct values unless the general direction of the side of unknown bearing is observed.

As the angle between the line of unknown length and the line of unknown bearing approaches 90°, the solution becomes weak. Also when the angle between the partially unknown lines is nearly 90°, it is impossible to distinguish which of the two determinations is the correct one, even though the general direction of the line of unknown bearing has been observed. When the partially unknown lines are perpendicular to each other, the problem is indeterminate; and the solution becomes strong as the angle between these lines becomes small.

251. Lengths of Two Sides Unknown.—This problem commonly occurs when angular observations are taken from two or more points in the main traverse to some landmark, the measurements being introduced as a check. It occasionally occurs on main traverse lines when there are obstacles to the direct measurement of length but angles are observed. The solution is so nearly identical with that for the case where the direction of one side and the length of another are unknown (Art. 248) that a detailed description will not be given here.

In Fig. 251, *ABCD* represents the portion of a closed traverse for which the courses are known in both direction and length, and the lines *DE* and *EA* are courses for which the directions are observed but the lengths are unknown. From the latitudes and departures of the known sides, the length and bearing of the closing line *DA* is computed and the angles *A*, *D*, and *E* in the triangle *ADE* are calculated from the known directions of the sides. *DE* and *EA* are determined through the relation

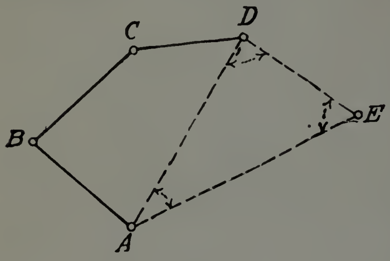


Fig. 251.

$$\frac{DE}{\sin A} = \frac{EA}{\sin D} = \frac{DA}{\sin E}$$

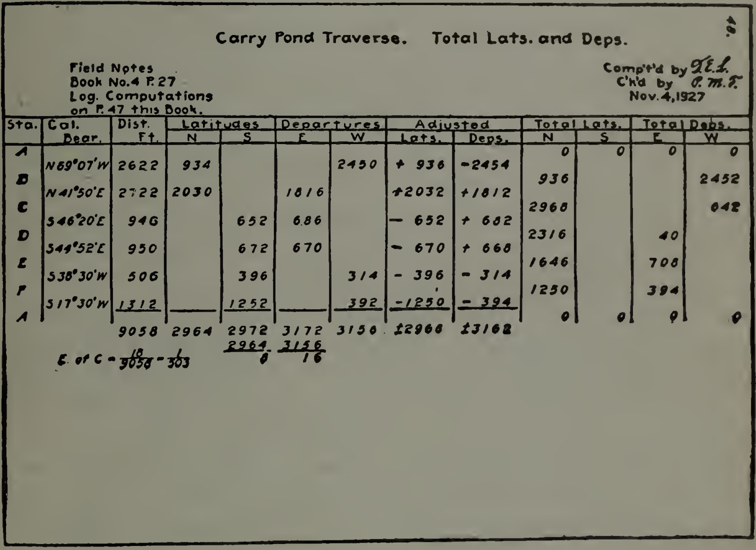
If the two lines are not adjacent to one another, the problem may be solved as though they were, as explained in Art. 249. If the partially unknown lines are parallel, the problem is indeterminate. As the angle between the lines approaches zero or 180°, a small error in angle produces a relatively large error in computed length; that is, the solution becomes weak. The solution is strongest when the angle between the partially unknown lines is 90°.

252. Directions of Two Sides Unknown.—The solution for this case is similar to that described in the preceding article. In all cases the general direction of at least one of the partially unknown lines must be observed, since the values of the trigonometric functions merely determine the shape of the triangle but do not fix its position.

When the partially unknown lines are parallel but not of the same length, their direction is that of the closing side of the figure formed by the known courses. When the partially unknown sides are parallel and of the same length the problem is indeterminate, since the length of the closing side of the figure formed by the known courses becomes a point.

253. Calculation of Coordinates.—When it is desired to plot points of horizontal control by the method of coordinates, the latitudes and departures of the control lines are computed and in the case of the closed traverse the survey is balanced. The origin is chosen, and the total latitudes and departures or coordinates of the several points in the survey are computed by summing the latitudes and departures of lines between that point and the origin.

The total $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right\}$ of any point equals the algebraic sum of $\left\{\begin{matrix}\text{latitudes}\\ \text{departures}\end{matrix}\right\}$ of lines lying between that point and the $\left\{\begin{matrix}\text{reference}\\ \text{reference}\end{matrix}\right.$ $\left.\begin{matrix}\text{parallel}\\ \text{meridian}\end{matrix}\right\}$ passing through the origin. $\left\{\begin{matrix}\text{North latitudes}\\ \text{East departures}\end{matrix}\right\}$ are considered positive. $\left\{\begin{matrix}\text{South latitudes}\\ \text{West departures}\end{matrix}\right\}$ are considered negative. $\left\{\begin{matrix}\text{Total latitudes}\\ \text{Total departures}\end{matrix}\right\}$ are positive or negative according to whether the corresponding points lie $\left\{\begin{matrix}\text{north or south of the reference parallel}\\ \text{east or west of the reference meridian}\end{matrix}\right\}$.



Carry Pond Traverse. Total Lats. and Deps.

Field Notes
Book No.4 P. 27
Log. Computations
on P. 47 this Book.

Comp't'd by J.E.L.
C'h'd by O.M.T.
Nov. 4, 1927

Sta.	Cal. Bear.	Dist. Ft.	Latitudes		Departures		Adjusted		Total Lats.		Total Deps.	
			N	S	E	W	Lats.	Deps.	N	S	E	W
A									0	0	0	0
	N69°07'W	2622	934			2450	+ 936	−2454				
B									936			2452
	N41°50'E	2722	2030		1816		+2032	+1812				
C									2968			642
	S46°20'E	946		652	686		− 652	+ 682				
D									2316		40	
	S44°52'E	950		672	670		− 670	+ 668				
E									1646		708	
	S38°30'W	506		396		314	− 396	− 314				
F									1250		394	
	S17°30'W	1312		1252		392	−1250	− 394				
A									0	0	0	0
		9058	2964	2972	3172	3156	±2968	±3162				
				2964	3156							
				8	16							

E. of C $= \frac{18}{9058} = \frac{1}{503}$

Fig. 253.—Calculation of coordinates.

Figure 253 is an example of the tabulations ordinarily made for a closed traverse. Station *A* is seen to be the origin. The adjusted latitudes and departures are used in calculating the coordinates. In a closed traverse the additions are verified if on making the complete circuit the total latitude and total departure of the initial point check these quantities, as assumed at the beginning. In the tabulations of Fig. 253 it will be noted that this check has been performed.

254. Use of Coordinates.—In surveying, rectangular coordinates are employed not only for plotting maps as described in the following article, but are also frequently used for calculating areas as is described in Art. 357.

As has before been stated, when angles are measured to a given point from a number of transit stations on a continuous traverse, these observations provide a means of checking the traverse. Any two radiating lines, together with the intercepted portion of the main traverse, form a closed figure. The unknown lengths may be computed as described in Art. 249. With the lengths and directions of the radial lines known, their latitudes and departures may be calculated and the coordinates of the given point *O* may be determined by as many independent solutions as there are triangles.

If the field work is without error the computed length of a given side will be the same by one calculation as by another. Also the coordinates of a point should be the same when based upon one solution as when based upon another.

Of course, in any actual traverse errors of greater or less magnitude occur in all measurements. Hence no two of the several determinations of the coordinates of a given point will exactly agree, but the closeness of agreement will serve as an index to the precision with which both angular and linear measurements are taken.

255. Plotting by Coordinates.—When a system of horizontal control is to be plotted by rectangular coordinates, the size of the enclosing rectangle is determined from the total latitudes and departures; this rectangle being one whose east and west sides are meridians passing through the most easterly and westerly points of the survey and whose north and south sides are parallels passing through the most northerly and southerly points of the survey. On the drawing paper the enclosing rectangle is drawn to the required scale. Let *HJKL* (Fig. 255) be the enclosing rectangle for the traverse of which the computations are tabulated in Fig. 253. The positions of the reference meridian and reference parallel are determined by scaling along the sides of the enclosing rectangle. For example, in Fig. 255, the reference meridian is found by scaling *JA* and *HM* equal to 2,452 ft., the total departure of the most westerly point of the survey. It is checked by scaling *ML* and *AK* equal to 708 ft., the total latitude of the most easterly point in the survey.

If the drawing is of large size and there are a considerable number of control points to be plotted, the work will be facilitated if other meridians and parallels are constructed so that the area is divided into squares the sides of which are less than the length of the scale used and represent some whole number of hundreds or thousands of feet at the given scale. Thus in the figure, meridians are drawn at intervals of 1,000 ft. west of the reference meridian, and parallels at intervals of 1,000 ft. north of the reference parallel. If the scale of the drawing is 100 ft. = 1 in., the actual length of the sides of the resulting squares is 10 in. Each of the lines thus drawn is numbered with its distance from the reference meridian or reference parallel.

Each point of horizontal control is located on the sheet by plotting its total latitude and departure, and its position is verified by scaling the length of the preceding course. For example, with the origin at *A* (Fig. 255) *F* has a total N latitude of 1,250 ft. and a total E departure of 394 ft. The point *F* is plotted by laying off above parallel 1,000N the distances *ab* and *cd* each equal to 1,250 − 1,000 = 250 ft., then drawing the line *bd*, and finally by laying off to scale the distance *bF* = 394 ft. As a check, the distance *EF* as scaled from the drawing should, within the scale of the map, agree with the length of this line as computed from the adjusted latitude and departure (*EF* = 506 ft. by notes of Fig. 253).

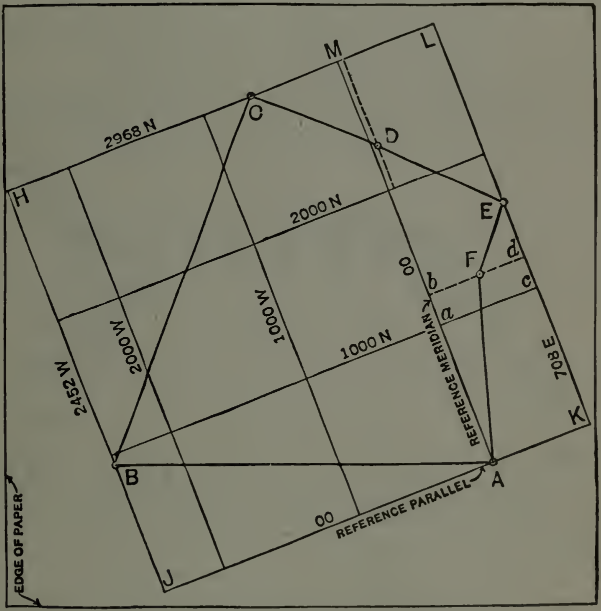


Fig. 255.—Plotting by coordinates.

256. The coordinate method is recognized as the most reliable method of plotting. Its single disadvantage as compared with the method of tangents or the method of chords is the increased amount of computation required preliminary to plotting. Often in the case of the closed traverse, latitudes and departures are necessarily computed for the purpose of determining the area, and with these values determined, the labor of computing coordinates is little more than that of computing tangent offsets or chord distances. The principal advantages of the coordinate method are: (1) that the size and shape of the drawing can be accurately determined beforehand; (2) that the accuracy of the location of any point does not depend upon the accuracy with which previous lines in the control system are plotted; (3) that the method of checking is simple and unlike the method of plotting; and (4) that for closed traverses the field measurements are checked and the survey is balanced before plotting is begun.

257. Plotting Details.—In general, the processes of plotting details on the map are not unlike those employed in making the field observations. The work of plotting details is less refined than that of plotting the horizontal control, the aim generally being to plot objects of definite shape and size so that their plotted positions shall be correct to the extent that subsequently scaled dimensions taken from the map will be correct within the allowable limit of error.

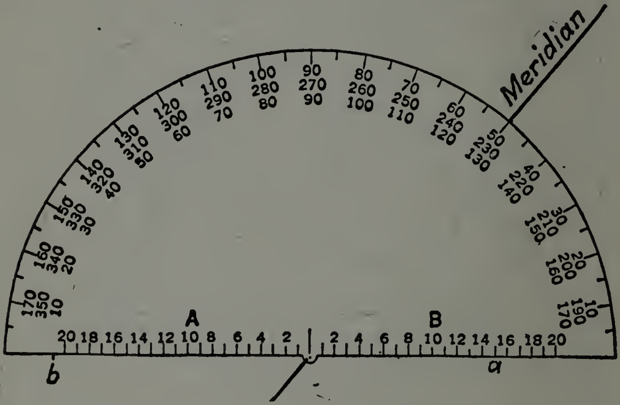

Meridian
A
B
b
a
20 18 16 14 12 10 8 6 4 2 2 4 6 8 10 12 14 16 18 20
10 170 190
20 160 200
30 150 210
40 140 220
50 130 230
60 120 240
70 110 250
80 100 260
90 90 270
100 80 280
110 70 290
120 60 300
130 50 310
140 40 320
150 30 330
160 20 340
170 10 350

Fig. 257.—Protractor for plotting details.

Angles are laid off with a protractor of which several varieties were described in Art. 40. If any considerable number of points are to be plotted from each control station, it is desirable that the protractor be of a type adapted to laying off both angle and distance in one operation. The protractor shown in Fig. 257 may perhaps be used more rapidly than any other type. The linear scale is so designed that distances are measured outward from the center along the diametrical edge. The angle numbers are so arranged that azimuths, bearings, or deflection angles may be laid off by rotating the protractor about the station as a center until the graduation on the protractor representing the given angle coincides with the reference line. The point is then plotted at the given distance on the diametrical scale. When the directions to details are given in azimuths or bearings measured from a given station, a reference meridian is drawn through the station and angles are laid off from this meridian. In plotting a detail of given azimuth (measured from north) the protractor is rotated until the graduation corresponding to the given azimuth coincides with a point on the meridian south of the station, and the point representing the detail is plotted at the given distance on scale *A* or *B* according as the azimuth is less or greater than 180°. Thus in the figure, the point *a* is plotted at an azimuth of 50° (measured from north) and *b* is plotted at an azimuth of 230°.

The location of all important details should be checked by actual map measurements, but the position of less important objects is checked simply by inspecting the drawing. It often happens that mistakes in the field work give an object more than one location, or that a part of the field notes is confusing. In such case it is best to proceed with the plotting of other portions, for these when mapped may help in clearing up the doubtful points.

258. Conventional Signs.—Objects are represented on a map by signs or symbols, many of which have such widely accepted usage that they may be said to be conventional in character. Some of these are shown in Figs. 258*a* to 258*d*. Of course, the kind or purpose of the map defines what objects shall be represented. For many purposes, even if the necessary data were available, it would partially destroy the usefulness of the map if all of the objects were shown for which conventional signs are given in the figures. On the other hand, maps made for special purposes often show features not included among the conventional signs here listed. Appropriate symbols for many such features will be found among the surveying and mapping publications of the various national and state surveys. Where the symbols are not in general use, a key to their meaning should be shown on the map sheet. The size of the symbols should be proportioned somewhat to the scale of the map.

Where the map is on tracing cloth from which prints are to be made, all features are ordinarily inked in black. When the map is made on paper, lakes, rivers, and other hydrographic features (Fig. 258*d*) are usually shown in blue. Often the conventional signs referring to the culture of the land are inked in green. When lines of horizontal control are left on the sheet, they are usually inked in black.

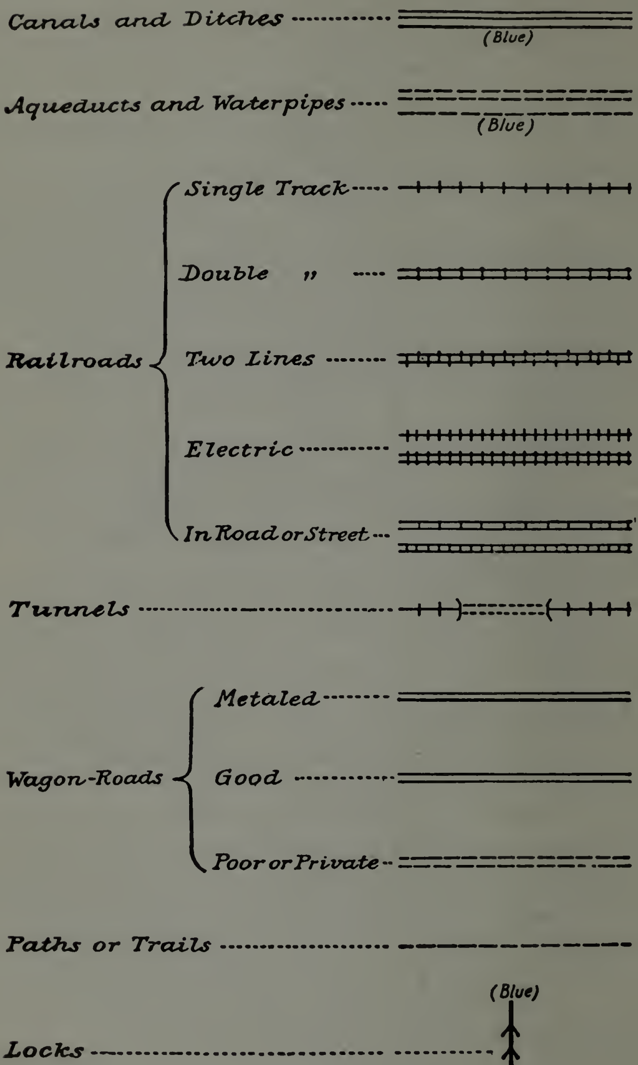

Canals and Ditches
(Blue)
Aqueducts and Waterpipes
(Blue)
Railroads
Single Track
Double "
Two Lines
Electric
In Road or Street
Tunnels
Wagon-Roads
Metaled
Good
Poor or Private
Paths or Trails
(Blue)
Locks

Fig. 258*a*.—Conventional signs; public and private works.

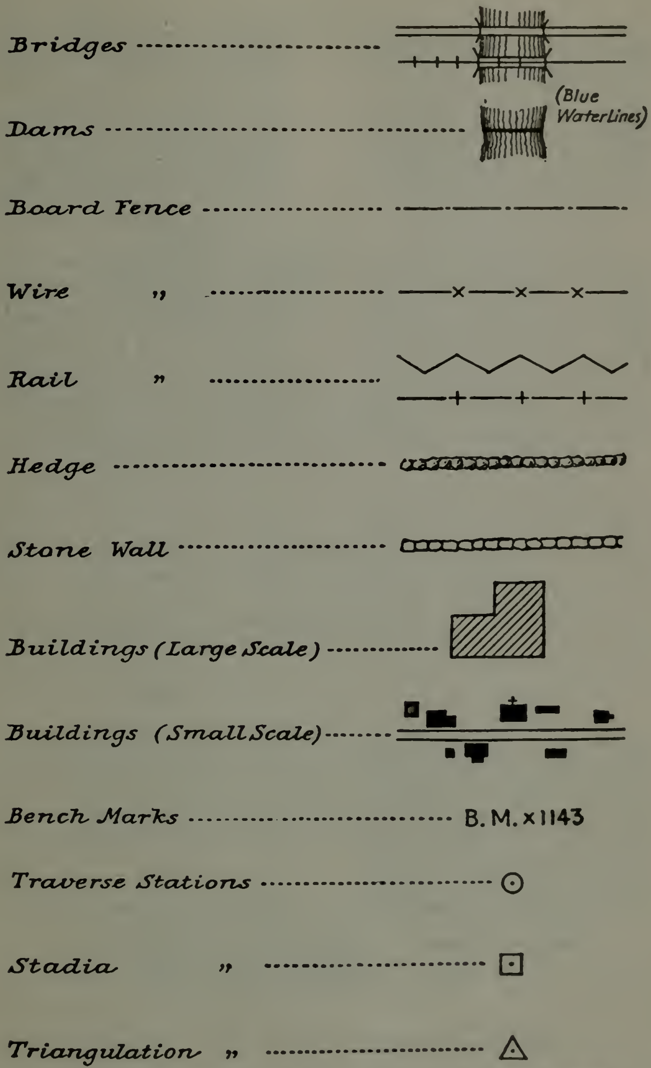
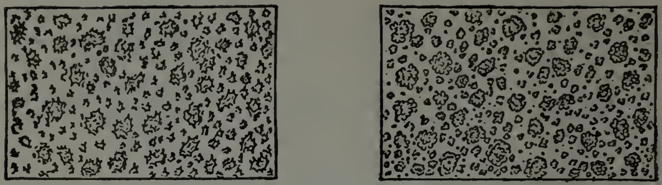
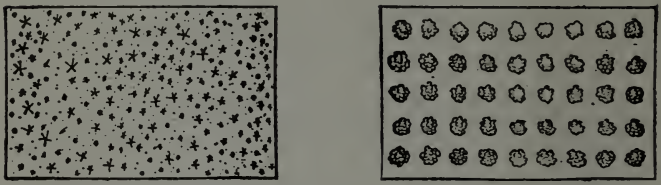

Bridges
Dams
(Blue Water Lines)
Board Fence
Wire „
Rail „
Hedge
Stone Wall
Buildings (Large Scale)
Buildings (Small Scale)
Bench Marks
B. M. x 1143
Traverse Stations
Stadia „
Triangulation „

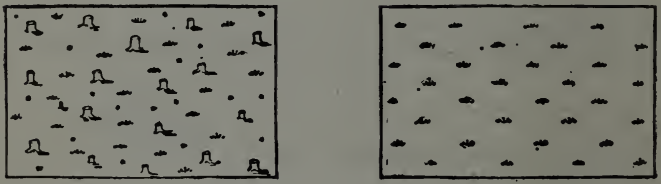
FIG. 258*b*.—Conventional signs; public and private works.



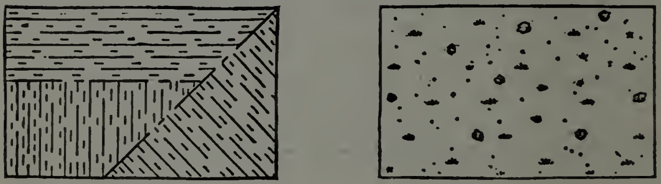
Deciduous Trees (Oak) Deciduous Trees (Round Leaf)



Evergreen Trees Orchard



Clearing Grass



Cultivated Land Pasture

Fig. 258c.—Conventional signs; culture.

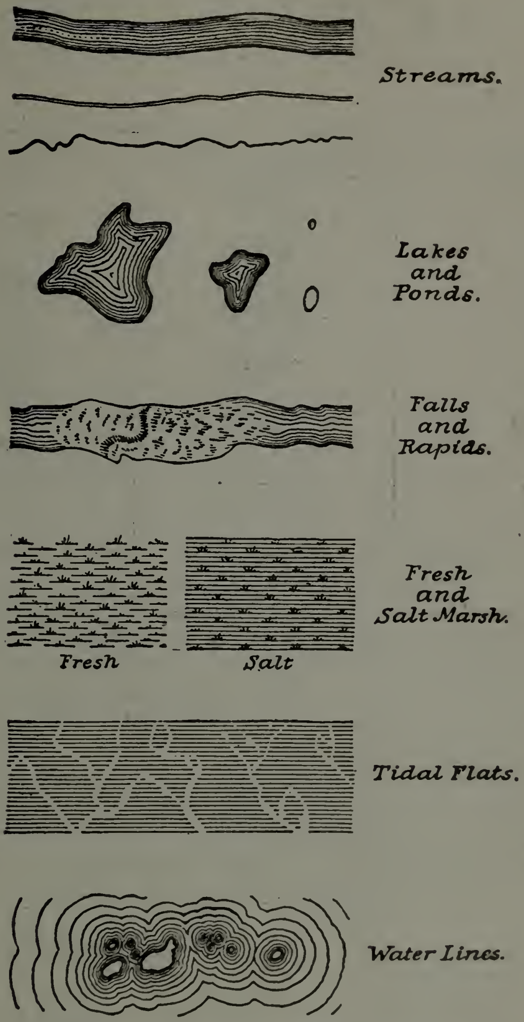

Streams.
Lakes and Ponds.
Falls and Rapids.
Fresh and Salt Marsh.
Fresh
Salt
Tidal Flats.
Water Lines.

Fig. 258*d*.—Conventional signs: hydrography.

259. Problems:

1. With the protractor plot to the scale of 1 in. = 200 ft., the closed deflection angle traverse for which notes are given below. Measure the error of closure and record it on the sheet. Distribute the error as suggested in Art. 236.

Station	Deflection angle	Length, feet
A	92°00′R	
		338
B	9°00′L	
		307
C	76°45′R	
		792
D	74°45′R	
		822
E	102°00′R	
		624
F	23°30′R	
		620
A		

2. Assuming the course *AB* in the traverse of problem 1 to be of zero azimuth, compute the azimuths of the remaining courses. Using a protractor, lay off the azimuths from a central meridian and plot the traverse at the scale of 1 in. = 200 ft., as described in Art. 232. Measure the error of closure and record it on the sheet. Distribute the error of closure as suggested in Art. 236.

3. Plot the following continuous traverse at the scale of 1 in. = 400 ft. by the method of tangents, using a 10-in. base. Lay off successive lines by deflection angles as described in Art. 234. Assume the direction of the first course to be North, and compute the bearings of other courses. Check the accuracy of the plotting by methods described in Art. 237.

Station	Deflection angle
118 + 75.0	
98 + 95.6	39°47′L
73 + 01.4	17°28′L
70 + 13.5	14°08′L
49 + 41.3	3°11′L
40 + 00	49°59′L
37 + 18.8	32°18′R
18 + 26.0	18°44′R
5 + 03.2	7°31′L
0 + 00	

4. Plot the following continuous azimuth traverse at the scale of 1 in. = 400 ft. by the method of chords, establishing the directions of the several lines from a centrally located meridian. From the azimuths calculate the deflection angles at the several points, and check the accuracy of the plotting as described in Art. 237.

Course	Azimuth	Distance, feet
AB	142°08′	815.3
BC	181°37′	1,146.0
CD	296°13′	520.8
DE	323°46′	816.5
EF	249°51′	726.4
FG	214°03′	1,862.0
GH	195°45′	2,795.5
HJ	191°28′	2,463.7
JK	138°42′	586.4

5. Given the notes of problem 1. Assuming the direction of *AB* to be North, calculate the bearings of the several courses. Compute the latitudes and departures, using four-place logarithms. Compute the error of closure and balance the survey by the compass rule (Art. 245).

6. Given the following notes of a survey with the transit. Calculate the latitudes and departures, using five-place logarithms. Compute the error of closure, and balance the survey by the Transit Rule (Art. 245).

Course	Bearing	Distance, feet
AB	N48°20′E	529.6
BC	N87°43′E	592.0
CD	S 7°59′E	563.6
DE	S82°12′W	753.4
EA	N48°12′W	428.2

7. Given the following notes for a closed traverse. (1) Calculate the latitudes and departures, using five-place logarithms, and compute the error of closure. Balance the survey by each of the two rules given in this text, and for each course find the change in length and in direction caused by the application of each of the rules.

Course	Azimuth	Distance, feet
AB	0°42′	1,221.2
BC	94°03′	541.3
CD	183°04′	795.4
DA	232°51′	646.8

(2) Add 40°00′ to each of the given azimuths, and make computations called for in (1). (3) Compare the results of (2) with those of (1), and explain reasons for variations.

8. Given the following data for a closed traverse. By the methods of Art. 247 compute the length and bearing of the unknown side. Make calculations with the slide rule.

Course	Bearing	Distance, feet
AB	N82°W	461
BC	unknown	unknown
CD	N68°15′E	829
DA	N80°45′E	441

9. Given the following data for a closed traverse, the length of *DE* and the azimuth of *EA* not being observed in the field. Determine the unknown quantities. The direction of *EA* is east of north, and zero azimuth is north.

Course	Azimuth	Distance, feet
AB	106°13′	1,081.3
BC	195°14′	1,589.5
CD	247°07′	1,293.7
DE	332°22′	unknown
EA	unknown	1,737.9

10. Given the following data for a closed traverse, the lengths of the courses *BC* and *DE* not having been measured in the field. Compute the unknown lengths, using five-place logarithms.

Course	Bearing	Distance, feet
AB	N9°30′W	689.32
BC	N56°55′W	unknown
CD	S56°13′W	678.68
DE	S2°02′E	unknown
EA	S89°31′E	1,082.71

11. Given the data of problem 4. Calculate the coordinates of the several points in the survey, assuming that the origin is at *A*. Plot the traverse by the coordinate method, using a scale of 1 in. = 400 ft.

12. Given a circular curve with total intersection angle $I = 40°00'$; $D = 10°00'$ (see Art. 412). Choose an initial tangent and P.C. Plot

this curve to a scale of 1 in. = 40 ft., using 50-ft. chords. Plot the angles by the method of tangents, laying off deflection angles from preceding chords produced. When the direction of the tangent at P. T. has been determined, it should be projected back to fix the position of the P. I. To check the work, scale the following quantities and compare with the computed values: radius, tangent distance, long chord, external distance, and mid-ordinate. All values should check within 2 ft.

CHAPTER XV

PRINCIPLES OF FIELD ASTRONOMY

260. Remarks.—The surveyor is directly interested in the science of astronomy in that it offers him a means of determining the absolute position of any point or the absolute position and direction of any line on the surface of the earth. The absolute position of a point is given by its longitude and latitude, and the absolute direction of a line is defined by the bearing which the line makes with the true meridian. The requirements of plane surveying, in general, will be met if sufficient data be available so that the true azimuth or bearing of the survey lines may be established with a degree of precision at least equal to that with which the relative direction of survey lines is determined by direct angular measurement. The azimuth of a line is established by angular observations to some celestial body. Incidental to the work of computing the azimuth of the line to the celestial body at the instant of sighting, the latitude of the place must be known. Also for certain observations it is essential that the longitude be roughly determined in order that the position of the star at a given instant may be computed with some accuracy. If the survey is through a territory for which there is a reliable map, latitude and longitude may ordinarily be determined with sufficient precision by scaling from the map, but for surveys in more remote regions, it is usually necessary to determine roughly at least the latitude.

261. The Celestial Sphere.—In making observations of the sun and stars, the surveyor is not interested in the distance of these heavenly bodies from the earth, but is interested merely in their angular position. In defining their position it is convenient to imagine that they are attached to the inside of a hollow sphere of infinite radius of which the earth is the center. This imaginary globe is called the *celestial sphere.* It is a helpful conception to imagine the earth as being fixed, and to consider the celestial sphere as rotating from east to west, its axis being the prolongation of that of the earth. The portion of the celestial sphere seen by the observer is the hemisphere above the plane of his own horizon.[1]

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[1] More properly speaking, the plane passes through the center of the earth parallel with the observer's horizon plane, but the radius of the

A vertical line at the position of the observer coincides with the plumb-line and is normal to the observer's horizon plane. The point where this vertical pierces the celestial sphere above the head of the observer is called the *zenith* and the corresponding point in the opposite hemisphere, directly below the observer, is called the *nadir*.

The *celestial equator* is the great circle formed by the intersection of the earth's equatorial plane with the surface of the celestial sphere.

The *celestial poles* are the points where the earth's axis prolonged pierces the celestial sphere.

Figure 261*a* represents the celestial sphere, the point O being the earth and $NES'W$ being the horizon of an observer, with letters standing for the points of the compass. Figure 261*b* may be taken as an enlarged view of the earth in the same position as that assumed in Fig. 261*a*. A is an observer in the Northern Hemisphere, the line NS being in his horizon plane and corresponding to NS' in Fig. 261*a*. Evidently he views everything above the horizon plane or that portion of the celestial sphere (Fig. 261*a*) which is shown by full lines. B is an observer in the Southern Hemisphere, $N'S'$ being in his horizon. Since the size of the earth is negligible as compared with that of the celestial sphere, it may be considered that $N'S'$ in Fig. 261*b* coincides

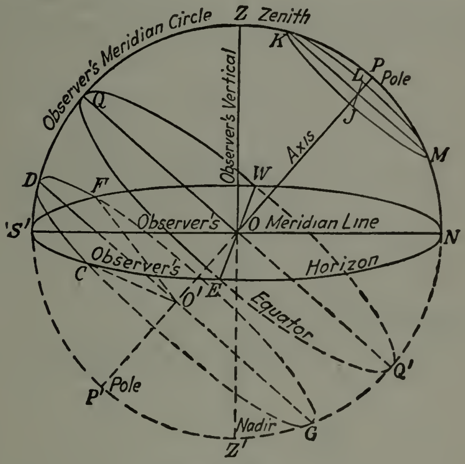

FIG. 261*a*.

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earth is so small in comparison with the distance to any of the celestial bodies here considered that the error introduced is negligible for rough observations. In case of the sun the error produced by this assumption is much larger than for any of the stars, amounting under certain conditions to about nine seconds of arc.

with NS' in Fig. 261*a*. Therefore if B is at a point on the earth diametrically opposite A, the portion of the celestial sphere which he views above his horizon plane will be the opposite hemisphere to that seen by A, or that portion of Fig. 261*a* which is shown by dotted lines.

Assuming the observer to be in the Northern Hemisphere (Fig. 261*a*), Z is the zenith and if P and P' mark the celestial poles, P is the *visible* or *elevated pole,* and $EQWQ'$ is the *celestial equator,* of which the portion EQW is visible to the observer.

Obviously no two points on the earth have the same celestial horizon, except they be diametrically opposite; in that case, the celestial hemisphere in view from one point will be opposite to that viewed from the other. Since we are, for the sake of simplicity, assuming that the celestial sphere is rotating and the earth remains stationary, N, E, S', W, and Z are regarded as fixed points with respect to any given station on the surface of the earth. If $S'N$ is a meridian line in the plane of the horizon passing through the station of the observer, then a vertical plane of which this line is an element cuts the celestial sphere in the great circle $S'ZPNZ'P'$, which is called the meridian circle or more often, simply the meridian. Since the meridian passes through the fixed points, it is, for any station, a fixed circle, but at a given instant the meridian for one station does not occupy the same position in the celestial sphere as does the meridian for another station, except the two stations be at the same longitude.

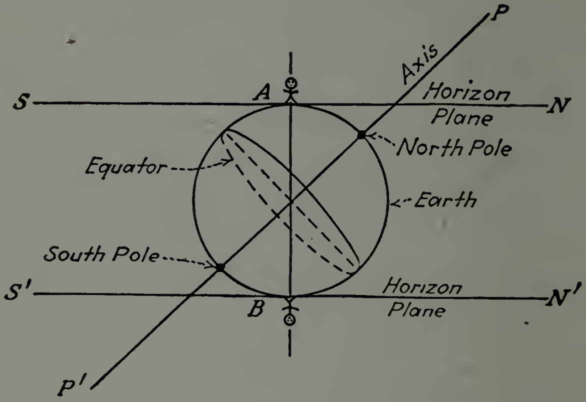


FIG. 261*b*.

Any celestial body which is on the celestial equator will first become visible at E; will pass over the meridian at Q, and will disappear from view at W, and the interval of time when the body is above the horizon will be the same as the interval when it is below, or the angle whose arc is EQW is 180°.

Any star which is below or south of the equator will follow some path as $CDFG$. It will become visible at C, will pass over the meridian at D, and will disappear from view at F. It will be above the horizon for a less length of time than it will be below, or the angle whose arc is CDF = angle $CO'F$ is less than 180°. From the figure it is evident that if any star is sufficiently far below the equator it will never appear above the observer's horizon.

Similarly, any star which is above or north of the equator will be above the horizon for a greater length of time than it is below. If it is far enough above the equator, it will be continually visible to an observer in a northern latitude and will, during the course of a single revolution of the celestial sphere, follow some path as $JKLMJ$.

262. Observer's Position; Latitude and Longitude.—The position of any point on the surface of a sphere may be fixed by angular measurement from two planes of reference at right angles to one another passing through the center of the sphere; these measurements are called the spherical coordinates of the point. The spherical coordinates of any station on the surface of the earth are designated as the latitude and longitude of the station. Figure 262 represents the earth, PP' being the axis and $QUVQ'$ being the equator. Let S be the station of an observer. Then $PSUP'$ is a meridian circle through the station. Also RSR' is a parallel passing through the station, the plane of RSR' being parallel to that of the equator.

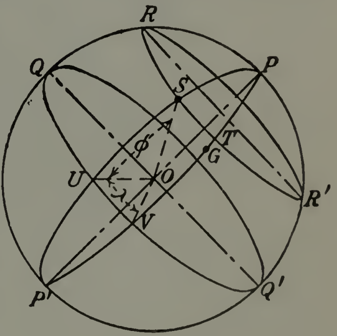


Fig. 262.

The *latitude* of a place may, for all practical purposes, be defined as the angular distance of the place above or below the equator. When the station is above the equator the latitude is north and its sign is positive; when below the equator, the latitude is south and its sign is negative. Hence in the figure the latitude of S is given by the angle ϕ or by the angular distance, measured along any meridian circle, between the equator and the parallel passing through S, such as US, and the latitude is north or positive. The latitude of a place is stated in degrees. Thus the latitude of the equator is 0° and that of the north pole is +90°, or 90°N.

The *longitude* of a place is defined as the angular distance measured along the arc of the equator between a reference meridian and the meridian circle passing through the station. The reference meridian is called the *primary meridian.* The primary meridian most generally used is that of Greenwich, England. Hence in the figure if the point G represents Greenwich, PGP' is the primary meridian, and the longitude of S is given by the angle λ or by the angular distance VU. Longitudes are expressed either in hours of time or in degrees of arc ($15° = 1$ hr.) and are measured east or west of the Greenwich meridian.

263. Position of a Celestial Body; Right Ascension and Declination.—Just as the position of any place on the earth is conveniently fixed by its latitude and longitude, so may the position of any heavenly body be similarly expressed by spherical coordinates referred to the celestial equator and a great circle normal thereto.

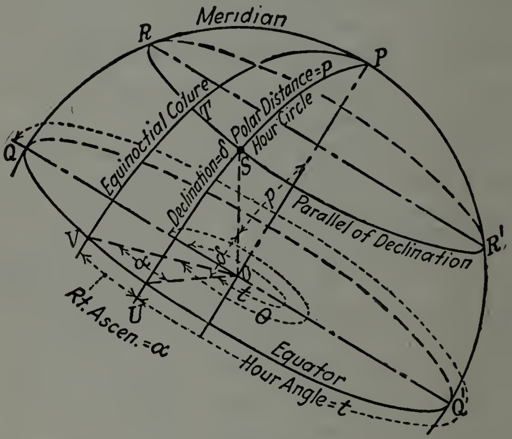


Fig. 263.

In Fig. 263 is shown the celestial sphere in a position similar to that of the earth in Fig. 262, S being a celestial body whose position is to be fixed by spherical coordinates. Comparable with the *meridian circles* or meridians of longitude of the earth are the *hour circles* of the celestial sphere, all of which converge at the celestial poles. The arc PSU is a portion of the hour circle passing through S. Comparable with the *parallels of latitude* of the earth are the *parallels of declination* of the celestial sphere. RSR' is the parallel of declination passing through S. And comparable with the prime meridian through Greenwich, is the *equinoctial colure* of the celestial sphere which passes through the *vernal equinox*, an imaginary point among the stars where the sun in its apparent upward path crosses the equator on March 21 of each year. In the figure, V represents the *vernal equinox* and PTV is the *equinoctial colure.* Comparable with the longitude and latitude of a station on the earth are the *right ascension* and *declination* of a heavenly body.

The *right ascension* of the sun or any star is defined as the angular distance measured along the celestial equator between the vernal equinox and the hour circle through the body. Right ascensions are measured eastward from the vernal equinox and may be expressed either in degrees of arc (from 0° to 360°) or in hours of time (0^h to 24^h). Thus, in the figure, the right ascension of S is given by the angle α in the plane of the equator or by the arc VU.

The *declination* of any celestial body is defined as the angular distance of the body above or below the celestial equator. If the body is above the equator its declination is said to be north and is considered as positive; if it is below the equator its declination is said to be south and it is considered as negative. Declinations are expressed in degrees and cannot exceed 90° in magnitude. Thus in the figure, the declination of S is given by the angle δ or by the arc of any hour circle between the equator and the parallel of declination RSR', such as US. Since S is above the equator, its declination is north and is considered as positive.

The *polar distance* of a star is the complement of the star's declination. In the figure, it is given by the angle $p = 90° - \delta$ or by the arc PS. In defining the position of a star near either pole, the polar distance is often given instead of the declination.

For present purposes it may be considered that the vernal equinox is a fixed point on the celestial equator, just as Greenwich is a fixed point on the earth. But while stations on the earth may be said to maintain practically an unvarying position with respect to the equator and the meridian of Greenwich, the coordinates of celestial bodies with respect to the celestial equator and the equinoctial colure change more or less with the passage of time. The fixed stars, or those outside the solar system, alter their positions in the celestial sphere but slightly from month to month and from year to year, the annual change being less than a minute of arc in either right ascension or declination. Since the earth actually travels about the sun, apparently the sun rapidly changes its position in the celestial sphere, making a complete circuit of the heavens once each year, its apparent path among the stars (called the ecliptic) twice cutting the equator during this interval. Its right ascension therefore changes 24^h (or 360°) during each year, being 0^h at the instant the sun's center apparently crosses the equator on March 21, and being 12^h when the sun in its downward path among the stars again cuts the equator on September 22. The annual change in the declination, caused by the axis of rotation of the earth not being at right angles to the plane of the earth's orbit about the sun, is nearly 47°, the declination varying from about N23½° on June 21 to about S23½° on December 21.

By means of astronomical observations and calculations, the positions of many of the heavenly bodies are predicted, and their right ascensions and declinations for various dates are available in various publications, the one most widely used by astronomers in the United States being the "American Ephemeris and Nautical Almanac," published two or three years in advance for each year by the Nautical Almanac Office, U. S. Naval Observatory. By means of the Ephemeris it becomes a very simple matter to determine for any instant of time the celestial position of any heavenly body therein catalogued

Two other publications which present most of the astronomical data used by the surveyor in more concise and somewhat more convenient form than that of the "American Ephemeris" are the "American Nautical Almanac," also published by the Nautical Almanac Office, and the "Ephemeris of the Sun and Polaris," published by the General Land Office.

264. Hour Angle and Declination.—In many of the problems of field astronomy it is not only necessary that a star's position in the celestial sphere be known, but also that its position with respect to the meridian through a given station on the surface of the earth be determined. In Fig. 263 let $QRPR'Q'$ represent the meridian of some station on the earth, say that of the observer, and let S be some heavenly body, say a star, whose position with respect to the meridian $QRPR'Q'$ and the equator $QQ'UV$ it is desired to establish. The spherical coordinates of the star are given by the angular distance of the star above or below the equator, which in the figure is given by the arc US, defined in the preceding article as the declination, and the angular distance measured along the equator between the meridian and the hour circle through the star. When this angular measurement is from east to west it is called an *hour angle.*

The *hour angle* of any heavenly body may then be defined as the angular distance measured from east to west along the equator from the meridian of reference to the hour circle through the body. Thus, in the figure, the hour angle of S is given by the angle t or by the angular distance $QQ'U$. Hour angles are expressed either in hours of time or in degrees of arc. In the figure the hour angle is more than 12^h or more than 180°. Without qualification, it is understood that an hour angle is measured from the upper branch of the meridian, that is, the branch above the station or above the observer's head, but in connection with the definition of civil time, hour angles are reckoned from the lower branch of the meridian. In the figure, if the hour angle were reckoned from the lower branch, it would be defined by the angular distance $Q'U$, and would be 12^{h} more or less than that given by the arc $QQ'U$, which is the hour angle reckoned from the upper branch. Sometimes the hour angles of stars east of the meridian are reckoned eastward from the upper branch of the meridian, rather than westward. When an hour angle is expressed in this way it is preceded by a minus sign. Thus if the hour angle of S (Fig. 263) were reckoned eastward, it would be given by the angular distance QU.

265. Equator Systems Compared.—The system of coordinates described in the preceding article is seen to be similar to that described in Art. 263, with this difference, that in the system last considered, the angular distance along the equator is measured from a fixed meridian to the hour circle through the star, while in the method described in Art. 263 angular distance along the equator is measured from the vernal equinox, which is a point on the celestial equator that rotates with the celestial sphere. Thus, while right ascensions of fixed stars have annual variations of but a few seconds, hour angles of the stars change as rapidly as the celestial sphere apparently rotates, which is 24^{h} or 360° for each $23^{\mathrm{h}}\ 56^{\mathrm{m}}$ of our civil time, and hour angles of the sun change approximately 24^{h} or 360° for each 24^{h} of our civil time.

The systems described in the two preceding articles are called equatorial systems of coordinates, being so named by reason of the fact that in each case the primary plane of reference is the celestial equator. The position of a heavenly body above or below the equator is given by the declination δ which is the same in one system as in the other. In the first system the angular distance along the equator is the right ascension α and is reckoned from the vernal equinox; in the second system the angular distance along the equator is the hour angle t and is measured from the meridian. Let θ be the hour angle of the vernal equinox represented in Fig. 263 by the angular distance $QQ'V$ measured along the equator. At any instant of time, if the hour angle of the vernal equinox with respect to a given meridian is known and if the right ascension of a heavenly body S is known, then the hour angle of the body may be computed, since by the figure

$$t = \theta - \alpha \text{ or } \theta = t + \alpha.$$

This equation is therefore an expression by means of which the coordinates of one system may be transposed to those of the other.

266. Horizon System.—In the ordinary operations of engineering, when observations are made with the engineers' transit, measurements are in the form of horizontal and vertical angles. It follows that, in astronomical field work where observations are made with the engineers' transit or similar instrument, angular measurement to celestial bodies must be of like character to those made upon terrestrial objects. The *horizon system* of spherical coordinates is such a system extended to include the celestial sphere. In the horizon system the coordinates of an object are given by its *altitude* and *azimuth*. The system is illustrated by Fig. 266 which represents a portion of the celestial sphere. $NES'W$ is the observer's horizon, Z is the zenith, and $S'ZN$ is the meridian; S is a heavenly body and BSZ is a great circle called a vertical circle, through the body and the zenith.

The *altitude* of a heavenly body is defined as the angular distance measured along a vertical circle, from the horizon to the body. It is expressed in degrees of arc. The altitude of S is given by the vertical angle h and also by the angular distance BS, the arc of a vertical circle passing through the zenith. Except in rare instances, celestial objects are observed when above the true horizon, when the sign of the altitude is positive. It is seen that positive altitudes may vary between 0° and 90°.

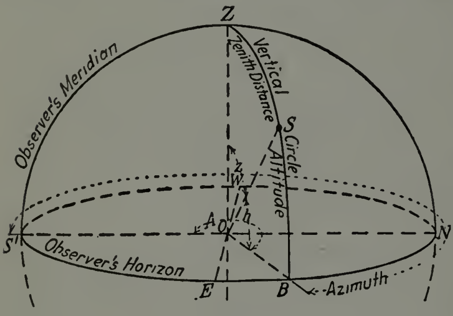


Fig. 266.

The *azimuth* of a heavenly body is defined as the angular distance measured along the horizon in a clockwise direction from the meridian to the vertical circle through the body. Azimuths may be reckoned either from the south or the north point of the meridian, but in astronomical work azimuths are customarily reckoned from south through 360°. An exception is often made in the case of circumpolar stars for which azimuths are reckoned from north. Also in trigonometric computations azimuths of stars west of north or east of south are often expressed as the counter-clockwise angles from the meridian and are considered as negative values. In Fig. 266 the azimuth of S reckoned in the customary manner is given by the angle A or by the angular distance $S'NB$, an arc of the horizon. If the azimuth of S were reckoned from north, it would be given by the angle $(A - 180°)$ or by the angular distance NB. The negative azimuth reckoned from south is given by the arc $S'EB$.

The complement of the altitude is called the *zenith distance*. It is the angular distance from the zenith to the celestial body measured along a vertical circle. In the figure the zenith distance is given by the angle z or by the angular distance ZS. Thus $z = 90° - h$. Since the celestial sphere is apparently rotating about its axis, while the meridian, horizon, and zenith are imagined as remaining fixed in position, it is clear that in general both the altitude and azimuth of a star undergo a maximum change in the course of one rotation of the sphere or in the course of one day. At a station intermediate between the pole and the equator, any circumpolar star is continually above the horizon, and since it crosses the meridian twice daily, once when above the pole and once when below, it follows that its azimuth (from south) is 180° twice daily. When it crosses the lower branch of the meridian, its altitude is a minimum; and when it crosses the upper branch, its altitude is a maximum. Obviously the mean of these two values (if variable refraction and slight changes in declination are to be neglected) is the altitude of the celestial pole. If any heavenly body arrives at a given altitude when it is east of the meridian, it will later arrive at the same altitude west of the meridian, and neglecting change in declination, the azimuth measured east of the meridian to the first position will equal the azimuth measured west of the meridian to the second position.

267. Relation between Horizon and Equator Systems.—Figure 267*a* represents a section of the earth through the poles and the station of an observer. Since the latitude of a place is its angular distance from the equator measured along a meridian of longitude, it is evident from the figure that the latitude of the observer is given by the angle ϕ between the equator and a vertical line through the observer's station, the angle being measured in the plane of the meridian. Also it appears, from similar triangles, that the latitude is

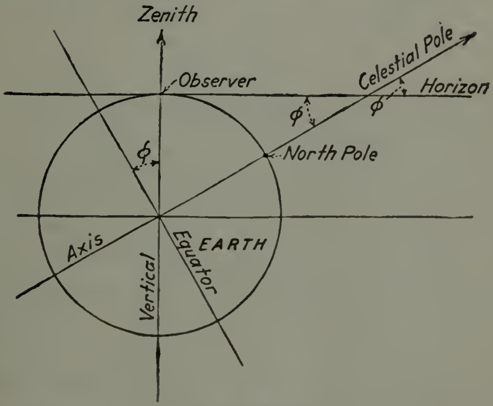


FIG. 267*a*.

given by the angle ϕ between the axis and the horizon, likewise measured in the plane of the meridian.

Similarly Fig. 267*b* represents a section of the celestial sphere through the celestial poles and observer's zenith. For reasons just explained,

$$\angle QOZ = \angle NOP = \phi = \text{latitude of place.}$$

Hence the latitude of a place is given by the angular distance *NP* which is the altitude of the pole, or by the angular distance *QZ* which is the zenith distance or co-altitude of the equator. The angular distance from the pole of the zenith is $90° - \phi = c$ which is the co-latitude of the place.

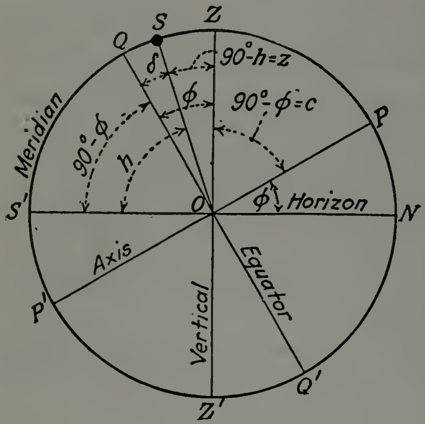


Fig. 267*b*.

In a northern latitude if any heavenly body *S* whose declination is δ is on the meridian and south of the zenith, then from Fig. 267*b*

$$\phi = 90° - h + \delta = z + \delta$$

and for any star north of the zenith

$$\phi = h \pm (90° - \delta) = h \pm p,$$

in which the sign of *p*, the polar distance, is positive or negative according as the star is below or above the pole. When a heavenly body *S* is on the meridian and south of the zenith, both its hour angle (reckoned from the upper branch of the meridian) and its azimuth (reckoned from south) are zero; but if a star is at upper culmination north of the zenith, while its hour angle (reckoned from the upper branch) is zero, its azimuth (reckoned from south) is 180°.

268. The relation between the coordinates of the horizon system and those of the equator system is shown for a star *S* not on the meridian, by Fig. 268. The place of observation is assumed to be north of the equator at a latitude ϕ, as shown by the angle between the horizon plane and the celestial axis, and the star is in a position east of the meridian and above the celestial equator, as in Figs. 263 and 266.

In the horizon system, the coordinates of *S* are *A*, the azimuth measured from the south point of the horizon, and *h*, the altitude measured above the horizon. In the equator system the angular position is fixed by *t*, the hour angle measured from the upper branch of the meridian, and δ, the declination measured above the equator. Most of the problems of field astronomy involve transposing from one system of spherical coordinates to the other, that is, certain coordinates in one or both systems are known or observed and calculations are made to determine the remaining unknowns; and this makes necessary the solution of the spherical triangle defined by the mutually intercepted arcs of the meridian, the vertical circle through the star, and the hour circle through the star. The vertices of this triangle are the pole P, the zenith Z, and the star S. The triangle is called the *astronomical triangle* or the *PZS triangle.*

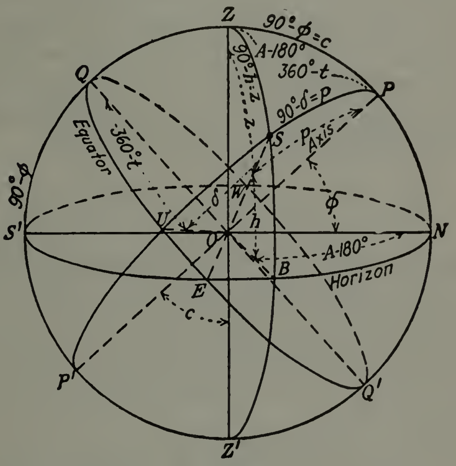


FIG. 268.

Referring again to the figure, the sides of the PZS triangle are seen to be the angular distances PS, PZ and ZS.

$$PZ = 90° - \phi = c = \text{co-latitude},$$
$$PS = 90° - \delta = p = \text{co-declination} = \text{polar distance},$$
$$ZS = 90° - h = z = \text{co-altitude} = \text{zenith distance}.$$

If the star is below the equator, the above expression for PS still holds true when due account is taken of the algebraic sign of δ.

When the star is east of the meridian, as in the figure, and azimuths are reckoned from the south point of the horizon and hour angles are reckoned fron the upper branch of the meridian, then the angles of the PZS triangle formed by the intersecting planes of the meridian, the hour circle, and the vertical circle are:

$$Z = A - 180° = (\text{azimuth} - 180°)$$
$$P = 360° - t = (360° - \text{hour angle})$$

If the star is west of the pole it can be readily shown by a sketch that the angles of the *PZS* triangle are

$$Z = 180° - A = (180° - \text{azimuth})$$
$$P = t = \text{hour angle}$$

269. Solution of PZS Triangle.—In surveying, the solution of the astronomical triangle is made in connection with determinations of azimuth. Usually observations are made upon the sun or upon some star that can be readily identified. In this case the altitude of the body is measured, its declination is determined from tables, and the latitude of the place of observation is known. Hence, the three sides of the astronomical triangle are known. The determination of azimuth of the heavenly body involves the computation of the angle at Z; and the angle at P, as a measure of the hour angle, must be computed for determinations of longitude or time.

By spherical trigonometry there is derived

$$\cos Z = \frac{\sin \delta}{\cos h \cos \phi} - \tan h \tan \phi \tag{1}$$

which is a general expression for determining azimuth when the three sides of the astronomical triangle are known, Z being the azimuth from the north point of the horizon, considered positive if the star is east of the meridian and negative if the star is west of the meridian, and being less or greater than 90° according as the sign of coz Z is positive or negative.

270. By a series of substitutions which will not be given here but which may be found in any treatise on spherical trigonometry, Eq. (1) may be changed to the form

$$\tan^2 \frac{1}{2}Z = \frac{\sin (s - h) \sin (s - \phi)}{\cos s \cos (s - p)} \tag{2}$$

which form possesses certain advantages over Eq. (1).

271. When azimuths are reckoned from the south point of the horizon, Eq. (1) takes the following form, A being the azimuth measured either clockwise or counter-clockwise from the south point:

$$\cos A = \tan h \tan \phi - \frac{\sin \delta}{\cos h \cos \phi} \tag{3}$$

272. Azimuth at Elongation.—The most favorable position for determining azimuth by observation on a circumpolar star (or for that matter any star which crosses the upper branch of the meridian north of the zenith) occurs when it is farthest east or farthest west of the pole, when the star appears to be traveling vertically for some time. In this position it is said to be at eastern or western *elongation* according as it is east or west of the meridian. At the instant of elongation, since the star appears to be traveling vertically, its apparent path in the celestial sphere is tangent to the vertical circle through the zenith, as is illustrated by Fig. 272. Therefore the angle between the plane of the hour circle and the plane of the vertical circle, or in other words, the angle at S, is 90°. For azimuth determinations of this sort, the latitude of the place of observation is known and the declination of the star for the given date is obtained from tables. At the instant of elongation, there are then known in the astronomical triangle the side $ZP = 90° - \phi$, the side $PS = 90° - \delta$, and the angle $S = 90°$. Remembering that the sine of S is unity when S is 90°, there is obtained

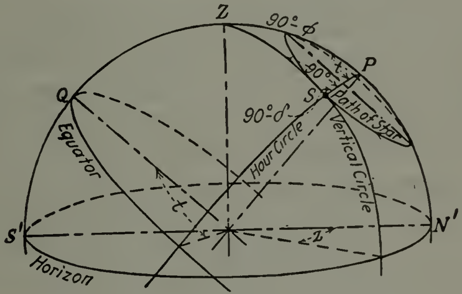


Fig. 272.

$$\sin Z = \frac{\sin (90° - \delta)}{\sin (90° - \phi)}$$

or

$$\sin Z = \frac{\sin p}{\cos \phi} \tag{4}$$

which is the expression employed for determining the azimuth of a circumpolar star when at elongation, the latitude of the place and the star's polar distance being known, Z being the azimuth reckoned east or west of north according as the star is at eastern or western elongation.

272a. Azimuth of Circumpolar Star.—The expression

$$\tan Z = \frac{\sin t}{\cos \phi \tan \delta - \sin \phi \cos t} \tag{5}$$

is commonly used in finding the azimuth of Polaris or any other circumpolar star when the star is not at elongation and the hour angle of the star is accurately known.

273. Time.—As the earth rotates about its axis in its travel through space, all celestial bodies apparently rotate about the earth (or about its axis) from east to west with regularity. We have imagined that the earth is the center of the celestial sphere—a hollow sphere of infinite radius to which are fastened the sun and stars—which rotates about the celestial axis regularly. It has been stated that owing to the fact that the earth in its orbit travels about the sun but does not travel about the fixed stars which are far outside its orbit, once each year the sun apparently encircles the celestial sphere along a path called the *ecliptic* which twice cuts the celestial equator during this interval. The point among the stars where the sun cuts the celestial equator on March 21 of each year is called the *vernal equinox*, and it has been stated that the vernal equinox is a point of reference whose position on the celestial sphere is unchanging.

Because the sun is apparently traveling from west to east among the stars, while the rotation of the celestial sphere about the earth is apparently from east to west, the angular velocity of the sun about the axis of the celestial sphere is less than that of the fixed stars or of the vernal equinox, just as the angular velocity of a person traveling towards the rear of a train upon a circular track is less than that of the locomotive. This being the case, at a given meridian the hour angle of the sun and of the vernal equinox will agree at some instant on March 21, but thereafter it will be less for the sun than for the vernal equinox. Six months later, on September 22, when the sun has covered one-half of its annual journey, the hour angle of the sun will be 180° or 12^h less than that of the vernal equinox; and 1 yr. later the hour angle of the sun will be 360° or 24^h less than that of the vernal equinox, and hence the hour angles will again numerically agree.

In the course of a tropical year as measured by the time taken by the sun apparently to make a complete circuit of the ecliptic, there actually occur 366.2422 revolutions of the earth, or there apparently occurs a like number of revolutions of the vernal equinox about the earth.

For reasons just explained, the sun during this interval will have traveled through a total hour angle 360° or 24^h less than that traversed by the vernal equinox, hence during a tropical year the sun apparently revolves about the earth 365.2422 times.

The interval of time occupied by one apparent revolution of the sun about the earth is called a *solar day*, the unit of time with which we are all familiar. The interval of time occupied by one apparent revolution of the vernal equinox is called a *sidereal day*, a unit of time that is much used by astronomers. Since 366.2422 sidereal days occupy the same period of time as 365.2422 solar days, the sidereal day is a shorter time interval than the solar day.

When any celestial body, real or imaginary, apparently crosses the upper branch of a meridian, it is said to be at *upper transit* or *upper culmination;* when any celestial body crosses the lower branch of the meridian it is said to be at *lower transit* or *lower culmination.*

The beginning of a sidereal day at a given place occurs at the instant the vernal equinox is at upper transit.

The solar day is sometimes considered as beginning at the instant of upper transit of the sun or at noon, but in general is considered as beginning at the instant of lower transit or at midnight.

Both sidereal and solar days are divided into 24 hr. each of 60 min. duration. The hours are reckoned consecutively from 0 to 24.

274. True and Mean Suns.—On account of the eliptical shape of the earth's orbit, the apparent angular velocity of the sun we see, or the true sun, is not constant, hence the days as indicated by the apparent travel of the true sun about the earth are not of uniform length. To make our solar days of uniform length, astronomers have invented the *mean sun,* a fictitious body which is imagined to move at a uniform rate along the celestial equator, making a complete circuit from west to east in one tropical year.

The true interval as measured by one apparent revolution of the *true sun* about the earth is called an *apparent solar day.* The time interval as measured by one revolution of the *mean sun* is called a *mean solar day.*

275. Apparent Solar Time.—Beginning with the 1925 volume, the *American Ephemeris and Nautical Almanac* has considered the apparent solar day at any place as the time interval between two successive lower transits of the true sun for the meridian of that place. The solar day therefore begins at midnight as does the civil day, and the *apparent solar time* at any place is given by the hour angle of the true sun plus 12^h. Thus, if the hour angle of the sun is $45° = 3^h$ at a given place and at a given instant, then the apparent solar time for the place is $3^h + 12^h = 15^h$. *Apparent time* has the same meaning as *apparent solar time.*

Local apparent time is the apparent solar time for the meridian of the observer. Apparent solar time for any other meridian is designated by name. Thus apparent solar time for the meridian of Greenwich is called *Greenwhich apparent time.*

276. Mean Solar Time.—Beginning with the 1925 volume, the *American Ephemeris and Nautical Almanac* has considered the mean solar day at any place to be the time interval between two successive lower transits of the mean sun for the meridian of that place. The mean solar day therefore begins at midnight, as does the civil day, and the *mean solar time* is given by the hour angle of the mean sun plus 12^h. Thus, if the hour angle of the mean sun is $-15° = -1^h$, the mean solar time is $-1^h + 12^h = 11^h$. *Mean time* has the same meaning as *mean solar time.*

Local mean time is the mean solar time for the meridian of the observer. Mean solar time for any other meridian is designated by name. Thus mean solar time for the meridian of Greenwich is called *Greenwich mean time.*

277. Equation of Time.—The difference between mean time and apparent time at any instant is called the *equation of time.* When the true sun is ahead of the mean sun, apparent time is faster than mean time, hence to get mean time from apparent time, the equation of time must be subtracted. If the true sun is behind, the equation of time must be added to apparent time to get mean time. The equation of time may be obtained from any solar ephemeris.

By inspecting the tabulated values of the equations given in the ephemeris it will be seen that in February the true sun is as much as 14^m behind the mean sun and in November the true sun is more than 16^m ahead of the mean sun, while on about the dates April 15, June 15, September 1, and December 25, the equation of time is zero, and hence the hour angle of the true sun is for an instant the same as that of the mean sun.

278. Sidereal Time.—The sidereal time at any place is the hour angle of the vernal equinox at that place; and the beginning of the sidereal day, occurring when the vernal equinox crosses the upper branch of the meridian, is called sidereal noon. Twenty-four-hour clocks, regulated to keep sidereal time are called sidereal clocks. The vernal equinox being an imaginary point can not be observed, as can the sun, but since it is the point of reference to which the right ascensions of stars are referred, the sidereal time may be obtained by determining the hour angle of any star the right ascension of which is known. Then if θ be the sidereal time, $\theta = t + \alpha$, as explained in Art. 265. The sidereal day is shorter than the mean solar day by $3^m55^s.9$ mean solar time, or $3^m56^s.6$ sideral time. The sidereal hour is shorter than the mean solar hour by $9^s.83$ mean solar time, or $9^s.85$ sidereal time. In a sidereal hour there are $60^m - 9^s.830 = 59^m50^s.170$ mean solar time; in a mean solar hour there are $60^m + 9^s.856 = 1^h00^m09^s.856$ sidereal time. Apparent right ascensions of the sun and stars are given in the *Nautical Almanac* and also in the *American Ephemeris.*

279. Relation between Longitude and Time.—It has been shown that as the earth rotates upon its axis, the hour angle of the sun referred to any meridian on the earth undergoes a change of 360° or 24^h during the interval of one solar day of 24^h. Since the longitudes of the earth change from 0° to 360°, it follows that the mean solar time interval taken for the mean sun apparently to travel from the meridian of one place to that of another is equal to the difference in longitude between the two places. The same statement applies equally well to the sidereal time interval and the vernal equinox. It is therefore evident that at any instant, the *difference in local time* between two places, whether the time under consideration be sidereal, mean solar, or apparent solar, is equal to the *difference in longitude* between the two places, expressed in hours. This relation makes it possible to determine the time when difference in longitude between two places is known, or to determine longitude when difference in time between two places is given.

Most of the solar ephemerides are for the meridian of Greenwich, and a problem of frequent occurrence is to find the local time corresponding to a given instant Greenwich time. As stated in Art. 262, the longitude of a place is reckoned east or west according to whether the place is east or west of Greenwich. The local time (L.T.) of a place at a given instant is obtained by adding to or subtracting from the Greenwich time (G.T.) the difference in longitude ($\Delta\lambda$) between the two places; if the place is east of Greenwich, the difference in longitude is added, and if the place is west, the difference in longitude is subtracted.

$$\text{L.T.} = \text{G.T.} \pm \Delta\lambda$$

The following examples illustrate the process of changing from the time of one place to that of another.

Example 1: An observation on the sun is taken at $9^h52^m56^s$ local apparent time (L.A.T.). The longitude of the place is $7^h12^m36^s$ west of Greenwich. What is the Greenwich apparent time (G.A.T.)?

$$\begin{aligned}\text{G.A.T.} = \text{L.A.T.} + \Delta\lambda &= 9^h52^m56^s + 7^h12^m36^s \\ &= 17^h05^m32^s\end{aligned}$$

Example 2: On Nov. 20, 1927, the mean sun crosses the lower branch of the Greenwich meridian at $3^h52^m2^s.80$, Greenwich sidereal time. At that instant it is desired to find the local sidereal time at a place whose longitude is $5^h12^m24^s.2$ west of Greenwich.

$$\begin{aligned}\text{L.S.T.} = \text{G.S.T.} - \Delta\lambda &= 3^h52^m2^s.8 - 5^h12^m24^s.2 + 24^h \\ &= 22^h39^m38^s.6\end{aligned}$$

Example 3: At the instant of $18^h48^m15^s$ Greenwich civil time, the local civil time of a place is $10^h37^m42^s$. It is desired to determine the longitude of the place with respect to Greenwich.

$$\Delta\lambda = 18^h48^m15^s - 10^h37^m42^s = 8^h10^m33^s \text{ west of Greenwich.}$$

280. Standard Time.—In order to eliminate the industrial confusion attendant upon the use of local time, the United States has been divided into longitudinal belts, in each of which the watches and clocks which control civil affairs all keep the same time. These belts are called standard time belts, and the time kept in each belt is *standard time.* The time in each belt differs from that of its neighbors by one mean solar hour, the standard time in the belt to the east being 1^h faster and that to the west being 1^h slower. The time in any belt is a whole number of hours slower or faster than Greenwich mean time.

On the North American continent there are five time belts, as follows:

Atlantic Standard.—Time kept in the Maritime Provinces of Canada, 4 hr. slower than Greenwich civil time. It is the local mean time of the meridian whose longitude is 60° west of Greenwich (the 60th meridian).

Eastern Standard.—Time kept in the Eastern States from Maine to Central Ohio, 5 hr. slower than Greenwich civil time. It is the local mean time of the 75th meridian.

Central Standard.—Time kept in Central States from Central Ohio to Central Nebraska, 6 hr. slower than Greenwich civil time. It is the local mean time of the 90th meridian.

Mountain Standard.—Time kept in Western States from Central Nebraska to Western Utah, 7 hr. slower than Greenwich civil time. It is the local mean time of the 105th meridian.

Pacific Standard.—Time kept in Pacific States west of Utah, 8 hr. slower than Greenwich civil time. It is the local mean time of the 120th meridian.

The exact boundaries of the time belts are irregular in character and can be determined only from a map.

The Greenwich mean time is found by adding to the standard time, the longitude (expressed in hours) of the meridian for which standard time is also local mean time.

Example 1: At a given instant the Central standard time is 9^h00^m a.m. It is desired to find the Greenwich mean time. The longitude of the meridian to which Central standard time is referred is 90° or 6^h west of Greenwich. The Greenwich mean time is $9^h00^m + 6^h00^m = 15^h00^m$, or 3^h00^m p.m.

If the longitude of a place is known, the standard time of the belt in which the place is situated may be determined by algebraically adding to the local mean time the difference in longitude (expressed in hours) between the given place and the meridian for which standard time is also local mean time.

Example 2: By observation on a star, the local mean time at a given instant is found to be $18^h37^m46^s$. The longitude of the place is $\lambda_l =$

$89°49'30'' = 5^h59^m18^s$. The standard time at the given instant is to be found. The place is evidently in the Central time belt for which the standard time (C.S.T.) is local time for the 90th meridian. The longitude of this meridian expressed in hours is $\lambda_s = \dfrac{90°}{15} = 6^h$.

$$\begin{aligned} \lambda_l &= 5^h59^m18^s \\ \lambda_s &= 6^h00^m00^s \\ \hline \Delta\lambda &= -0^m42^s \end{aligned}$$

$$\text{C.S.T.} = \text{L.M.T.} + \Delta\lambda = 18^h37^m46^s + (-0^m42^s) - 12^h = 6^h37^m04^s \text{ p.m.}$$

281. Parallax and Refraction Corrections.—The origin of the several systems of coordinates for locating the position of a heavenly body has been taken as the center of the earth, and in the previous discussion it has been assumed that the celestial sphere was of infinite radius. In point of fact, angular measurements to a heavenly body are made from stations on the surface of the earth, and the distance to any celestial body is not infinite. In Fig. 281 $S_1S_2S_3$ represents the path of any celestial object, O being the center of the earth and T being the station of an observer on the earth's surface. Then, h' represents the true altitude of S_1 above the observer's horizon and h represents the true altitude measured above the celestial horizon. The difference between these two angles which is $\angle OS_1T = C_p$ is called the *parallax correction*. Hence the true altitude above the celestial horizon $h = h' + C_p$.

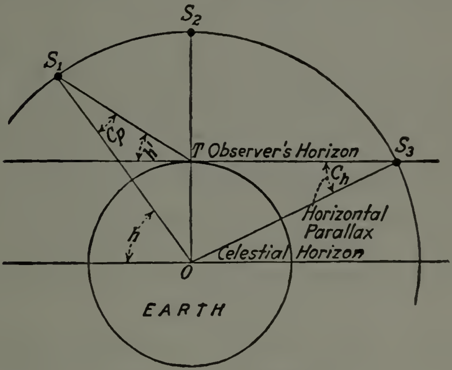


Fig. 281.—Parallax.

From the figure it is evident that the parallax correction can be calculated, if the distance to the heavenly body is known. Also it is obvious that the correction depends upon the altitude, being zero for S_2 when the body is directly overhead, and being a maximum when the body is on the observer's horizon at S_3. When the body is in the latter position, the correction is called the *horizontal parallax.* If the horizontal parallax be designated by C_h, it can be readily demonstrated that the parallax correction for any observed altitude is

$$C_p = + C_h \cos h.$$

For the fixed stars the parallax correction is too small to be of consequence. For the sun the parallax correction is sufficiently large to be of consequence in other than rough observations. The horizontal parallax of the sun (given for each day of the year in the *American Ephemeris*) varies somewhat from month to month, being always slightly less than 09″.

282. When a ray of light emanating from any heavenly body enters the atmosphere of the earth, it is bent downward. Hence the sun and stars appear to be higher above the observer's horizon than they actually are. The angle of deviation of the ray from the direction that it takes on entering the earth's atmosphere, to the direction it takes at the surface of the earth, is called the *refraction* of the ray. Since celestial objects appear too high, their observed altitudes above the observer's horizon are greater than their true altitudes above the observer's horizon. The *refraction correction* C_r is applied to the observed altitude (h'') to determine the true altitude above the observer's horizon (h'). Since the observed altitude is always too great, the refraction correction is always subtracted.

$$h' = h'' - C_r$$

The magnitude of the refraction correction depends upon the temperature and barometric pressure of the atmosphere and upon the altitude of the ray. It does not depend upon the distance to the body from which emanates the ray. Under normal conditions the refraction correction amounts to about 34′ when the sun or star is on the observer's horizon, but for low altitudes it varies greatly with changes in atmospheric conditions. Table II gives values of refraction corrections for a barometric pressure of 29.5 in., for various temperatures, and for altitudes between 10° and 90°. Between these altitudes it can be shown that, other things remaining constant, the refraction correction varies as the cotangent of the altitude. Hence $C_r = 0$ when $h = 90°$.

Owing to the uncertainties of the refraction correction for small altitudes, observations for accurate determinations are never taken upon a body which is near the horizon.

283. Refraction and parallax corrections for the sun are usually considered together. Table I gives corrections to be subtracted from observed altitudes of the sun (for both refraction and parallax) to determine the true altitudes above the celestial horizon.

284. Problems.

1. When the local apparent time is $8^h17^m12^s$ at a place whose longitude is 96°15′10″W, what is the Greenwich apparent time?

2. On a given date 0^h Greenwich civil time occurs at $4^h17^m32^s$ Greenwich sidereal time. At that instant what is the local sidereal time at a place whose longitude is $7^h17^m43^s$?

3. When it is $15^h31^m12^s$ Greenwich civil time, it is $10^h16^m37^s$ local civil time. What is the longitude of the place?

4. What is the Greenwich civil time when it is 3^h15^m p.m. Central standard time?

5. If the local civil time at a place is $16^h23^m22^s$ and the longitude of the place is 78°36′20″, what is the Eastern standard time?

6. From an ephemeris find the equation of time for the instant of $4^h15^m0^s$ Pacific standard time on April 21 of the current year. If the longitude of the place is $7^h46^m3^s$W, calculate the local mean and local apparent times.

7. From an ephemeris, find the equation of time for the instant of $3^h19^m30^s$ p.m. apparent time July 4 of the current year, at a place whose longitude is $6^h15^m30^s$. Calculate the corresponding local mean time.

8. At a given place the hour angle of the true sun at 11^h30^m p.m. Greenwich civil time on January 12 of the current year is 42°36′30″. What is the local sidereal time?

9. The mean radius of the earth is 3,956 miles and the mean distance to the sun is 92,900,000 miles. What is the sun's mean horizontal parallax? What is the parallax correction when the altitude is 30°?

10. The observed altitude of a star is 23°15′20″. The temperature is 90°F. By Table II, find the refraction correction, and compute the true altitude of the star.

11. The observed altitude of the sun's center is 15°07′30″. The temperature is 15°F. By Table I find the parallax and refraction correction and calculate the true altitude of the sun.

CHAPTER XVI

DETERMINATION OF AZIMUTH, LATITUDE, LONGITUDE, AND TIME

285. General Remarks.—The field methods of determining azimuth, latitude, longitude, and time described in the succeeding articles are the methods that are in common use on surveys of ordinary precision within latitudes of the United States, where the engineers' transit or the repeating theodolite is the instrument with which angular measurements are made. The two heavenly bodies upon which the surveyor takes most observations are the sun and Polaris, the pole star. For this reason, the following discussion is concerned with these two bodies, but the principles involved are the same for any star, as is clear from the preceding chapter. Measurements to the sun can not be made with the same degree of precision as can measurements to the stars, hence the probable error in computed values is larger than when a fixed star is chosen. The sun may be viewed at times convenient to the surveyor, however, and it is suitable for determinations of azimuth, latitude, and longitude which, while in general of relatively low precision, are sufficiently accurate for the majority of surveys. Polaris, being near the pole, changes its position slowly. It is the bright star among all others which is most favorably located for precise determinations of latitude and azimuth, but owing to its slow change in azimuth it is not suitable for longitude or time observations.

286. Angular Measurements.—The effect of instrumental errors upon the precision of horizontal angles was discussed in Art. 206. Whenever observations are made to determine azimuth, a part of the field work consists of measuring the horizontal angle between the celestial body and a reference mark on the earth's surface. Since the sights to the celestial body are, in general, steeply inclined, it is highly important that the horizontal axis be in adjustment with respect to the vertical axis and that the transit be carefully leveled. Attention is again called to the fact that, even though the horizontal axis be in perfect adjustment, it will be inclined unless the vertical axis is truly vertical, and the error due to such inclination will in general not be eliminated by a reversal of the telescope between sights. For accurate observations the transit should be equipped with a sensitive striding level by means of which the horizontal axis may be leveled prior to each sight. With the ordinary transit not so equipped, the plate may be leveled with greater refinement by means of the telescope bubble than it can be by means of the plate levels, and this method of leveling the plate should be employed when other than rough observations are being made. Also sights should be taken with the telescope in both the direct and reversed positions in order that the mean of horizontal angles may be free from other instrumental errors.

When altitudes are observed, the index error of the vertical circle should be determined just after each observation. It is to be noted that the error due to the line of sight not being parallel with the axis of the level tube will be eliminated by taking the mean of altitudes with the telescope in the direct and in the reversed positions, but any error due to the inclination of the vertical axis will not be eliminated by this procedure. When the transit has a full vertical circle, the index error may be most accurately found by taking vertical-circle readings to some well-defined point in the direction of the star or sun with the telescope in both the direct and reversed positions. The difference between the two readings is double the index error. This method of determining the index error does not take into account the error due to the vertical axis not being truly vertical, but does determine the errors due to the vernier being displaced and to the line of sight not being parallel to the axis of the level tube.

287. Prismatic Eyepiece.—When altitudes are large, it is impossible to view the image by looking directly into the eyepiece of the transit telescope. The *prismatic eyepiece* is a device which, when attached to the telescopic eyepiece, reflects the rays through an angle of 90° with the axis of the telescope. By means of one type the sun and stars may be observed when at altitudes of more than 80°. The reflecting medium is a prism; hence the image appears upside down from that seen through the telescopic eyepiece but is not reversed horizontally.

288. Observations on the Sun.—The sun has an angular diameter of about 32′. Since the sun is so large, observations with the ordinary transit can not be made with any degree of precision by sighting the intersection of the cross-hairs at the estimated position of the sun's center, and it is the practice to bring the cross-hairs tangent to the sun's image. When the horizontal cross-hair is brought tangent to the lower edge of the sun, the sight is said to be taken to the sun's *lower limb*, and this is indicated in the notes by the symbol $\underline{\odot}$. Similarly the symbol $\overline{\odot}$ indicates a sight to the sun's *upper limb;* $\odot\!|$ a sight with the vertical cross-hair to the sun's *right limb;* and $|\!\odot$ a sight to the sun's *left limb.*

When a single observation is taken it is necessary to correct the observed vertical angle to the sun's limb by the amount of the sun's semidiameter. The *American Ephemeris* and other solar ephemerides give values of the semidiameter of the sun for each day of the year. For rough calculations the semidiameter may be taken as 16′ at all seasons of the year. The true altitude of the sun's center is found by correcting the observed altitude of the sun's limb by the amount of the semidiameter. To reduce to the sun's center an observed horizontal angle taken to one of the sun's limbs, a correction equal to the semidiameter times the secant of the altitude is applied.

For most solar observations, an equal number of sights is taken to opposite limbs of the sun, in which case the average of observed horizontal and vertical angles gives the horizontal and vertical angles to the sun's center at the mean of the times, and no correction for semidiameter is necessary.

The sun is too bright to be viewed directly through the telescopic eyepiece. Some transits are equipped with a piece of colored glass, called the sun glass, which, when screwed to the eyepiece end of the telescope cuts off many of the light rays and makes it possible to sight at the sun with comfort. All prismatic eyepieces are equipped with such a glass.

In case a prismatic eyepiece is not available and the sun is not at too high an altitude, very good observations can be made by bringing the sun's image to a focus on a white card held several inches in the rear of the telescopic eyepiece. A rough pointing on the sun is made by sighting over the telescope. The eyepiece is then drawn out and the objective is focused until the sun's image and the cross-hairs are clearly seen on the card. If the eyepiece of the telescope is erecting, the image on the card will be inverted; if the eyepiece is inverting, the image will appear erect. The cross-hairs are visible only on the image of the sun. All three horizontal hairs are not at the same time visible on the sun's image, and a common blunder is to mistake one of the stadia hairs for the middle cross-hair.

289. Sun's Declination.—In connection with the determinations of azimuth, latitude, or longitude by solar observations, it is necessary to determine the declination of the sun at a given instant, and this is done by interpolating between values given in a solar ephemeris for the year in which the observation occurs. There are two types of ephemeris in common use, each of which is designed to be employed under certain conditions, but either of which it is possible to use under any circumstances. The first of these types gives the apparent declination for each day of the year at the instant of 0^h *Greenwich civil time* and is especially adapted for use when the standard time, Greenwich civil time, or Greenwich mean time is known. The solar ephemeris for the Greenwich meridian as given in the *American Ephermis and Nautical Almanac* is an example of an ephemeris of this kind. The second type gives the apparent declination for each day of the year at the instant of *Greenwich apparent noon* and is especially adapted for use when the longitude of the place and the local apparent time of the observation are known. The *Ephemeris of the Sun and Polaris* published by the General Land Office contains an ephemeris of this sort.

Example 1: An observation is taken on the sun at 10^h00^m a.m. Eastern standard time, on Dec. 15, 1927. It is desired to determine the declination at the given instant. By ephemeris for 0^h Greenwich civil time the declination for 0^h on Dec. 15 is $\delta_0 = -23°12'28''$. The Greenwich civil time is $10.00 + 5.00 = 15^h.00$. The average hourly variation is $-8''.8$. The change in declination since 0^h G.C.T. is $-8.8 \times 15 = 02'12''$. Declination at instant of observation $= -23°14'40''$.

Example 2: An observation is taken on the sun as it crosses the meridian on Nov. 16, 1926, at a place whose longitude is 87°49′30″ west of Greenwich. It is desired to determine the apparent declination at the given instant. G.A.T. $= \lambda = \dfrac{87°49'30''}{15} = 5^h51^m18^s$ (after apparent noon) $= 5^h.86$. From the *Ephemeris of the Sun*, General Land Office, the declination at Greenwich apparent noon is S18°37′21″. The average difference for $1^h = -37''.8$. The change in declination since Greenwich apparent noon is $-37''.8 \times 5.86 = 03'42''$. Declination at local apparent noon at place $=$ S18°41′03″.

290. Latitude by Observation on Sun at Noon.—The latitude of a given station may be determined with a fair degree of precision by observing with the engineers' transit the altitude of the sun at local apparent noon. If the longitude of the place is roughly known, it is unnecessary to observe the time, but if the longitude is unknown, the standard time of the observation must be taken. The problem consists of determining the true altitude h of the center of the sun above the celestial horizon and computing the apparent declination δ of the sun at the instant of sighting. Then as explained in Art. 267, the latitude is

$$\phi = 90° - h + \delta$$

The accuracy obtainable ordinarily depends upon the precision of the instrument. Since the maximum rate of change of declination is about 01′ per hour, a considerable error in time will affect but slightly the declination. With the ordinary transit having a vertical circle reading to single minutes the latitude may be determined in this manner with an error not greater than 01′. The mean of a series of observations on different days will, of course, render a much closer result.

When the time of observation is observed, an ephemeris giving values of apparent declination of 0^h Greenwich civil time is preferable. When the direction of the meridian is unknown, which is the usual case, the procedure is as follows: The transit is set up and carefully leveled; the horizontal cross-hair is sighted either on the lower or upper limb of the sun and kept continually in this position until a maximum altitude has been reached and the sun begins its apparent descent. At that instant the watch time is observed. The altitude and time are recorded. With the telescope still in the meridian plane it is plunged and its index error is determined, preferably by the method of double-sighting on an index mark as described in Art. 286. The watch is compared with a clock keeping correct standard time and the error is noted. The Greenwich civil time of the observation is calculated, and the declination is found in the solar ephemeris as illustrated by Example 1, Art. 289. The true altitude of the sun's center is determined by applying to the observed altitude, corrections for semidiameter (see ephemeris) and refraction and parallax (see Table I). The latitude is then determined by the equation given above. It should be noted that the sign of the refraction and parallax correction as given in Table I is always negative. The sign of the declination is negative from September to March, and is positive for the remainder of the year.

When desired, a second sight may be taken on the opposite limb of the sun with the telescope reversed. The mean of the two vertical angles is taken as the altitude of the sun's center at the mean of the two times of observation, no correction for index error being necessary. If time between sights does not exceed 3 or 4 min., the mean altitude may be considered as the altitude at apprent noon. The latitude is then calculated as described in the preceding paragraph.

291. Azimuth by Direct Solar Observation.—The azimuth of a line may be determined by a single observation of the sun at any time when it is visible, provided the latitude of the place is known.

At a known instant of time the sun is observed, and the altitude of the sun and the horizontal angle from the sun to the line are measured. The declination of the sun at the given instant is found from a solar ephemeris. With the declination δ, latitude ϕ, and altitude h known, the *PZS* triangle may be solved as described in Art. 269, the azimuth of the sun from north being given by any of several expressions, one of which is

$$\cos Z = \frac{\sin \delta}{\cos h \cos \phi} - \tan h \tan \phi$$

Knowing the azimuth of the sun and the horizontal angle from its position to the line, the azimuth of the line is readily computed.

The accuracy with which azimuths may be determined depends, of course, upon the precision of field observations, but also is affected by the shape of the astronomical triangle and closeness with which corrections to the observed altitude can be determined. In the light of previous discussion, it is evident from an inspection that the *PZS* triangle becomes weak as the sun approaches the meridian and the solution becomes indeterminate at the instant of apparent noon. On the other hand, although the strength of the *PZS* triangle increases as the hour angle measured in either direction from the meridian approaches $90° = 6^h$, the refraction correction becomes large and very uncertain for low altitudes, particularly for those less than 10°. For these reasons, when possible, observations within the latitudes of the United States are usually taken between the hours of 8 to 10 a.m. and 2 to 4 p.m. In no case should reliable results be expected for altitudes of less than 10°.

When the azimuth of a given line is to be determined by this method, the transit is set up and carefully leveled over one end of the line. The *A* vernier is set at zero on the horizontal circle, and the reading of the *B* vernier is noted. A sight is taken along the given line with the telescope, say, in the direct position, and the lower motion is clamped. The upper motion is loosened and a sight is taken at the sun, the vertical and horizontal cross-hairs being brought tangent in the upper left-hand quadrant of the field of view if the observation is in the morning; or in the upper right-hand quadrant if the observation is in the afternoon. At the instant of tangency, the time is observed. As quickly as convenient, vertical and horizontal circle readings are taken on all verniers. The upper motion is then loosened, the telescope is plunged, and a second pointing is made, this time with the sun in the diagonally opposite quadrant from that of the first observation; that is, the vertical and horizontal cross-hairs are brought tangent to the sun in the lower right-hand quadrant if in the forenoon, and in the lower left-hand quadrant if in the afternoon. The time of tangency is observed and the vertical and horizontal angles are measured as before. The upper motion is loosened, and the field work is completed by again sighting along the line and reading the horizontal circle, this time with the telescope in the reversed position. The watch is compared with a timepiece keeping correct standard time, and the correction is noted. The mean of the two observed altitudes is taken to be the apparent altitude of the sun's center at the mean of the two observed times.

Similarly the mean of the two horizontal angles is taken as the horizontal angle to the sun's center at the mean of the observed times.

292. The process of bringing the cross-hairs tangent to the sun is illustrated by Fig. 292, which shows the position of the sun in the field of view of the telescope just prior to each observation taken in the morning. The portion of the field of view covered by the sun in the position shown depends upon the angle of the field of view. For telescopes of low magnifying power, the full disc may be visible. If the position of the sun is determined by a card held in the rear of the eyepiece, the image on the card will include that portion of the sun's disc shown in the figure, and the cross-hairs on the shaded portion of the field of view will not be visible.

During the process of taking the first observation, the horizontal cross-hair is sighted a short distance above the sun's lower limb as illustrated. Since the altitude of the sun is increasing, the horizontal cross-hair approaches tangency due to the sun's apparent movement. At the same time, the vertical cross-hair is kept continuously on the sun's western limb by means of the upper-motion tangent screw. The instant when the vertical and horizontal cross-hairs are simultaneously tangent to the sun's disc is the instant of observation, and the horizontal and vertical circle readings are taken with the direction of the line of sight unchanged from the position at that instant. For the second observation, in the lower right-hand quadrant, the vertical cross-hair is set a short distance to the right of the sun's eastern limb. Since the sun is traveling westward, the vertical cross-hair approaches tangency due to the sun's apparent movement. At the same time the horizontal cross-hair is kept continually on the sun's upper limb. As before, observations are taken for the instant when both horizontal and vertical cross-hairs are simultaneously tangent to the sun's disc. It will be noted that the procedure is such that the final

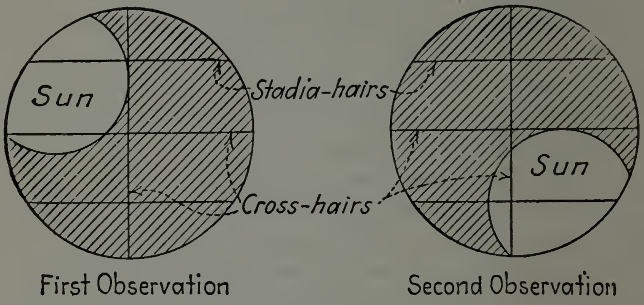
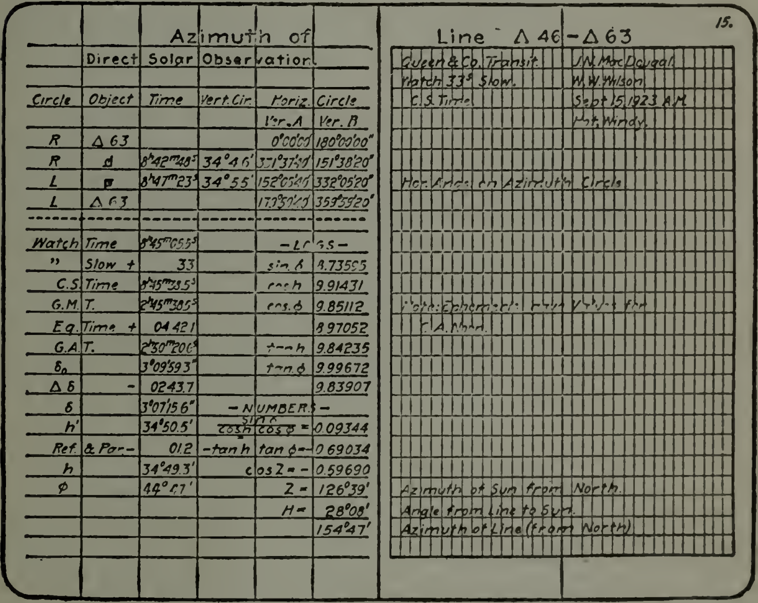


Fig. 292.—Position of sun in field of view just prior to tangency in morning.

setting for either observation requires the manipulation of only one tangent screw, the cross-hairs always being visible upon the sun's disc.

A similar demonstration may be made to show the advantage of sighting at the sun, first in the upper right-hand quadrant and then in the lower left-hand quadrant for afternoon observations in northern latitudes.

293. To the mean of the observed altitudes are applied corrections for parallax and refraction (Table I). The sun's apparent declination is found from a solar ephemeris as described in Art. 289. Attention is again directed to the two types of solar ephemeris, one giving declinations for 0^h Greenwich civil time, the other giving declinations for Greenwich apparent noon. In case the former type (such as the *American Ephemeris*) is employed, the mean time interval since 0^h G.C.T. must be determined (see Example 1, Art. 289). In case an ephemeris of the latter type (such as *Ephemeris of the Sun*, General Land Office) is used, the apparent time interval since Greenwich apparent noon must be found, and this involves computing the equation of time. The azimuth of the sun at the given instant can be computed from Eq. (1), either by using a calculating machine or by logarithms. When observations are of ordinary precision, as those taken with a transit reading to minutes, five places are sufficient for



Azimuth of Direct Solar Observation

Circle	Object	Time	Vert. Cir.	Horiz. Ver. A	Circle Ver. B
R	Δ 63			0°00'00"	180°00'00"
R	☉	$8^h42^m48^s$	34°46'	331°37'40"	151°38'20"
L	☉	$8^h47^m23^s$	34°55'	152°05'40"	332°05'20"
L	Δ 63			179°59'40"	359°59'20"
Watch	Time	$8^h45^m05.5^s$		— LOGS —	
"	Slow +	33		sin δ	8.73595
C.S.	Time	$8^h45^m38.5^s$		cos h	9.91431
G.M.	T.	$2^h45^m38.5^s$		cos φ	9.85112
Eq.	Time +	04 42.1			8.97052
G.A.	T.	$2^h30^m20.6^s$		tan h	9.84235
δ_0		3°09'59.3"		tan φ	9.99672
Δ δ	−	02 43.7			9.83907
δ		3°07'15.6"	— NUMBERS —		
h'		34°50.5'	$\frac{\sin\delta}{\cos h \cos\phi}$		= 0.09344
Ref.	& Par. −	01.2	−tan h	tan φ =	−0.69034
h		34°49.3'		cos Z =	−0.59690
φ		44°07'		Z =	126°39'
				H =	28°08'
					154°47'

Line Δ 46 – Δ 63

15.

Queen & Co. Transit.
Watch 33ˢ Slow.
C.S. Time

J.N. MacDougal
W.W. Wilson
Sept 15, 1923 A.M.
Hot, Windy.

Hor. Angles on Azimuth Circle

Note: Ephemeris gives values for G.A. Noon.

Azimuth of Sun from North.
Angle from Line to Sun.
Azimuth of Line (from North)

Fig. 294.—Azimuth by observation on sun.

calculations. In solving the above equation it should be noted that when declinations are south, the sign of sin δ is negative. The azimuth of the line is calculated by algebraically subtracting the mean of the observed horizontal angles from the azimuth of the sun, angles taken in a clockwise direction from line to sun being considered positive.

294. Figure 294 illustrates a suitable form for notes, the observations being taken with a transit having a vertical circle reading to 01′ and a horizontal circle reading to 20″. It is seen that horizontal angles are read from the azimuth circle. An ephemeris giving declinations for Greenwich apparent noon was used, hence the central standard time had to be transformed into Greenwich apparent time in order that the change in declination since Greenwich apparent noon could be determined.

295. Great care should be taken in leveling the instrument, for reasons explained in Art. 286. Unless it is equipped with a striding level, the final test prior to observing the sun should be to see that the telescope bubble remains centered as the instrument is revolved about the vertical axis. While reversal of the telescope between observations will eliminate from the mean of the observed altitudes errors due to lack of parallelism between line of sight and axis of telescope level tube, and will eliminate from the mean of the horizontal angles the error due to the horizontal axis not being perpendicular to the vertical axis, it will not eliminate the effect of the inclination of the vertical axis.

When it is assumed that at the mean of the two times the sun is at a position given by the mean of the vertical and the mean of the horizontal angles, it is equivalent to saying that the sun is apparently traveling in a straight line, which, of course, is not true. Within a period of 10 or 15 min., however, the error introduced is so small as to be of no consequence.

296. Time by Observation on the Sun at Noon.—If the longitude of a station is known and the direction of the meridian has been established, the standard time may be determined quite accurately by observing the sun as it crosses the meridian at local apparent noon. The instant each limb crosses the meridian is observed, and the mean of the two times thus determined is the watch time of upper transit of the sun's center, which is local apparent noon.

If the longitude of the place is expressed in hours, then for reasons explained in Art. 279, this is the Greenwich apparent time reckoned from noon. From the *Solar Ephemeris* of the General Land Office, or other ephemeris giving values for Greenwich apparent noon, the equation of time at the instant of observation is determined. Since the equation of time is the difference between mean and apparent time at the instant of observation, it is clear that 12^h plus or minus the equation of time is the local mean time of local apparent noon, the sign being plus or minus according as mean time is faster or slower than apparent time. Then if T be the observed time, E the equation of time, and $\triangle T_m$ be the correction of the timepiece to local mean time,

$$\triangle T_m = 12^h \pm E - T$$

The sign of the equation of time is given in the ephemeris. Since the difference between standard time and local mean time at a given place is the difference in longitude in hours between the place of observation and the meridian at which standard time and local mean time agree, it follows that the correction to give standard time is

$$\triangle T_s = \triangle T_m + \triangle\lambda$$

in which $\triangle\lambda$ is the difference in longitude between the given meridian and the standard meridian. If the place of observation is east of the meridian, the sign of $\triangle\lambda$ is negative; and if west, positive.

297. Longitude by Observation on the Sun at Noon.—If the standard time is accurately known and the meridian has been established, the longitude of a place may be determined by an observation upon the sun at local apparent noon, the field procedure being in all respects identical with that for finding time.

With the standard time of passage of the center of the true sun (local apparent noon) known, the Greenwich civil time of local apparent noon can be computed, and the equation of time at the instant of local apparent noon may readily be found from an ephemeris giving values for 0^h Greenwich civil time. Then if E is the equation of time, T_m is the standard time of local mean noon, and T_a is the standard time of local apparent noon, the difference in longitude (expressed in time) between the given place and the meridian at which mean time and standard time agree is

$$\triangle\lambda = (T_a \pm E) - 12^h = T_m - 12^h;$$

in which the sign of the equation of time is positive or negative according as apparent time is faster or slower than mean time, indicated by the ephemeris. If the result is negative, the meridian of the place is east of the standard meridian; if the result is positive, the meridian of the place is west of the standard meridian. Following is an example illustrating the method.

Example: The sun's center is observed to pass the meridian at a given place at $11^h30^m12^s.2$ a.m. Pacific standard time, Dec. 2, 1927. The longitude of the place is desired.

The difference between Pacific standard time and Greenwich civil time is 8^h, hence the G.C.T. is $19^h30^m12^s.2 = 19^h.503$. From ephemeris giving values for 0^h G.C.T., the equation of time at 0^h is $10^m59^s.63$. The difference for one day is $-22^s.64$. The change since 0^h is $-22.64 \times \frac{19.503}{24} = -18^s.30$. The equation of time at the instant of the local apparent noon is $10^m59^s.63 - 18^s.30 = 10^m41^s.33$. The ephemeris indicates that apparent time is faster than mean time, hence local mean noon occurs at a standard time later than that for local apparent noon by an amount equal to the equation of time, and the Pacific standard time of local mean noon is $T_m = T_a + E = 11^h30^m12^s.2 + 10^m41^s.3 = 11^h40^m53^s.5$. Then $\triangle\lambda = 11^h40^m53^s.5 - 12^h = -19^m6^s.5 = -4°46'38''$. Pacific standard time is local mean time for the 120° meridian, hence, the longitude of the place is $120° - 4°46'54'' = 115°13'22''$.

298. Solar Attachments.—A solar attachment is a device which, when attached to the transit, furnishes a means of determining the direction of the meridian by mechanically solving the *PZS* triangle. There are several varieties of solar attachment differing widely in appearance, but alike in principle. Each type has a polar axis and a line of collimation. The solar attachment may be rotated about the polar axis just as the transit may be rotated about its vertical axis. Means are provided for laying off the co-latitude of the place and the declination of the sun. When these quantities have been laid off, and the line of collimation of the solar attachment is directed to the sun, the line of sight of the transit telescope is in the plane of the meridian.

The use of a solar attachment makes it possible to determine the azimuth of a line more quickly than it can be found by direct observation and numerical computation, but the precision of the azimuth determination is likely to be much lower when an observation is made with a solar attachment than when the azimuth is computed from direct measurement of the sun's altitude. At the present time the solar attachment is little used except on certain mining and public land surveys of moderate precision, where several azimuth determinations must be made daily. Most surveyors agree that its cost and the bother of adjusting it more than offset any advantage gained, unless it is to be used very frequently. Because of the inaccuracies of the device, a solar attachment is quite useless for the more refined surveys, and it seems not improbable that before many years its use will be no more widespread than is that of the surveyors' compass. The three types of solar attachment which have been most popular are the Saegmuller, the Burt, and the Smith, the latter of which will be briefly described.

All that has previously been said regarding favorable hours for taking direct observations, applies equally well to observations with a solar attachment. Also errors in latitude and declination settings will have the same effect as when the azimuth is computed algebraically.

299. The Smith Solar Attachment.—This attachment consists of a telescope of low magnifying power, called the solar telescope, mounted in collar bearings which are attached to a graduated vertical arc, called the latitude arc. This assembly is mounted on a horizontal axis, called the axis of the latitude arc, which is attached to one of the standards of the transit as shown in Fig. 299. The solar telescope can be rotated about its own axis in the collar bearings and can also be revolved about the axis of the latitude arc. The latitude arc is read by a vernier which is fixed to the standard of the transit

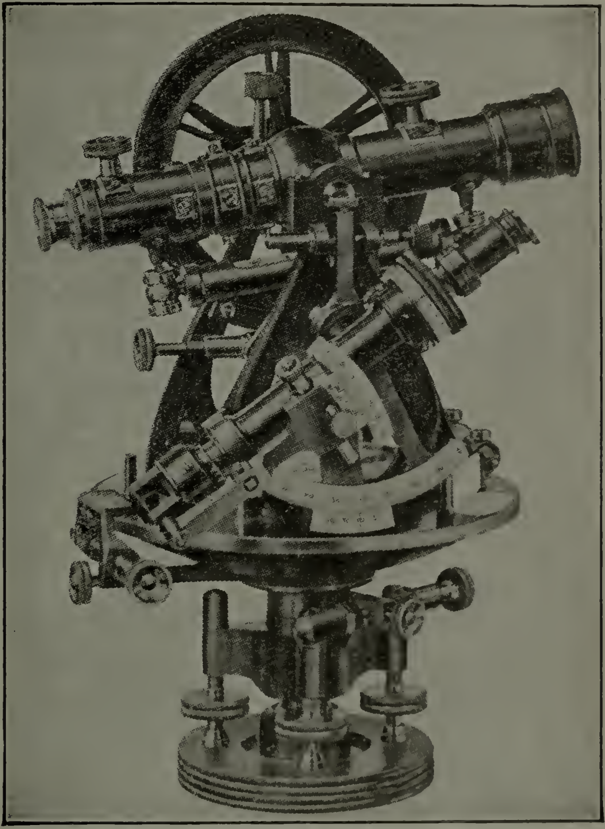


Fig. 299.—Smith solar attachment.

In front of the objective is a plane mirror, called the reflector, which has its axis normal to the line of sight of the solar telescope and which can be rotated (in bearings) about its axis. From the axis of the mirror an arm carrying a vernier leads to a graduated arc, called the declination arc, attached to the barrel of the telescope. Movements of both latitude and declination arcs are controlled by clamps and tangent screws. Near the eyepiece end of the solar telescope is an hour circle, the index of which is on one of the collars.

When the latitude is laid off on the latitude arc and the main telescope is pointed in the direction of north, the axis of the solar telescope points towards the pole. Further, for the position just specified, if the declination of the sun, corrected for refraction, is laid off on the declination arc, the mirror is so tilted that the course of the sun may be followed by rotating the solar telescope about its own axis, the image of the sun being reflected from the mirror to the objective, and thence to a focus at the cross-hairs of the solar telescope. When the sun is viewed in this manner, the index of the hour circle reads the local apparent time.

The advantage of the Smith solar attachment lies in a construction which permits the use of the main telescope without disturbing the latitude and declination settings. Since the latitude is constant for a given locality and the declination changes but slowly, this is a decided advantage when frequent observations are necessary, and for this reason the Smith solar may be regarded as superior to either of the other attachments previously mentioned. It has been adopted by the General Land Office as the standard instrument for use on the public land surveys.

300. Declination Settings.—If a solar attachment is to be used frequently, a table is prepared giving the declinations to be laid off when using the solar attachment at various hours of the day. The time may be either local apparent or standard time, but usually is the former. The apparent declination for a given time is found from a solar ephemeris, as described in Art. 289. Since the sun appears to be higher than it really is, due to atmospheric refraction, it is evident from a study of the *PZS* triangle (Fig. 268) that the declination setting for a sight to the apparent position of the true sun at a given instant must be algebraically greater than the true declination of the true sun, which is the value obtained from the ephemeris. If the refraction correction in altitude is known, the refraction in declination for a given latitude, hour angle, and declination may be computed by solving the spherical triangle. Table III gives such values throughout the year for latitude 40°. Any number in the second column gives the hour angle of the true sun on either side of the meridian — or in other words, gives the local apparent time before or after local apparent noon—to which the refraction correction given in the third column applies. For a latitude other than 40°, the correction of Table III is multiplied by the appropriate latitude coefficient of Table III*a*

301. Azimuth with Smith Solar Attachment.—In using the Smith solar attachment, the latitude and declination settings are made on the appropriate arcs. The transit is set up and carefully leveled, and the horizontal circle is set at zero. The local apparent time is then set off on the hour circle, this being done by rotating the solar telescope in its collar bearings. The transit is revolved on the lower motion until the image of the sun appears between the equatorial hairs of the solar telescope, when the line of sight of the transit telescope lies in the plane of the meridian, and the axis of the solar telescope points towards the celestial pole. Since the declination changes slowly, if no mistake has been made, it will be possible to keep the image of the sun between the equatorial hairs for some time by simply rotating the telescope in its collar bearings, keeping the hour circle set at the apparent time. When a sight is taken along the line with the transit telescope, the reading of the clockwise scale gives the azimuth.

This type of solar attachment is particularly convenient for checking azimuth traverses, since the latitude and declination settings may be maintained without interfering with the normal functions of the transit telescope. At any station the azimuth can be checked simply by setting the horizontal circle back to zero, rotating the solar telescope until the hour circle reads the apparent time, and making any slight change in the declination setting that may have occurred in the declination since the preceding observation.

302. Observations on Stars.—In general, the methods of determining azimuth, latitude, and time by direct solar observations are, with slight modifications, applicable to observations upon the stars. Usually it is expected that a higher degree of precision will be obtained by stellar than by solar observations. Consequently a corresponding degree of refinement is necessary and special care is taken to eliminate systematic errors. Since there are fixed stars in all parts of the heavens, it is an easy matter to select a star or stars in a celestial region favorable to an accurate determination of the quantity sought.

Thus, conditions favorable to an accurate determination of latitude by measuring the altitude of a star at culmination are a fairly high altitude, in order that the uncertainty of the refraction correction be small, and a rate of apparent movement that is small, in order that a series of observations may be taken without an appreciable change in the altitude. Within the latitudes of the United States, stars near the pole satisfy these conditions.

Likewise for accurate determination, an observation for azimuth by measured altitude and known declination and latitude should be taken upon a star in the east or west far enough above the horizon to eliminate the uncertain refraction, but not so near the meridian as to produce a weak astronomical triangle; and an observation for azimuth, knowing the hour angle, declination, and latitude, should be taken upon a circumpolar star, the nearer the pole the better, since its azimuth changes more slowly in a given length of time than does the azimuth of a star near the equator, and hence any error in time will have less effect. Also, for longitude or time, stars should be chosen near the equator because they are apparently traveling more rapidly than those near the pole.

The right ascensions and declinations for a large number of stars are given in the *American Ephemeris* and also in the *Nautical Almanac.* Since for the fixed stars these coordinates change very slowly, it is not necessary to determine values for the hour of observation, as with the sun. In the ephemeris of the sun the sidereal time of 0^h G.C.T. is given, from which the sidereal time corresponding to any given solar time can be found, and the hour angle can be computed by the expression $t = \theta - \alpha$ as explained in Art. 278.

Stars may be identified by means of charts which show the various constellations. For many stellar observations, however, the direction and altitude of the star can be laid off with sufficient accuracy so that the star will be brought into the field of view at a given time, and to be able to distinguish the star from among its neighbors is not essential. In fact, observations may be and frequently are taken on stars before dark, when they are still invisible to the naked eye.

During the hours of darkness, artificial illumination is required to make visible the cross-hairs of the transit. Some instruments are equipped with a reflector sleeve which slips over the objective as does the sun shade. When a flash light is held to one side of the reflector, the field of view is faintly illuminated and both the cross-hairs and the star can be seen. With the transit not so equipped, the cross-hairs can be illuminated sufficiently by holding the light a few inches in front of the objective and a little to one side of the telescope barrel, thus causing the rays to enter the telescope diagonally. There will be found a position of the light where both cross-hairs and star are visible.

The position of any terrestrial mark that may be used in observing must, of course, be indicated by a light. Often the mark is a slit in the side of a box in which there is a lamp. The mark may be a strongly illuminated target, the source of illumination being shielded from the observer.

In sighting on a star, the objective should be focused until the star appears as a fine, brilliant point of light.

303. Polaris.—The pole star, Polaris, is the star more than all others, upon which observations for latitude and azimuth are taken in the latitudes of the United States. Its distance from the pole is only a little more than a degree. Its annual change in declination is less than a minute of arc, and its maximum daily change in declination is less than half a second of arc. It is a second-magnitude star whose position is readily identified by the neighboring constellations of Ursa Major and Cassiopeia. Figure 303 shows the position of Polaris with respect to the pole and to these constellations. The seven most brilliant stars in the constellation of Ursa Major are known as the Great Dipper; and the two stars forming the part of the bowl farthest from the handle are called the *pointers* because a line through these stars points very nearly to the celestial north pole. It will be noted that the constellation of Cassiopeia is on the same side of the pole as Polaris, so that when Cassiopeia is above the pole, Polaris is near upper culmination; when Cassiopeia is west of the pole, Polaris is near western elongation; and so on. The position of Polaris relative to the pole may be quite closely estimated by noting the positions of δ Cassiopeia and ζ Ursa Major. A line joining these two stars passes nearly through the pole and Polaris. The line is nearly vertical when the star is at either culmination, and nearly horizontal when the star is at either elongation.

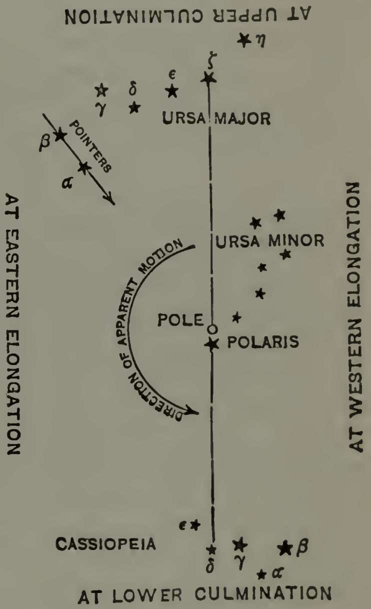


Fig. 303.—Positions of constellations near the North Pole when Polaris is at culmination and elongation.

Data concerning the position of Polaris can be found from a variety of sources. In the *American Ephemeris* the declination and right ascension for each day are given in the table of circumpolar stars for the upper transit at Washington.

The *Ephemeris of the Sun and Polaris*, General Land Office, also gives values of the declination of Polaris and the Greenwich mean time of culmination at the meridian of Greenwich.

In Table IV of this book are given data by means of which the approximate standard time of culmination may be determined for any place and date. Accompanying the table is an explanation of its use. The reasons for the various steps are quite clear when it is remembered that the hour angle of Polaris is changing at a faster rate than that of the mean sun, this gain being approximately the gain of sidereal time on mean time.

304. Latitude by Observation on Polaris at Culmination.—As shown in Art. 267, if h is the true altitude of any circumpolar star as it crosses the meridian, then the latitude is

$$\phi = h \pm p$$

in which the sign preceding p is positive or negative according to whether the star is at lower or upper culmination. By this method the latitude of a station is determined by measuring the altitude of Polaris when either at upper or lower culmination, and by applying to this altitude, corrected for refraction, the star's polar distance as given in Polaris tables. Inasmuch as the star is apparently traveling in a horizontal line when at either of these two positions, it is not essential to the accuracy of the latitude determination that the time of culmination be found, but in any case it facilitates the work of observing if the approximate time is known. For all ordinary observations the time of culmination may be found with sufficient accuracy from Table IV.

Further, it is not essential that the altitude of the star be observed at the instant it crosses the meridian. For some minutes before and after culmination the star travels in so nearly a horizontal line that with the ordinary transit, vertical movement can not be detected. Within the period 6^{m} before to 6^{m} after culmination, the maximum change in altitude is only $01''$, and within a half-hour of culmination the change in altitude is only about $\frac{1}{2}'$.

The procedure to be employed in making an observation depends somewhat upon the precision with which the latitude is to be determined, and the accuracy with which time is known. For an observation with the ordinary transit having a vertical circle reading to minutes, when the watch time or the longitude of the place may be in doubt by a few minutes, the time of culmination at the given station is roughly determined by Table IV of this book, or by use of an ephemeris as described in the preceding article. A few minutes before the estimated time of culmination the transit is set up and very carefully leveled. (As a final test the telescope bubble should remain centered as the transit is revolved about the vertical axis.) The star is found with the naked eye by noting its position with respect to the neighboring constellations shown in Fig. 303. The objective is focused for a star. If the latitude is approximately known, its estimated value, plus or minus the star's polar distance, is set off on the vertical circle, and the telescope is sighted at Polaris. When it has been brought within the field of view, the cross-hairs are illuminated and the star is continually bisected with the horizontal cross-hair. When after an interval of 3 or 4 min., the star no longer appears to move away from the hair but moves directly along it, Polaris is practically at culmination. The vertical angle is read with dispatch, the transit is carefully releveled, the telescope is plunged, and a second observation on the star is taken with the telescope in the reversed position. The mean of the two observed altitudes, corrected for refraction (Table II), is taken as the true altitude of the star. The polar distance can be found approximately by Table VIII of this book where there is given the mean polar distance for each year, or more exactly for the given date from any ephemeris giving declinations of Polaris for days of the year in which the observation occurs. Finally the latitude is computed by applying to the true altitude, the polar distance with proper sign.

305. Azimuth by Observation on Polaris at Elongation.—As shown in Art. 272, the azimuth of any star when at elongation is given by the expression

$$\sin Z = \frac{\sin p}{\cos \phi}$$

in which Z is the angle east or west of north according as the star is at eastern or western elongation, p is the star's polar distance, and ϕ is the latitude of the place. By this method, it is necessary that the latitude of the place and the polar distance of the star be known, in order that the azimuth of the star may be computed. In the field, the direction of Polaris from a given station is established by projecting a vertical plane from the star to the earth at the time of either eastern or western elongation. The terrestrial line thus established has the same azimuth as does the star at the given position, hence the azimuth of any connecting line can be found if the horizontal angle between the two lines is measured. The star's polar distance is found approximately by Table VIII or more exactly by the *American Ephemeris* or other ephemeris giving values of the declination of Polaris for the days of the year in which the observation is made. The latitude is determined by observation, as explained in preceding articles. Inasmuch as the star is apparently traveling in a vertical line when at either elongation, it is not essential to the accuracy of azimuth determination that the time of elongation be found, but in any case it facilitates the work of observing if the approximate time is known. For all ordinary observations, the time of elongation may be found with sufficient accuracy from Table IV of this book. The time of elongation may be determined with greater accuracy from an ephemeris for the current year.

Further, it is not essential that the direction to the star be observed at the instant of elongation. For some minutes before and after elongation the star travels in so nearly a vertical line that with the ordinary transit, horizontal movement can not be detected. For latitudes of the United States, within the period 4^m before elongation to 4^m after elongation, the maximum change in the azimuth of Polaris is less than 01″ and within 10^m of elongation the change in azimuth is not more than 06″.

The procedure to be followed depends somewhat upon the precision with which azimuth is to be determined and the accuracy with which the time of elongation is known. When the watch time or the longitude of the place may be in doubt by a few minutes, the standard time of elongation is roughly determined by Table IV. A few minutes before the estimated time of elongation the transit is set up over a given point and very carefully leveled. The telescope is focused for a star, the latitude of the place is laid off on the vertical circle to facilitate finding the star, and the transit is revolved about the vertical axis until Polaris comes within the field of view. The transit is clamped, the cross-hairs are illuminated, and the star is continuously observed until during a period of 2 or 3 min. Polaris no longer appears to move horizontally but moves vertically directly along the cross-hair.

The telescope is depressed, and the position of the line of sight is marked on a stake or other reference monument 300 ft. or more away. The telescope is then plunged and another sight is taken upon Polaris. The line of sight is again depressed, and a second point is set on the stake beside the first. Usually the transit is releveled and another set of observations such as just outlined is made, so that there are at least four points on the stake beneath the star.

By daylight the mean of the points of each set is found, and the point marking the average of these is established on the stake. The line joining the occupied station with the established mean point defines the direction of Polaris when at elongation. Its azimuth is either computed by the expression given in the early part of this article, or is found directly from tables. It is given within a few seconds by Table V of this book.

The precision of azimuth determination by this method necessarily depends upon the quality of the instrument, the care and skill of the observer, and the number of observations, but for the procedure described, under ordinary conditions the error should not exceed 10″. It should be noted that a given error in latitude produces a relatively small error in azimuth. For latitudes of the northern part of the United States, an error of 01′ in latitude produces an error of about 02″ in azimuth, and for lower latitudes the effect is less. For reasons explained in Art. 206, if precise observations are to be obtained, it is important that the transit be leveled with great care, and in order that the error may be of an accidental rather than a systematic nature, the instrument should be releveled at least for each set of two observations. When the transit is equipped with a striding level for the horizontal axis, it should be centered prior to each sight upon the star.

306. Azimuth by Observation on Polaris at Any Time.—Although elongation is the most favorable time for accurate determination of the azimuth of polaris, it is often inconvenient or impossible to view the star when in this position. Under these circumstances, if the standard time and the longitude of the place are accurately known, the hour angle of the star at any instant can be found and the azimuth of the star at any instant can be determined, by the expression

$$\tan Z = \frac{\sin t}{\cos \phi \tan \delta - \sin \phi \cos t}$$

in which Z is the azimuth east or west of north, and t is the hour angle reckoned from 0° to 180° before or after upper culmination. It should be noted that for angles between 90° and 180° the sign of $\cos t$ is negative. Table VII gives the azimuths for a declination of 88°51′ and various hour angles. By interpolation the azimuth for any hour angle and declination can be found.

The field observation consists of measuring the horizontal angle between a terrestrial mark and the position of the star. For a rough determination correct say within ½′, a single set of two observations, one with the telescope direct and the other with it reversed, should be taken, the time of the passing of the star across the vertical cross-hair at each setting being noted. The hour angle of the star is then found for the mean of the two times of observation, and the azimuth of the star (for this hour angle and the proper declination) is computed by means of the preceding equation or is found in Table VII. This azimuth, combined with the mean of the two observed horizontal angles, gives the azimuth of the reference line.

307. To find the hour angle of Polaris at any observed watch time, the watch is compared with a timepiece keeping correct standard time, and the observed time is corrected accordingly. The correct standard time is changed to local mean time by adding or subtracting the difference in longitude expressed in hours between the place of observation and the standard meridian (meridian where standard time is also local mean time), adding, if the place of observation is east of the meridian and subtracting, if west. The local mean time of the culmination nearest the time of observation is determined from the ephemeris or is found with less precision by Table IV, this book. The difference between this value and the local mean time of observation gives the mean solar time interval before or after upper culmination. For reasons previously explained (Art. 278), Polaris is gaining on the mean sun at practically the same rate that sidereal time is gaining on mean solar time. Since in a mean solar hour there are $60^m + 9^s.856$ of sidereal time, it follows that the hour angle of the star expressed in time is greater than the mean solar time interval by $9^s.856$ or nearly 10^s per hour.

Example 1: What is the hour angle of Polaris Dec. 8, 1927 at $10^h30^m15^s$ p.m. Eastern standard time at a place whose longitude is $5^h15^m45^s$W? The standard meridian is 5^hW. The place is west of the standard meridian and hence local time is slower than E.S.T. by the difference in longitude. The local mean time of observation is therefore

$$10^h30^m15^s - 15^m45^s = 10^h14^m30^s \text{ p.m.}$$

$$\underline{12^h}$$

$$\text{The civil time from } 0^h = 22^h14^m30^s.$$

By Art. 303, the local civil time of U.C.
is found to be $= 20^h28^m59^s$
Mean time interval since U.C. $= 1^h45^m31^s = 1^h.76$
Gain of sidereal on mean $= 1.76 \times 9.86 = 17^s$
Hour angle of Polaris $= 1^h45^m48^s$ west of meridian.

Example 2 shows the use of Table VII for finding the azimuth of Polaris.

Example 2: By use of Table VII find the azimuth of Polaris for the observation of Example 1 and latitude 50°N.

From ephemeris the declination of the star for this date (Dec. 8, 1927) is 88°55′15″.

For $t = 1^h45^m$ and $\delta = 88°51'$, $Z = 0°48'31''$

Change in Z in $48^s = \dfrac{48 \times 378''}{60 \times 15} = 20''$

Azimuth for declination of 88°51′ $= 0°48'51''$

Change in Z for change in $\delta = 4.25 \times 43'' = 03'03''$

Azimuth west of North $= 0°45'48''$

In *Ephemeris of the Sun and Polaris*, General Land Office, azimuths of Polaris at all hour angles are given with argument in mean time interval before or after upper culmination. When using this table it is unnecessary to determine the actual hour angle of the star.

308. Observations on Other Stars.—Latitude and azimuth by observation on any circumpolar star can be determined by methods identical with those for Polaris. The other stars near the pole are of lesser magnitude than Polaris and are therefore not as readily identified, but if the approximate direction of the meridian is known and the hour angle and declination of the star are known, the approximate altitude can be laid off and the telescope can be pointed so that the star will come within the field of view.

Latitude, azimuth, longitude, and time can be determined by observation on stars distant from the pole by methods similar to those described for the sun.

309. Problems.

1. In connection with a solar observation, sights are taken to an index mark in the direction of the sun. The vertical angles to the mark are −3°17′00″ with telescope direct and −3°18′30″ with telescope reversed. With the telescope still in the reversed position, a sight is taken to the sun for which the observed vertical angle is +36°02′30″. Correct the observed angle for index error of the vertical circle.

2. Find the apparent declination of the sun for the instant of 9^h0^m Central standard time on February 15 of the current year, using an ephemeris giving values of 0^h Greenwich civil time.

3. Find the apparent declination of the sun for the instant of local apparent noon at a place whose longitude is $5^h52^m54^s.1$W for the date of July 21 of the current year, using an ephemeris giving values for Greenwich apparent noon.

4. Find the apparent declination of the sun at 3^h18^m Pacific standard time on November 4 of the current year, using an ephemeris giving values for Greenwich apparent noon, and taking into consideration the equation of time. Determine the effect of neglecting the equation of time.

5. The observed altitude of the lower limb of the sun as it crosses the meridian at a given place is 55°31′30″. The observation is made at $11^h34^m20^s$ a.m., Eastern standard time on May 16 of the current year. Calculate the latitude of the place.

6. The observed altitude of the upper limb of the sun as it crosses the meridian at a given place is 52°13′. The longitude of the place is 7^h32^mW. The date is March 4 of the current year. What is the latitude of the place?

7. At a place whose latitude is 41°58′10″N the observed altitude of the sun's center at 3^h12^m p.m. Central standard time on October 21 of the current year is 20°04′30″. The horizontal angle (measured on the azimuth circle) from a reference line to the sun is 81°32′20″. The

temperature is 40°F. What is the azimuth (measured from south) of the sun at the given instant and what is the azimuth of the reference line? Compute the azimuth of the sun by Eq. (3), Art. 269.

8. On August 1 of the current year the observed altitude of the sun at a given place is 30°51′45″ at $7^h42^m20^s$ a.m. local apparent time. The latitude of the place is 37°18′20″N and the longitude is 102°17′30″W. The temperature is 75°F. The horizontal angle (measured clockwise) from reference line to sun is 89°39′15″. What is the azimuth of the sun measured from north, and what is the azimuth of the reference line? Compute the azimuth of the sun by Eq. (1), Art. 269.

9. Compute the changes in azimuth of sun due to a 01′ change in latitude, in declination, and in altitude for latitude 50°, declination 0°, and altitudes of 15° and 30°. Use Eq. (1), Art. 269, as a basis for calculations.

10. Same as problem 9 but for latitude 30°.

11. When an azimuth observation is taken on the sun, the vertical axis of the transit is inclined 01′ to the true vertical, the inclination being in a plane normal to the sight plane when the line of sight is directed towards the sun. The altitude of the sun is 60°. Due to this inclination, what error will be introduced in the horizontal angle from reference line to sun?

12. At Orono, Maine, on December 5 of the current year the sun's center is observed to cross the meridian at a watch time of $11^h24^m21^s$. The longitude of the place is $4^h34^m40^s.3$W. What is the watch correction to give local mean time? What is the watch correction to give Eastern standard time?

13. At Denver, Colorado, on May 5 of the current year the sun's eastern limb crosses the meridian at a watch time of $11^h57^m48^s$. The longitude of the place is $6^h59^m47^s.7$. What is the watch correction to give local mean time? What is the watch correction to give Mountain standard time?

14. At a given place the center of the true sun crosses the meridian at $11^h41^m37^s$ a.m. Pacific standard time on September 15 of the current year. What is the longitude of the place?

15. Compute the declination settings for a solar attachment at local apparent times of 1^h, 2^h, 3^h, and 4^h, after noon on October 13 of the current year for a place whose latitude is 59°06′N and whose longitude is 118°36′45″W.

16. On September 7 of a given year, upper culmination of Polaris at the meridian of Greenwich occurs at $2^h35^m29^s$ Greenwich civil time. What is the Eastern standard time of upper culmination on September 10 of the same year at a place whose longitude is 78°30′15″W?

17. On January 12 of the current year the observed altitude of Polaris at upper culmination at a given place is 44°36′25″. The temperature is 15°F. What is the latitude of the place?

18. From Table IV find the Central standard time of upper culmination of Polaris on December 7, 1928, at Des Moines, Iowa, (longitude $6^h14^m30^s.6$W).

19. The altitude of Polaris is observed 20^m after the time of upper culmination and found to be 48°32′20″. The polar distance is 1°04′35″. What is the latitude of the place?

20. Compute the azimuth and hour angle of Polaris when at elongation, the polar distance being 1°04′35″ and the latitude of the place being 43°00′49″N.

21. What is the time of western elongation of Polaris at a given place when upper culmination occurs at $2^h15^m20^s$ p.m. Eastern standard time and the latitude is 42°22′47″.6?

22. By Table IV find the Pacific standard time of eastern elongation of Polaris on August 18 of the current year for latitude 37°52′24″N and longitude 8^h9^m3W.

23. By Table V find the azimuth of Polaris when at elongation on August 18 of the current year for a place whose latitude is 37°52′24″N.

24. By Table V*a*, determine the azimuth correction to be applied to an observation on Polaris 15^m after elongation to reduce to elongation, the azimuth at elongation being 1°34′12″. Compute the corresponding perpendicular offset at the reference monument beneath the star when the monument is 600 ft. from the station occupied.

25. At a given place upper culmination of Polaris occurs at $3^h15^m20^s$ p.m. Central standard time on a given date. On the same date an azimuth observation is made at $7^h0^m20^s$ p.m. The latitude of the place is 41°15′30″N and the polar distance of the star is 1°04′12″. Compute the hour angle and azimuth of the star. Check the azimuth by Table VII.

CHAPTER XVII

STADIA SURVEYING

310. Remarks.—The stadia method of measuring distances is adapted to a wide variety of conditions. It is extensively employed on topographic and hydrographic surveys and other surveys conducted for the purpose of securing data for the plotting of maps. It is far more rapid than chaining, and under certain conditions is as accurate. It is a very useful means of checking more precise measurements.

In this chapter the principles of the stadia are explained, and stadia surveys are described.

311. The Stadia Method.—The equipment for stadia measurements consists of a telescope with two horizontal hairs, called stadia hairs, and a graduated rod, called a stadia rod. The telescopes are furnished with stadia hairs in addition to the regular cross-hairs, one stadia hair being above and the other being an equal distance below the horizontal cross-hair. The rod is usually graduated in decimals of a foot.

The process of taking a stadia measurement consists of observing through the telescope the apparent interval between the two stadia hairs, the rod being held in a vertical position. The interval thus determined, called the *stadia interval* or *stadia reading,* is a direct function of the distance from the instrument to the rod, as will shortly be demonstrated. Thus, the distance from the instrument to any given point is determined by observing the stadia interval on the rod held at the point.

312. Stadia Hairs.—Stadia hairs are usually mounted on the same ring and in the same plane as the horizontal and vertical cross-hairs. Under these conditions the stadia hairs are not adjustable with respect to one another and hence the distance between hairs remains unchanged. Both stadia and cross-hairs are simultaneously visible and in focus.

313. Stadia Rods.—Any leveling rod of the self-reading type may be used as a stadia rod, but the common leveling rod graduated in hundredths of feet is suitable only for short sights, say less than 400 ft. For longer sights a rod with heavier graduations is necessary. For stadia work alone, rods of the type shown in Fig. 313 are suitable, the finer graduations being usually omitted and numbers indicating feet and tenths often not being shown.

For general stadia work where the length of sight may be 1,500 ft. or greater, the rods are usually 3 or 4 in. wide and 10 to 15 ft. long, and the finest division is 0.05 ft.

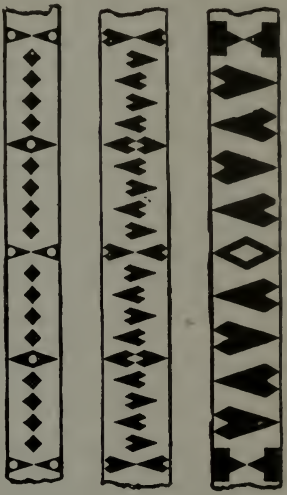


Fig. 313.—Stadia rods.

314. Principle of the Stadia.—Figure 314 illustrates the principle upon which the stadia method is based. In the figure the line of sight of the telescope is horizontal and the stadia rod is vertical. The stadia hairs are indicated by the points a and b, the distance between the hairs being i. The apparent positions of the hairs on the rod are A and B, and the interval apparently intercepted by the stadia hairs on the rod is s.

In optics it is shown that a ray of light passing through the optical center of a lens remains undeviated in direction, and further, that rays which are parallel on one side of the lens are all brought to a focus at a fixed point on the optical axis. This point is called the principal focus, and its distance from the optical center is called the *focal length* of the lens.

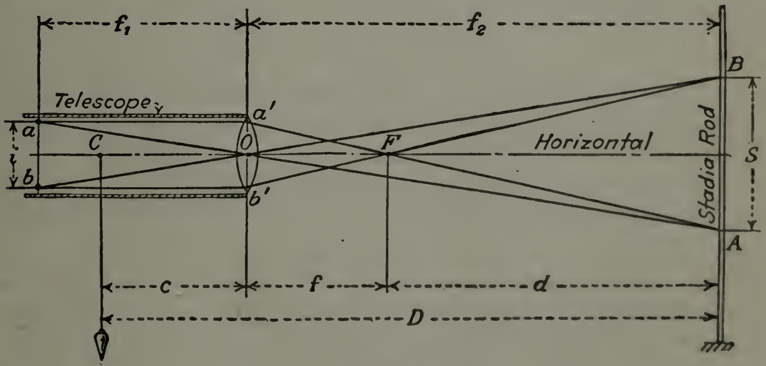


Fig. 314.

In the figure imagine that aa' and bb' are parallel rays emanating from the stadia hairs a and b. Then F is the principal focus through which these rays pass after emerging from the objective, f is the focal length of the objective, and the emerging rays take the positions $a'FA$

and $b'FB$. Also imagine that aOA and bOB are rays emanating respectively from a and b which pass undeviated through the optical center O.

Since all rays from a given point are brought to a focus on the opposite side of the objective, it follows that if rays from a passing through O and F are brought to a focus at A, the reverse is true, namely that rays from A which pass through F and O are brought to a focus at a. These two points a and A, or any other corresponding points as b and B, are called *conjugate foci* and their distances from the optical center of the objective measured along the optical axis, designated as f_1 and f_2 in the figure, are called the *conjugate focal distances.*

Since $ab = a'b'$, by similar triangles

$$\frac{f}{i} = \frac{d}{s}$$

Hence the horizontal distance from the principal focus F to the rod is $d = \frac{f}{i}s = Ks$, in which $K = \frac{f}{i}$ is a coefficient called the *stadia interval factor*, which for a particular instrument is a constant so long as optical conditions remain unchanged. Thus, for a horizontal sight it is clear that, as measured from the principal focus (the principal focus being at a fixed distance in front of the objective), the horizontal distance to the rod is obtained by multiplying the stadia interval factor by the stadia reading.

The horizontal distance from the center of the instrument to the rod is then

$$D = Ks + (f + c) \tag{1}$$

in which f is a constant for a given instrument and c, though a variable depending upon the position of the objective, may for all practical purposes be considered a constant. This expression is employed in transforming stadia readings to horizontal distances when sights are horizontal and the rod is held vertically.

315. Stadia Constants.—The value of $(f + c)$ is usually determined by the manufacturer and is placed on the inside of the instrument box. The focal distance f may be determined with all necessary accuracy by focusing the objective on a distant point and then measuring the distance from the cross-hair ring to the objective. Similarly a mean value of c can be determined by measuring the distance from the vertical axis to the objective when the latter is focused for an average length of sight. Usually $(f + c)$ is nearly 1 ft., and under ordinary circumstances may be considered as 1 ft. without error of consequence.

316. Interval Factor.—As before stated, the nominal value of the stadia interval factor $K = \frac{f}{i}$ is usually 100. With an instrument having fixed stadia hairs the interval factor may be determined by observation. The usual procedure is to set up the instrument in a position where a horizontal sight can be obtained, and with a tape to lay off from a point distant $(f + c)$ in front of the instrument, distances of 100 ft., 200 ft., etc., up to perhaps 1,000 ft., stakes being set at these points. The stadia rod is then held on each of the stakes thus established, and the stadia interval is read. The interval factor for each distance is obtained by dividing the distance from the principal focus ($(f + c)$ in front of the instrument) to the stake on which the rod was held, by the corresponding stadia interval read on the rod.

317. Inclined Sights.—The principle of the stadia was explained in Art. 314 and there was derived the expression

$$D = \frac{f}{i}s + (f + c)$$

giving the horizontal distance from the center of the instrument to the rod when the line of sight is horizontal and the rod is held vertically. In stadia surveying horizontal sights are the exception rather than the rule, and, in general, where inclined sights are taken, not only are horizontal distances determined, but vertical distances or differences in elevation as well. The problem therefore resolves itself into finding the horizontal and vertical projections of an inclined line of sight, the terminals of which are the center of the instrument and a point on the rod. For convenience in field operations the rod is always held vertically.

Figure 317 illustrates an inclined line of sight, AB being the stadia interval on the rod in its vertical position and $A'B'$ being the corresponding projection normal to the line of sight. Evidently the inclined length of the line of sight from center of instrument is

$$D_i = \frac{f}{i} \cdot A'B' + (f + c) \qquad (1)$$

For all practical purposes the angles at A' and B' may be assumed to be 90°. Then letting $AB = s$, $A'B' = s \cos \alpha$. Making this substitution in Eq. (1), and letting $K = \frac{f}{i}$, the inclined distance is

$$D_i = Ks \cos \alpha + (f + c) \qquad (2)$$

The horizontal component of this inclined distance is

$$H = Ks \cos^2 \alpha + (f + c) \cos \alpha \qquad (3)$$

which is the general expression for determining the horizontal distance from center of instrument to rod, when the line of sight is inclined.

The vertical component of the inclined distance is

$$V = Ks \cos \alpha \sin \alpha + (f + c) \sin \alpha$$

or

$$V = \tfrac{1}{2} Ks \sin 2\alpha + (f + c) \sin \alpha \qquad (4)$$

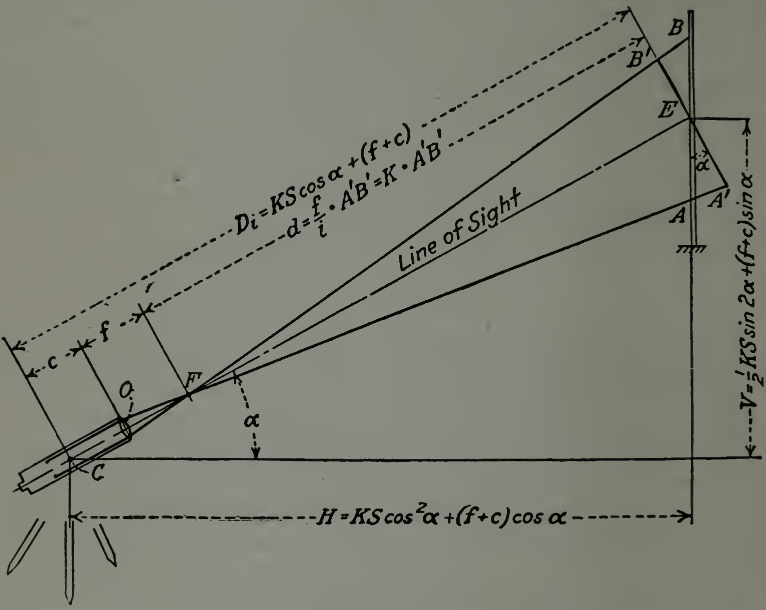

B
B'
E
A'
A
α
$Di = KS\cos\alpha + (f+c)$
$d = \frac{f}{i} \cdot A'B' = K \cdot A'B'$
Line of Sight
c
f
O
F
C
α
$V = \frac{1}{2} KS \sin 2\alpha + (f+c) \sin\alpha$
$H = KS\cos^2\alpha + (f+c)\cos\alpha$

FIG. 317.

which is the general expression for the difference in elevation between the center of the instrument and the point where the line of sight cuts the rod. Equations (3) and (4) are known as the stadia formulas for inclined sights.

318. Stadia Reductions.—In practice horizontal distances and differences in elevation are not ordinarily computed by actually solving the two stadia formulas, but are obtained by the use of tables, diagrams, slide rules, etc., which are based upon these two equations. Table IX gives for each 02′ of vertical angle up to 30°, the horizontal distances and differences in elevation for $Ks = 100$ ft. For any other value of Ks, the tabular quantities are multiplied by the value in hundreds of feet. The table also gives the horizontal distances and differences in elevation for several values of $(f + c)$ (indicated as c in table).

By far the most rapid and convenient means of calculating horizontal distances and differences in elevation is by means of a stadia slide rule of which there are several patterns. The type which for ordinary use seems preferable to all others is constructed as is the ordinary slide rule, except that on the slide are given values of $\cos^2 \alpha$ and $\frac{1}{2} \sin 2\alpha$, these quantities being plotted to a logarithmic scale.

319. Permissible Approximations.—More approximate forms of the stadia formulas than those just derived are sufficiently precise for most stadia work. Generally distances are calculated to feet and elevations are calculated to tenths of feet.

Under these conditions for side shots where vertical angles are less than 3°, Eq. (3) for horizontal distances may properly be reduced to the form

$$H = Ks + (f + c) \tag{5}$$

Owing to unequal refraction and accidental inclination of the rod, observed stadia intervals are, in general, slightly too large (see Art. 327). To offset the systematic errors from these sources the $(f + c)$ constant is generally neglected except on surveys of more than ordinary precision.

Hence in any ordinary case Eq. (3) may with sufficient accuracy be expressed in the form

$$H = Ks \cos^2 \alpha \tag{6}$$

Also for small angles $(f + c) \sin \alpha$ is approximately equal to $\frac{1}{2}(f + c) \sin 2\alpha$. Hence Eq. (4) may generally be expressed in the form

$$V = \tfrac{1}{2}[Ks + (f + c)] \sin 2\alpha \tag{7}$$

Differences in elevation obtained by using this approximate formula will be correct to the nearest 0.1 ft. for angles of less than 25°.

The determination of horizontal distances and differences in elevation by algebraic, graphical, or mechanical methods is considerably simplified by making Eqs. (6) and (7) the basis of calculations; hence these forms of the stadia formulae are the ones generally employed. When K is 100 the common practice is mentally to multiply the stadia interval by 100 at the time of observation, and to record this value in the field notebook. This distance is often called the stadia distance. Thus, if the stadia interval were 7.37 ft., the stadia distance recorded would be 737 ft.

320. Uses of the Stadia.—Uses of the stadia are as follows:

1. In differential leveling the backsight and foresight distances are conveniently equalized or balanced if the level is equipped with stadia hairs.

2. In profile leveling or cross-sectioning the stadia is a convenient means of finding distances from level to points where rod readings are taken.

3. In rough trigonometric or indirect leveling with the transit, the stadia method is more rapid than any other. The line of trigonometric levels is run as described in Art. 80, except that stadia intervals are observed and differences in elevation are calculated by stadia formula. Stadia trigonometric leveling is described in Art. 322.

4. In transit surveys of low precision where only horizontal angles and distances are required, stadia is more rapid than chaining. It may be used either in running traverse lines or in locating details from such lines. Stadia intervals are observed as each point is sighted. Horizontal angles are measured, but vertical angles are observed only when of sufficient magnitude to make the horizontal distance appreciably different from the stadia distance (say when greater than 3°) and then are estimated without reading the vernier, this being sufficiently accurate for the determination of horizontal distances. The transit-stadia method of running such surveys is described in Art. 323.

5. On transit surveys of low precision, particularly topographic surveys, where not only the relative positions of points in a horizontal plane, but also the elevations of these points are desired. Both horizontal and vertical angles are measured and the stadia interval is observed as each point is sighted. The transit-stadia method of making observations when both horizontal position and elevation are desired is described in Art. 323.

6. Where the plane table is used, stadia observations are made with the telescope alidade in the same manner as with the transit, but calculations of horizontal distances and differences in elevation are made in the field and are immediately plotted instead of being recorded in the form of notes.

321. Observation of Stadia Interval.—On transit or plane-table surveys the stadia interval is determined by setting the lower hair on a foot mark and reading the position of the upper hair. The stadia interval is then mentally calculated with much less chance of error than would be the case if the lower hair were allowed to take a random position on the rod. When the vertical angle is taken to a given mark on the rod, the corresponding stadia interval is observed with the lower hair on the foot mark which renders a minimum displacement of the horizontal cross-hair from the mark to which the vertical angle is referred.

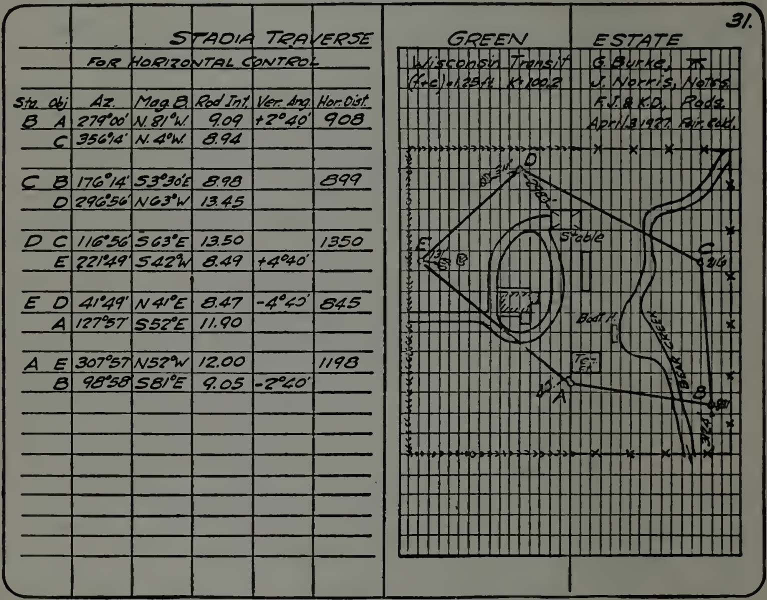
322. Indirect Leveling by Stadia.—Where the required precision is low and the country is rolling or rough, the stadia method of indirect leveling offers a rapid means of determining differences in elevation. The instrument used is the engineers' transit, which should preferably be provided with a sensitive control level for the vertical vernier in order that index error may be readily eliminated. With the ordinary transit having a vertical circle reading to single minutes, differences in elevation are usually calculated only to the nearest 0.1 ft. In general, the average length of sight in stadia surveying is considerably greater than in differential leveling.

In running a line of levels by this method, the transit is set up in a convenient location. A backsight is taken on the rod held at the initial benchmark, the stadia interval being observed first and then the horizontal cross-hair being set on some arbitrarily chosen rod reading, and the vertical angle to this mark being measured. A turning point is then established in advance of the transit, and similar observations are taken, the vertical angle being measured with the horizontal cross-hair set on the same index mark as before. The transit is moved to a new position in advance of the turning point, and the process is repeated. The stadia distances and vertical angles are recorded, as is also the rod reading which is used as an index when vertical angles are measured. If it is impracticable to sight at this chosen index reading, the vertical angle is measured with the line of sight directed to some other graduation, usually a whole number of feet above or below the index, and this rod reading is given in the notes.

For any set-up the difference in elevation determined from either the backsight or foresight observations is the difference in elevation between the index mark on the rod and the center of the instrument, and the algebraic sum of the backsight and foresight difference is the total difference in elevation between the two positions of the index mark. It is clear that so long as the index mark to which vertical angles are taken is unchanged, the difference in elevation determined as just described gives also the difference in elevation between the two points adjacent to the instrument on which the rod is held, the actual value of the index reading being of no consequence.

323. Transit-stadia Surveying; Elevations Not Required.—For certain reconnaissance or preliminary surveys, rough surveys for the location of boundaries, and detailed surveys for maps, where only the horizontal position of objects and lines is desired, the transit-stadia method is sufficiently accurate and considerably more rapid and economical than corresponding surveys made with transit and tape. The surveying procedure depends somewhat upon the nature and purpose of the survey but in a general way parallels that which would be employed were distances measured with tape instead of by stadia. The field party for transit-stadia work consists of a transitman, usually a recorder, and sometimes one, but usually two, and occasionally three, rodmen. The field work consists of determining horizontal angles and directions by any of the methods described in Chaps. XI to XIII, and in addition the observation of stadia intervals together with approximate vertical angles when they are of sufficient magnitude to make the horizontal distances appreciably different from the observed stadia distances. Horizontal distances are expressed to the nearest foot.

When the survey consists of a traverse, it is customary to observe the stadia interval both forward and back from each set-up of the transit. In this way two independent stadia observations are made for each line or distance in the traverse. The closeness of agreement between the two values for each line is a check against mistakes, and the mean is taken as the most probable value. Figure 323 is a sample page of notes for a closed traverse.



STADIA TRAVERSE

FOR HORIZONTAL CONTROL

Sta.	Obj.	Az.	Mag. B.	Rod Int.	Ver. Ang.	Hor. Dist.
B	A	279°00′	N. 81° W.	9.09	+2°40′	908
	C	356°14′	N. 4° W.	8.94		
C	B	176°14′	S3°30′E	8.98		899
	D	296°56′	N63°W	13.45		
D	C	116°56′	S63°E	13.50		1350
	E	221°49′	S42°W	8.49	+4°40′	
E	D	41°49′	N41°E	8.47	-4°40′	845
	A	127°57′	S52°E	11.90		
A	E	307°57′	N52°W	12.00		1198
	B	98°58′	S81°E	9.05	-2°40′	

31.

GREEN ESTATE

Wisconsin Transit
(f+c)=1.25ft. K=100.2

G. Burke, ⊼
J. Norris, Notes.
F.J. & K.D., Rods.
April 3 1927. Fair, Cold.

Fig. 323.—Stadia traverse notes.

Where details are to be located by stadia measurements from transit stations, the observations may be made at the same time that the transit station is established or may be made later.

324. Transit-stadia Surveying with Vertical Angles.—This method is extensively employed on topographic and similar surveys where the elevations of points as well as their location in a horizontal plane are desired. The field procedure of locating points consists of observing directions usually by azimuths, and distances by stadia, as described in the preceding article. In addition, differences in elevation are determined either by direct leveling when it is practicable to do so, or more usually from observed vertical angles and stadia distances. The field party consists of an observer, one or more rodmen, and usually a recorder.

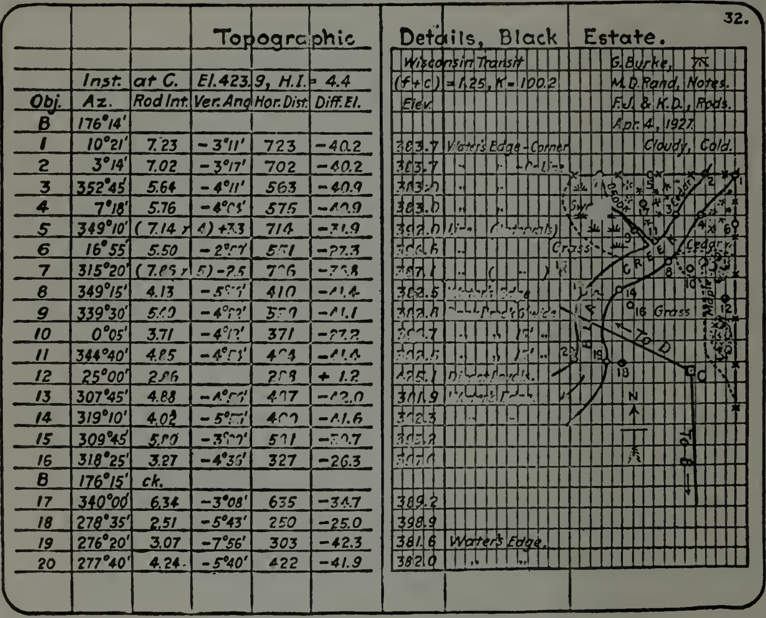
In topographic surveying this method may be employed merely for the collection of details, the horizontal and vertical control being established by other means, or it may be utilized for establishing control as well as for details.

325. In the former case the instrument is set up at a triangulation or traverse station, the elevation and position of which are known. The height of the instrument (H.I.) above the station over which it is set is measured with a rod or tape. The transit is oriented by taking a backsight to a station whose azimuth is known, this azimuth being set off on the horizontal circle. The upper motion is loosened, and sights to desired points are taken. Figure 325 is a page of notes showing observations taken from station *C* of a traverse, the elevation of the station having been previously determined as 423.9. The H.I. is 4.4 ft. The transit is oriented by sighting to *B*, the azimuth of the line *CB* being set off on the horizontal circle prior to taking the sight. In the first column are given the numbers of the side shots, their positions being shown on the sketch. In the second column are the azimuths of the several points sighted; in the third column are the rod intervals. In the fourth column are the vertical angles, and the following columns show respectively horizontal distances, differences in elevation, and elevations.

In measuring vertical angles it is customary, when practicable, to sight at a rod reading equal to the height of instrument above the station over which the transit is set. In this way, the difference in elevation between center of instrument and H.I. on the rod is the same as the difference in elevation between station over which the transit is set and the point on which the rod is held. When the line of vision is obstructed so that the H.I. can not be sighted upon, the horizontal cross-hair is set on some other graduation of the rod, usually a whole number of feet above or below the H.I., and the rod reading is recorded in the notes. When the corresponding difference in elevation is calculated, proper allowance is made for the difference between the H.I. and the recorded rod reading. Thus, in the case illustrated by the notes, had the vertical angle of $-3°17'$ for Object 2 been taken to a rod reading of 6.4 ft. instead of the H.I. = 4.4, the difference in elevation would have been $-40.2 - 2.0 = -42.2$ ft.

When the detail to be observed is at nearly the same elevation as the point over which the transit is set, there is a marked advantage in determining difference in elevation by direct leveling. The notes of Fig. 325 show that Object 12 was observed in this manner. The observed rod reading 3.2 might have been shown, but instead, the difference in elevation has been calculated by mentally subtracting 3.2 from 4.4 giving 1.2, which is recorded in the notes.

Generally where measurements are made solely for plotting a map, horizontal angles are estimated to 05′ without reading the vernier. Vertical angles are usually measured to minutes, and differences in elevation are calculated to tenths of feet. Where elevations to the nearest foot are sufficiently accurate, the vertical angles, except for long shots, may properly be read by estimation without use of the vernier.



Topographic Details, Black Estate. 32.

Inst. at C. El. 423.9, H.I. = 4.4

Obj.	Az.	Rod Int.	Ver. Ang	Hor. Dist.	Diff. El.
B	176°14′				
1	10°21′	7.23	−3°11′	723	−40.2
2	3°14′	7.02	−3°17′	702	−40.2
3	352°45′	5.64	−4°11′	563	−40.9
4	7°18′	5.76	−4°[illegible]	575	[illegible]
5	349°10′	(7.14 r	4) +3.3	714	[illegible]
6	16°55′	5.50	−2°[illegible]	[illegible]	−27.3
7	315°20′	(7.85 r	5) −2.5	[illegible]	[illegible]
8	349°15′	4.13	−5°[illegible]	410	−41.4
9	339°30′	[illegible]	−4°[illegible]	[illegible]	−41.1
10	0°05′	3.71	−4°[illegible]	371	[illegible]
11	344°40′	4.85	−4°[illegible]	[illegible]	−41.4
12	25°00′	2.86		288	+ 1.2
13	307°45′	4.88	−4°[illegible]	[illegible]	[illegible]
14	319°10′	4.02	−5°[illegible]	[illegible]	−41.6
15	309°45′	[illegible]	−3°[illegible]	[illegible]	[illegible]
16	318°25′	3.27	−4°35′	327	−26.3
B	176°15′	ck.			
17	340°00′	6.34	−3°08′	635	−34.7
18	278°35′	2.51	−5°43′	250	−25.0
19	276°20′	3.07	−7°56′	303	−42.3
20	277°40′	4.24	−5°40′	422	−41.9

Wisconsin Transit
(f + c) = 1.25, K = 100.2
Elev.

G. Burke, ⊼
M.D. Rand, Notes.
F.J. & K.D., Rods.
Apr. 4, 1927.
Cloudy, Cold.

383.7 Water's Edge - Corner
389.2
398.9
381.6 Water's Edge.
382.0

Fig. 325.—Stadia notes.

The usual procedure of observing is to sight on the rod, the lower stadia hair being set on a foot mark such that the horizontal cross-hair falls somewhere near the H.I., and to read the stadia interval. The horizontal cross-hair is then set on the H.I., the horizontal angle is read without clamping the upper motion, the instrument is rotated about the vertical axis until the vertical circle is in a convenient position for reading, and finally the vertical angle is observed.

326. Where the required precision is not high, the stadia traverse with elevations of traverse stations determined by vertical angle and stadia distance is a very rapid means of establishing horizontal and vertical control. The procedure is the same as that already described in Art. 323, except that in addition, vertical angles are observed both for the backsight and for the foresight from each station, the telescope being sighted at a rod reading equal to the height of instrument above the transit station over which the transit is set up.

327. Errors in Stadia Surveying.—Many of the errors of stadia surveying are obviously not peculiar to this work alone, but are common to all similar operations of surveying. The errors of direct leveling, most of which also have their effect upon the determination of difference in elevation by methods described in this chapter, are discussed in Art. 124. Sources of error in the measurement of horizontal angles with the transit are discussed in Arts. 206 to 208. Sources of error in calculated distances, both horizontal and vertical, based upon observed stadia intervals, are as follows:

1. *Stadia Interval Factor Not That Assumed.*—This produces a systematic error in all calculated distances, the error being proportional to the stadia interval. If the stadia hairs are fixed, the interval factor is not likely to change appreciably, but may change slightly with variations in natural conditions. When the value of the interval factor is closely determined by observations, as described in Art. 316, and the stadia measurements are taken under conditions paralleling those existing when the interval factor was determined, the error from this source may be reduced to a negligible quantity.

2. *Rod Not Standard Length.*—If the spaces on the rod are uniformly too long or too short, a systematic error, proportional to the stadia interval, is produced in each computed distance. Errors from this source may be kept within comparatively narrow limits if the rod is standardized and corrections for erroneous length are applied to observed stadia intervals. Except for stadia surveys of more than ordinary precision, errors from this source are usually of no consequence.

3. *Incorrect Stadia Interval.*—This refers to the inability of the instrumentman to observe exactly the rod interval between the stadia hairs. Under usual conditions the error is accidental in character. Errors from this source are the ones principally affecting the precision of computed distances. They may be kept to a minimum by proper focusing to eliminate parallax, by taking observations at a time of day when atmospheric conditions are favorable, and by care in observing. Where a high precision is required, stadia measurements may be taken by sighting on a rod with two targets, one

fixed, upon which the upper stadia hair is set, and the other movable, it being brought to the position of the lower stadia hair.

4. *Rod Not Plumb.*—This produces an error in the vertical angle, since in measuring vertical angles, the horizontal cross-hair is set on a rod reading equal to the instrument H.I., but this error is of small consequence. It also produces an error in the observed stadia interval, and hence in computed distances, this error for a given distance and for a given inclination of the rod being greater for large vertical angles than for small ones. It may be eliminated by using a rod equipped with a plumbing level. Where the rod is not so equipped, the error from this source is likely to be large for steeply inclined sights.

5. *Unequal Refraction.*—It has been determined by experiment that unequal refraction of light rays in layers of air close to the earth's surface introduces systematic positive errors in stadia measurements. While errors from this source are of no consequence in ordinary stadia surveying, they may be of sufficient magnitude to be of importance on the more precise surveys. The periods most favorable for equal refraction are at times when it is cloudy or during the early morning or late afternoon when the sun is shining. On precise stadia surveys where it is necessary to work under a variety of atmospheric conditions, it is proper to determine the stadia interval factor for each condition and to apply the proper factor to all observations taken under a given condition.

328. While the sources just listed produce errors in both horizontal distances and differences in elevation, errors in vertical angles also have their effect upon these computed values. From a study of the stadia formulas it is evident that errors in vertical angles, within the usual range of values, are relatively unimportant in their effect upon computed horizontal distances. On the other hand, the errors in vertical angles are relatively important in their effect upon the precision with which differences in elevation are determined. For example, an error of 01′ in any vertical angle within the usual range produces an error in elevation of nearly 0.1 ft. in a distance of 300 ft.

Under normal conditions where vertical angles are small, the effect of observational errors in vertical angles may be expected to be somewhere near the same as that due to observational errors in stadia intervals; but when vertical angles are large the errors in observed stadia intervals are likely to be the ones which most affect the precision of computed differences in elevation. Thus to maintain a given precision in computed values of difference in elevation, stadia intervals must be observed with much greater refinement when vertical angles are large than when they are small.

329. Precision of Stadia Surveying.—In transit-stadia surveying, say for a traverse, where horizontal and vertical angles and stadia intervals are observed, the precision with which the relative position of points, in plan and elevation, is determined is dependent upon errors from each of these three sources. It has been shown that the principal errors of both horizontal and vertical angular measurements are likely to be accidental in character. The principal error in stadia measurements may be either systematic or compensating, depending upon conditions. If the stadia rod is standardized and proper corrections are applied for erroneous length, and if the interval factor is accurately determined and the rod is carefully plumbed, the principal error is that of observing the stadia interval. Under such circumstances, the errors which mainly control the computed position of points, both in plan and elevation, are accidental in character, and hence it is to be expected that these errors will in the long run tend to vary as the square root of the distance. This marks one of the important advantages of surveying with the transit and stadia over surveying with the transit and tape, and it explains why the precision obtained on extensive transit-stadia surveys often compares favorably with the precision obtained on similar surveys for which measurements are made with the transit and tape.

Unless care is taken to eliminate the systematic errors mentioned, the resultant error in plan is likely to vary more nearly as the distance, as is the case with tape measurements. Under these circumstances the advantage just mentioned is lost and the transit-stadia survey, regardless of its extent, would in general be considerably less precise than the surveys made with transit and tape. If observed vertical angles are small, large systematic errors in stadia observations or reductions will have comparatively small effect upon computed differences in elevation, and the resultant error may be expected to vary more nearly as the square root of the distance; on the other hand, if vertical angles are large, systematic stadia errors are likely to be the chief contributors to the resultant errors in elevation, and the error may be expected to vary directly with the distance.

The factors which influence the precision of stadia surveying are numerous. The quality of the instrument, the accuracy of graduation of the rod, the character of country, the skill of observer, the care with which the rod is held, the length of sight, and the condition of the weather, all affect the results. Following are estimates which it is believed are fairly representative of several classes of stadia work, these estimates being based upon the results secured on surveys run under a variety of conditions. In accepting these estimates it should be borne in mind that the conditions surrounding no two surveys are alike and that a definite statement of the precision that can be obtained with a given course of procedure is impossible.

1. For side shots where a single observation is taken with sights steeply inclined and no particular care taken to insure the rod being plumb, horizontal distances may have a precision of less than 1/100 and individual differences in elevation may be in error 2 ft. or more per 1000 ft. of horizontal distance.

2. Under the same conditions as in (1) but with small vertical angles and reasonable care used in approximately plumbing the rod and with lengths of sights not less than 200 ft. nor more than 1,500 ft., the precision of horizontal distances should, with ordinary surveying equipment and ordinary skill on the part of the transitman, be not less than 1/200; if vertical angles are observed to minutes, differences in elevation should not be in error more than 0.3 ft. per 1,000 ft. of horizontal distance and if vertical angles are estimated to 05′, the error in difference in elevation need not exceed 1 ft. per 1,000 ft. of horizontal distance.

3. For a rapid stadia traverse of considerable length run through rough country with numerous long sights, angles being measured to minutes but without special precaution to eliminate systematic errors, the error of closure may be as great as 25 ft. per mile in plan and 3 ft. per mile in elevation.

4. For conditions as in (3) but country fairly level so that all vertical angles are small, the error of closure ought not to exceed 15 ft. per mile in plan and 0.5 ft. times square root of distance in miles in elevation.

5. For rough country with vertical angles up to 15°, angles to minutes, rod standardized, rod plumbed with level, sights limited to 1,500 ft. and taken forward and back from each transit station, and interval factor accurately determined, the error may be less than 15 ft. times the square root of the distance in miles in plan and 1 ft. times the square root of the distance in miles in elevation.

6. For the same conditions as in (5), but level country so that all vertical angles are small, the error may be as small as 6 ft. times the square root of the distance in miles in plan and 0.3 ft. times the square root of the distance in miles in elevation.

7. For conditions as in (5) but stadia intervals determined by use of a target rod with two targets and observations made during cloudy days, the error of closure in plan should not exceed 4 ft. times the square root of distance in miles.

330. Problems.

1. To determine the stadia interval factor, a transit is set up the distance $(f + c)$ in the rear of the zero end of a level base line 800 ft.

long, the base line being marked by stakes set every 100 ft. A rod is then held at successive stations along the base line. The stadia interval and each half-interval observed at each position of the rod are tabulated below.

Calculate the lower, upper, and full interval factor for each distance, and find the mean values.

Distance $-(f+c)$, feet	Lower interval, feet	Upper interval, feet	Full interval, feet
100	0.49	0.50	0.99
200	0.98	0.99	1.97
300	1.47	1.48	2.95
400	1.97	1.98	3.96
500	2.46	2.47	4.94
600	2.95	2.97	5.92
700	3.45	3.47	6.91
800	3.94	3.96	7.89

2. A transit for which the stadia interval factor is 98.5 and $(f+c)$ is 1 ft. is used in making the following observations.

Observation	Stadia interval	Vertical angle
a	12.15	+ 3°18′
b	10.92	+11°31′

By means of Eqs. (3) and (4), Art. 317, compute the horizontal distances and differences in elevation. Check the computed values by means of Table IX.

3. The following observations are taken with a transit for which the interval factor is 100.0 and $(f+c)$ is 1 ft.

Observation	Stadia interval	Vertical angle
a	10.00	+ 0°30′
b	10.00	+10°00′
c	10.00	+25°00′

By means of Eqs. (3) and (4), Art. 317, compute the horizontal distances and differences in elevation. By Eqs. (6) and (7), Art. 319, determine the same quantities and note the errors introduced by the approximations.

4. If the observations of problem 3 were taken with a 12-ft. rod which was unknowingly 0.5 ft. out of plumb with top leaning towards the transit, what would be the amount and sign of the error introduced in each computed horizontal distance and difference in elevation?

5. If the observations of problem 3 were taken with the top of the rod leaning 0.5 ft. away from the transit, what would be the amount and sign of the error introduced in each computed horizontal distance and difference in elevation?

6. What conclusions may be drawn from the results of problems 4 and 5?

7. For the observations of problem 3, if each vertical angle contains an error of 01′ what error will be introduced in each computed horizontal distance and difference in elevation?

8. For the observations of problem 3, if each stadia interval is in error 0.1 ft. what error will be introduced in each computed horizontal distance and difference in elevation?

9. What conclusions may be drawn from the results of problems 7 and 8?

10. Following are the notes for a line of stadia levels:

Station	Backsight		Foresight	
	Vertical angle	Stadia interval	Vertical angle	Stadia interval
B.M.1	−4°36′	3.37		
T.P.1	−1°39′	2.95	+ 2°17′	2.74
T.P.2	−3°48′	4.11	+ 5°04′	1.47
T.P.3	−5°39′	2.07	+ 0°34′	3.47
B.M.2	+0°32′	4.76	+ 0°55′	4.16
T.P.4	+1°18′	1.42	− 7°42′	2.10
T.P.5	+2°19′	3.42	−10°08′	3.12
T.P.6	+3°43′	1.88	− 4°24′	3.61
B.M.3	....	....	− 2°00′	1.41

The elevation of B.M. 1 is 641.12. The interval factor is 100.0 and $(f + c) = 1.2$ ft. By use of Table IX calculate the elevations of all B.M.s and T.P.s.

11. Following are vertical angles and stadia measurements for a transit-stadia traverse. The elevation of Station *A* is 301.7 ft. The stadia interval factor is 100, and $(f + c) = 1$ ft. Calculate the horizontal lengths of the several courses and the elevations of the several transit stations.

Station	Object	Stadia interval	Vertical angle
B	*A*	4.84	−3°12′
	C	5.95	+4°41′
C	*B*	5.92	−4°40′
	D	2.46	−1°05′
D	*C*	2.47	+1°05′
	E	8.45	−0°49′
E	*D*	8.50	+0°48′
	F ~~*A*~~	4.37	+8°13′
F	*E*	4.34	−8°14′
	G	12.45	+2°22′
G	*F*	12.40	−2°21′
	H	7.18	+0°31′

CHAPTER XVIII

LAND SURVEYING—RURAL AND URBAN

331. Remarks.—Land surveying deals with the laying off or measurement of the lengths and directions of lines forming the boundaries of real or landed property. Land surveys are made for the following purposes:

1. To secure the necessary data for writing the legal description and finding the area of a piece of land, the boundaries of the property being defined by visible objects;
2. To reestablish the boundaries of a tract of which a survey has previously been made and for which the description as defined by the previous survey is known;
3. To subdivide a tract into two or more smaller units in accordance with a definite plan which predetermines the size, shape, and location of the units.

The functions of the land surveyor are to carry out field surveys as suggested above, to calculate dimensions and areas, to prepare maps showing the lengths and directions of boundary lines and areas of lands, and to write descriptions by means of which lands may be legally conveyed, by deed, from one party to another.

In the present chapter, land surveying practices as applied to both rural and urban properties are described. In the succeeding chapter methods of calculating areas are developed, and in Chap. XX the United States system of subdividing the public lands is outlined.

332. Instruments and Methods.—At the present time nearly all land surveys are run with the transit and tape, by methods already described in Chaps. VI, XII, and XIII. Distances are ordinarily measured to feet and decimals, and angles are measured to minutes or fraction thereof. On the U. S. Public Land Surveys all distances are in Gunter's or 66-ft. chains, as prescribed by law, measurements being taken with a tape graduated to read chains and links. The directions of lines are usually referred to the true meridian, and angular measurements are transformed to bearings.

Formerly the surveyors' compass and 66-ft. link chain were extensively used, particularly in rural surveying, and the lengths and directions of boundary lines contained in many old deeds are given as Gunter's chains and magnetic bearings. In retracing old surveys of this character, due allowance must be made for change in magnetic bearing since the time of the original survey.

Whenever possible, the field procedure is such that the lengths of boundary lines and the angles between boundaries are obtained by direct measurement. In general, the land survey is therefore a traverse, the transit stations being at corners of the property and the traverse lines coinciding with property lines. When obstacles render direct measurement of boundaries impossible, traverse stations are established and a traverse is run as near the property lines as is practicable. By use of the data thus collected, supplemented by measurements from traverse lines to property corners, the lengths and directions of the property lines are computed. Distances are usually measured horizontally. On urban surveys where the desired precision is high, the distances are frequently measured on the slope and are then reduced to the horizontal. Where the boundary is irregular or curved, the traverse is established in a convenient location and offsets are taken from the traverse line to points on the boundary, after which the length of the boundary is calculated.

333. Kinds of Land Surveys.—In accordance with the purposes listed in Art. 331, surveys may be divided into the following classes:

1. Original surveys, or those that are made for the purpose of measuring the unknown lengths and directions of boundaries already established and in evidence. Surveys of this character are usually of rural lands.

2. Resurveys, or those that are run for the purpose of relocating the boundaries of a tract for which a survey has previously been made. The surveyor is guided by a description of the property based upon the original survey and evidence on the ground.

3. Subdivision surveys, or those which are run for the purpose of subdividing land into more or less regular tracts according to a prearranged plan.

334. Monuments and Reference Marks.—It is customary to mark the corners of landed property by visible monuments. In the case of original surveys, many of the corners were natural objects such as trees and stones already marking the borders of a tract before the survey was actually made. In general, however, the corner monuments are established by the surveyor either to mark the intersections of boundaries already in existence or to define new boundaries. In the great majority of cases monuments thus established are of temporary character, often consisting of wooden stakes driven in the ground. Examples of markers of a more permanent character are an iron pipe or bar driven in the ground, a concrete or stone monument with drill hole, cross or metal plug marking the exact corner, a stone with identifying mark placed below the surface. Subsurface stones are commonly used for corner monuments in localities where roads follow section lines.

A corner is said to be *witnessed* or *referenced* when measurements to nearby witness marks have been completed. A corner may be witnessed as described in Art. 226. Trees, stones, and buildings are examples of objects which are often utilized as witness marks. In many large cities systems of permanent monuments are established, and to these all surveys are referred. On public land surveys, the bearing and distance from a corner to a tree are taken when possible, the tree being blazed and so marked as to identify the section on which the tree stands, the mark terminating with the letters "B.T." signifying bearing tree.

Where a corner falls in such location as to make it impossible or impracticable to establish a monument in its true position, it is customary to set a point on one or more of the boundary lines leading to the corner. A point thus established is called a *witness corner*. Everything that has been said concerning monuments at the true corners also applies to witness corners.

335. Descriptions of Rural Lands.—Nearly all of the original land grants in the older portions of the United States were of irregular shape, the boundaries often following stream and ridge lines. In the process of subdivision, units were taken here and there without much regard for regularity, and it was thought sufficient if lands were specified as bounded by natural or artificial features of the terrain and if the names of adjacent property owners were given.

336. As the country developed, land became more valuable, many of the original boundaries ceased to exist, and land litigations were numerous. It then became the general practice to determine the lengths and directions of the boundaries of land and permanently to fix the locations of corners by monuments. The lengths were ordinarily given in rods or chains and directions were expressed as bearings, usually referred to the magnetic meridian. At the present time, surveys of this character are usually made with the transit and tape, distances being recorded in feet or chains, and directions being given in true bearings computed from angle measurements. In describing a tract surveyed in this manner the lengths and bearings of the several courses are given in order. Also the objects marking the corners are made known, and in the event of any boundary being coincident with some prominent feature of the terrain, it is so stated, and the calculated area of the tract is given. When the bearings and lengths of the several sides are thus given, a tract is said to be described by *metes and bounds.* Within the limits of the precision of the original survey, it is possible to relocate the boundaries of a tract if its description by metes and bounds is available, provided at least one of the original corners can be identified and the true direction of one of the boundaries can be determined.

The following is illustrative of a description by metes and bounds typical of rural lands in the Eastern States and of isolated grants in the Western States where the subdivision of lands has been outside the rectangular system of the public land surveys:

"Beginning at an iron pipe sunk in the ground thirty-three and no tenths (33.0) feet west of the center line of the concrete highway extending from Fryburg to North Conway and twelve hundred and seventy-three and four-tenths (1,273.4) feet south of the north face of the south abutment of the bridge over Jordon Brook in the town of Redstone, New Hampshire, at the southeast corner of land now or formerly belonging to William Bancroft, and running by fence and land of said Bancroft south 89°26′ west a distance of fourteen hundred and seventy-nine and four-tenths (1,479.4) feet to a cross cut in the surface of granite ledge . . . thence turning and running along the west boundary of said road north 2°15′ west a distance of nineteen hundred and eighty-nine and two-tenths (1,989.2) feet to the point of beginning; all bearings being referred to the true meridian, and the parcel of land containing a calculated area of fifty-nine and sixty-seven hundredths (59.67) acres more or less."

337. A third type of description is employed for lands which have been divided in accordance with the rectangular system of the General Land Office. This method of subdividing the public domain is described in detail in Chap. XX.

Following is an example of the legal description of a 40-acre tract comprising a full quarter-quarter section:

"The north-east quarter of the south-west quarter of section ten (10), Township four (4) South, Range six (6) East, of the Initial Point of the Mount Diablo Meridian, containing forty (40) acres, more or less, according to the United States Survey."

For reasons which are discussed in Chap. XX, certain subdivisions called *fractional sections* do not contain the nominal acreage of that subdivision. Following is a description of a fractional half of a fractional quarter-section:

"The west fractional half of the northwest fractional quarter of Section seven (7), Township four (4) North, Range twelve (12) East of the Initial Point of the . . . Meridian, comprising lots one and two and containing sixty-three and seventy-nine hundredths (63.79) acres, according to the official plat of the United States Survey."

338. An Original Survey of Rural Land.—The need for an original survey usually arises when one person desired to transfer to another a tract of land which has not been previously surveyed but which is defined by certain natural or artificial features of the terrain.

After having been shown the boundaries of the land, the surveyor establishes monuments of one kind or another at the corners, and proceeds to run a closed transit traverse about the property, measuring the lengths of lines and the angles between intersecting lines. Where boundaries are not straight, offsets from transit line to curved boundary are measured at known intervals, and where obstructions make direct measurement along boundaries impossible, the traverse is run as close to the boundary as convenience will allow and measurements are taken from transit stations to corners of the tract, so that the length and bearing of the boundary line may be calculated. Also the direction of the true meridian should be determined, this usually being done by a solar observation. The bearings of the sides are then calculated, with respect to the true meridian.

A description of the tract, similar in arrangement to that given in Art. 336 is prepared. Usually a plat is drawn, details being shown as suggested in Art. 27. The area is computed as described in Chap. XIX. In the process of computation, the error of closure of the traverse is determined, and thus a check on the reliability of the survey is obtained. A copy of the description and a tracing of the plat are submitted to the person for whom the survey is made.

339. A Resurvey of Rural Land.—The resurvey of lands is attended with greater difficulty than is usually appreciated by those inexperienced in work of this character. Particularly is this true in the older sections of the United States where the early surveys were not of the rectangular system and were not under the control of the U. S. General Land Office. The proper relocation of old lines doubtless calls for greater ingenuity and broader experience on the part of the surveyor than any other kind of surveying.

The purpose of a resurvey is to reestablish boundaries in their original positions. The information that the surveyor has to guide him is a description, contained in the deed or obtained from old records. Descriptions of adjacent property may also be obtained.

If the description of the property were without error and one or more of the original corners were in evidence, and further, if the resurvey could be run without error, the problem would be as simple as running the original survey. When the lengths and directions given in the description had been laid off, the surveyor could say with assurance that the reestablished corners were in their original position. The facts are, however, that the original survey did contain errors. Further complications may be added by directions in the description being given by magnetic bearings and the declination at the time of the original survey being unknown, or by no statement as to whether the bearings of the original survey were referred to the magnetic or to the true meridian. Not infrequently large mistakes are made in transposing from one record to another or are present in the measurements of the original survey. The loss of corners, the lack of witness measurements, the removal or alteration of physical boundaries, the conflicting testimony of persons having knowledge concerning the position of boundaries, and numerous other factors may add to the uncertainties of the problem.

As a first step the surveyor usually calculates the latitudes and departures of the several courses given in the description, determines the error of closure, and plots the boundaries of the tract to scale. If original bearings are magnetic, the declination at the time of the original survey is sought, and true bearings are calculated.

340. If true bearings can not be found in this manner, and one or more boundaries can be positively identified, observations are made to determine the true bearings of these known lines, and by a comparison of true and original magnetic bearings, the declination at the time of the original survey is estimated and the true bearings of other lines are computed.

If a single boundary line can be identified by means of described corners which are found in place or by reliable reference marks, a comparison is obtained between the length of the chain or tape used on the original survey and that to be used on the resurvey, and the proportionate lengths of other sides of the tract are computed.

With these calculated directions and proportionate lengths, the surveyor, starting from a known corner, re-runs the courses, and at each estimated position of a corner seeks evidence of the position of the original monument. At any point where the surveyor finds what he regards as positive evidence as to the original location of the corner and this location does not agree with the relocation measurements derived from the description of the property, a monument is set at the original location and new measurements of angles and distances are made to refer to the mark thus established. At any point where physical evidence as to the original location of the corner is entirely lacking, the corner is temporarily located by measurements derived from the description of the property. The survey is then continued until positive evidence of the location of a succeeding corner is found, or until the traverse is brought to a closure at the initial point.

If no further evidence of the original location of corners is found, the survey is run to the point of beginning, and the linear error of closure is measured. The total error in latitudes and in departures is then computed; the corrections in latitudes and in departures of the preceding lines are computed as described in Art. 243; and these corrections are applied by moving the preceding temporary monuments. Finally the lengths and bearings of the adjusted courses are measured in the field.

341. Where only a single corner can be found, the process of reestablishng boundaries is not so simple, particularly when the bearings of the original survey were observed with a compass and were referred to the magnetic meridian without the declination being given. Usually an estimate of the amount of the magnetic declination at the time of the original survey can be obtained by consulting old records, but often the date of the survey from which the description is derived is unknown and can not be closely determined.

By means of the estimated declination, the magnetic bearings are changed to true bearings, the latitudes and departures are calculated, and the error of closure of the original survey is determined. If this is reasonably small (say not greater than $1/300$ if the old survey was run with a compass) it indicates that there are no mistakes in the lengths and bearings given in the description. About the only course then open to the surveyor is to rerun the survey in accordance with the old description, first establishing the direction of the true meridian, and to consider the corners as being relocated to the best of his ability if the error of closure of the resurvey is no larger than that of the original survey. This error of closure is distributed proportionally among the several courses, as described in the preceding article.

342. When a tract is to be resurveyed, a description of the tract being accessible but all evidence of the location of original corners having been lost, the surveyor will find it expedient to search the records for descriptions of adjoining property and by means of these descriptions to reestablish by measurement as many corners of the tract in question as seems feasible. The locations of the corners thus determined are, of course, not without error. Often one corner may be reestablished by measurements from several different sources, each of which results in a different location. In cases of this kind the surveyor is called upon to exercise his judgment as to the most probable location of the original corner.

Having thus tentatively fixed the location of one or more corners, the surveyor attempts to reconcile these locations with the description of the given tract, the resurvey being conducted somewhat as described in one or another of the last two articles, and readjustments of the position of the tentatively located corners being made to conform to the judgment of the surveyor in light of the information he obtains as the survey progresses.

343. When a resurvey has been completed, it is the duty of the surveyor to render a report to his client stating exactly what he found and what course of procedure he employed in attempting to reestablish missing corners. The report should be accompanied by a plat showing the observed lengths and directions of the sides of the tract and other data similar to that shown on the plat of an original survey (see Art. 338). In addition, it should indicate which are original monuments and which are monuments established at the time of the resurvey. Mistakes in the original description should be pointed out, but the surveyor should clearly understand that it is his function to reestablish boundaries of a given tract in as nearly as possible their original position. Often resurveys are run to settle controversies between adjoining property owners. The surveyor should understand that while he may act as an arbiter in such a case, it is not within his power to fix boundaries without the mutual consent of all interested parties. In general, his functions are those of the expert witness, rather than the judge; and in case of dispute he may voice his expert opinion, but the final decision rests with the court.

344. Surveys for Subdivision of Rural Lands.—A subdivision survey implies a survey which is conducted for the purpose of subdividing into two or more tracts an area whose boundaries are already established, all in accordance with some prearranged plan. In the United States, surveys of this kind made in connection with rural lands may be classified as follows:

1. *U. S. Surveys of Public Lands.*—The public lands are divided into townships, sections, and quarter-sections by United States land surveyors, in a manner prescribed by law. The work of surveying the public lands is carried out under the direction of the General Land Office. The system is described in Chap. XX.

2. *Surveys for Fractional Subdivision of Sections.*—Surveys of this class are conducted for the purpose of further subdividing lands by an extension of the U. S. system. In general, the U. S. surveys establish the boundaries of sections and establish quarter-section corners on section lines; and any further subdivision is made after the lands have passed into the hands of private individuals, the work being carried out by surveyors in private practice. Subdivisions of this kind are described in detail in Art. 383. They include the quarter-section (160 acres), the half-quarter or eighth-section (80 acres), the quarter-quarter or sixteenth-section (40 acres), the half-

quarter-quarter or thirty-second-section (20 acres), and the quarter-quarter-quarter or sixty-fourth-section (10 acres).

3. *Surveys for Irregular Subdivisons.*—Surveys of this class are conducted for a variety of purposes. The following examples serve to illustrate the procedure for certain cases:

Example 1: A railroad is to traverse the land belonging to Mr. Black and the railroad company desires to secure title to a right-of-way of a definite width on either side of the center line which has already been surveyed and marked with stakes. A description of Black's tract has been secured.

The right-of-way surveyor reruns the boundaries of Black's tract that are intersected by the railroad line, establishes the directions of the right-of-way boundaries parallel with the center line, and sets monuments at the intersection of these boundaries with those of Black's tract. He then makes a survey of the tract thus defined, securing sufficient data so that the lengths and directions of the boundaries of the right-of-way tract now within Black's tract are obtained. He also ties right-of-way corners which he has established to the nearest old corners of Black's property.

With these data the area of the right-of-way tract is calculated, and a description of the tract is prepared as for an original survey (see Art. 336). The point of beginning is referred to one of the old monuments marking the original tract, and not to the center line of the railroad.

Example 2: It is stipulated in the will of Mr. Green that his New England farm is to be divided equally among his three sons, each to have an equal frontage on the highway which forms one of the boundaries of the tract. The farm is of irregular shape and has not been surveyed for many years.

In a case of this kind, the surveyor first makes a resurvey of the entire tract, angles being measured probably to minutes, and distances being measured probably to tenths of feet. From the data thus obtained, the area is computed. In connection with the resurvey, subdivision corners are established on the highway.

The simplest division is one for which the subdividing lines are straight, each cutting off the required area from the given tract. With the area of the entire tract known, the area each son is to receive is calculated, and the length and bearing of each of the lines rendering the subdivision is calculated as described in Art. 366.

Finally, from each of the two subdivision corners already established on the highway frontage, the surveyor lays off the calculated direction of the subdividing line through that point, and establishes the remaining unknown corner at the point where this subdividing line intersects the opposite boundary. The distances from this latter corner to adjacent corners are measured, and the survey is considered as checked, if these measured distances agree closely with the computed lengths of the same courses. A plat is drawn as for an original survey, the lengths and bearings of all lines and the area of each subdivision being shown; and a description of each of the three subdivisions is prepared.

345. Surveys for the Subdivision of Urban Lands.—As a city or town develops, unimproved lands are subdivided into lots which are placed on sale as residential or business property. In most instances such extensions are the result of the activities of real-estate operators who acquire a tract of rural land of considerable area, develop a plan of subdivision which is approved by the authorities of the municipality to which the tract is to be attached, and cause surveys to be made for the purpose of establishing the boundaries of individual lots. A tract thus divided according to an acceptable plan is known as an *addition* or *subdivision*.

For large and important developments the work of originating the general plan is often carried out by persons specializing in city planning and landscape architecture, under whose direction the surveyor works. Such developments require a great deal of care and skill on the part of the designer, and usually extensive surveys (particularly in hilly sections) are carried out before the actual plan of subdivision can be decided upon.

For the ordinary real-estate development, the owner usually calls for the services of an engineer or surveyor who has had experience in such work. The surveyor confers with the owner and they discuss a general plan. The surveyor makes a resurvey of the entire property, and if the character of the topography is irregular, he usually makes certain preliminary surveys for the purposes of finding the location and elevation of the governing features of the terrain. In some cases a complete topographic survey may be made. With the general plan fixed, the surveyor works out a detailed plan on paper, there being shown on the drawing the names of all streets and numbers of all blocks and lots, the dimensions of all lots, the widths of streets, the length and bearing of all street tangents, and the radius and length of all street curves. The surveyor then proceeds to execute the necessary subdivision surveys, including the laying out of roads, walks, blocks, and lots. When the surveys are completed, the map of the subdivison is revised to show minor changes made during the survey, together with the location and character of permanent monuments. A tracing is submitted to the municipality, which tracing, when duly signed by those in authority, becomes the official map of the subdivison. It then becomes a part of the public records and is usually filed in the Registry of Deeds of the county in which the municipality lies.

346. Descriptions of Urban Lands.—The manner of legally describing the boundaries of a tract of land falling within the corporate limits of a city or town depends upon conditions attached to the surveys by which the boundaries of the tract were first established, as indicated by the following classifications:

1. *By Lot and Tract.*—When the boundaries of the tract exactly conform with a lot which is a part of a subdivision or addition for which there is recorded an official map, a statement giving the lot and block numbers and the name and date of filing of the official map constitutes a definite and legal description of the tract. Most city property is described in this way. Following is a description of this character occurring in a deed:

"Lot 15 in Block No. 5 as said lots and blocks are delineated and so designated upon that certain map entitled *Map of Thousand Oaks, Alameda County, California,* filed August 23, 1909, in Liber 25 of Maps, page 2, in the office of the County Recorder of the said County of Alameda."

2. *By Metes, Bounds, and Lots.*—When the boundaries of a given tract within a subdivision for which there is a recorded plat do not exactly conform to boundaries shown on the official plat, the tract is described by metes and bounds, with the point of beginning referred to a corner shown on the official plat. The numbers of lots of which the tract is composed is also given. The following is an example of a description of this kind:

"Beginning at the intersection of the Northern line of Escondido Avenue, with the Eastern boundary line of Lot No. 16, hereinafter referred to; running thence Northerly along said Eastern boundary line of Lot 16, and the Eastern boundary line of Lot 17, eighty-nine (89) feet; thence at right angles Westerly, fifty-one (51) feet; thence South 12°6′ East, seventy-five (75) feet to the Northern line of Escondido Avenue; thence Easterly along said line of Escondido Avenue, fifty-three and $^{13}/_{100}$ (53.13) feet, more or less, to the point of beginning.

"Being a portion of Lots 16 and 17, in Block No. 5, as said lots and block are delineated and so designated upon that certain Map entitled *Map of Thousand Oaks, Alameda County, California,* filed August 23, 1909 in Liber 25 of Maps, page 2, in the office of the County Recorder of the said County of Alameda."

3. *By Metes and Bounds to City Monuments.*—A number of the larger and older cities of the United States have, by precise surveys, established an elaborate system of reference monuments and have determined their coordinates with respect to some arbitrarily assumed initial point. When the tract cannot be defined by descriptions such as the preceding, the point of beginning may be definitely fixed by stating its direction and distance from an official reference monument and by describing the character of the monument by which the point of beginning is marked. The boundaries of the tract may then be described by metes and bounds.

The location of corners may also be defined by rectangular coordinates referred to the origin or initial point of the system.

When the tract is within a city or town not so monumented, the point of beginning of the boundary description may be referred by direction and distance to the intersection of the center lines of streets. It is not good practice to refer to the intersection of sidewalk or curb lines, for these are apt to be changed from time to time. In sections of the country within the rectangular system of U. S. surveys, the point of beginning of a boundary description may properly be referred to section lines and corners.

347. Boundary Records.—There are three general sources from which the descriptions of the boundaries of real property may be found, *viz.*, deeds, official plots or maps, and notes of original surveys.

Throughout the United States, records of the transfer of land from one person to another are kept either in the office of the city or town clerk or in the county registry of deeds, but usually in the latter place, exact copies of all deeds of transfer being filed in deed books. These files are free to be examined by anyone and are a frequent source of information for the land surveyor in search of boundary descriptions when it is inconvenient or impossible to secure permission of the owner to examine the original deed.

As the United States public lands are subdivided, official plats are prepared showing the dimensions of subdivisions and the character of monuments marking the corners. The notes of the U. S. surveys are transmitted to the U. S. Surveyor General of the state or territory in which the lands lie, and together with the official plats prepared from the notes, are kept as a part of his records until all public lands within the State are surveyed, when all records are given to the State.

In most cases the deeds of transfer of city lots give only the lot and block number and the name of the addition or subdivision. The official map of the subdivision on which are shown the dimensions of all lots and the character and location of permanent monuments is on file either in the office of the city or town clerk of the municipality in which the subdivision lies or in the county registry of deeds. Copies of the official plats giving dimensions and ownership of all lots are also on file in the offices of city and county assessors.

348. Legal Authority of the Surveyor.—Often resurveys are run to settle controversies between adjoining property owners. The surveyor should understand that while he may act as an arbiter in such cases, it is not within his power to fix boundaries without the mutual consent and authority of all interested parties. In the event of a dispute involving court action he may present evidence and argument as to the proper location of a boundary, but he has no authority to establish such a boundary against the wishes of either party concerned. A competent surveyor by his wise counsel will prevent most litigation; but if he can not bring his clients to an agreement, the boundaries in dispute become valid and defined only by a decision of the court. In boundary disputes the surveyor is therefore an expert witness but not a judge.

349. Legal Interpretation of Deed Description.—As indicated in the preceding pages, the descriptions of the boundaries of a tract include the objects which fix the corners, the lengths and directions of lines between the corners, and the area of the tract. It frequently happens that a deed description will contain errors of measurement or calculation, or mistakes of record which introduce inconsistencies which, when retracement becomes necessary, it is impossible to harmonize completely.

In such cases, where uncertainty has arisen as to the location of property lines, it is a universal principle of law that the endeavor is to make the deed effectual rather than void, and to execute the intentions of the contracting parties. The following general rules are pursuant of this principle:

1. *Monuments.*—It is presumed that the visible objects which marked the corners when a conveyance of ownership was made, indicated best the intentions of the parties concerned, hence it is agreed that a corner is established by an existing material object, or by conclusive evidence as to the previous location of the object. A corner thus established will prevail against all other conflicting evidence, whatsoever may be its character. The kinds of evidence which are valid in relocating obliterated corners are stated in Art. 398.

2. *Calls for Distance and Direction.*—In the case of discord between the described courses and the calculated area, deed-description calls for distances or directions of courses will prevail against the call for area of a tract; again on the assumption that the boundary lines are more visible and actual evidence of the intentions of the parties than the calculated area of the tract.

3. *Mistakes.*—It is a well-established principle that a deed-description which taken as a whole plainly indicates the intentions of the parties concerned, will not be invalidated by evident mistakes or omissions. For example, such obvious mistakes as the omission of a full tape length in a dimension, or the transposition of the words *Northeast* for *Northwest* will have no effect on the validity of a description, provided it is otherwise complete and consistent, or provided its intention is plainly manifest.

4. *Purchaser Is Favored.*—In the case of description capable of two or more interpretations, that one will prevail which favors the purchaser.

5. *Ownership of Highways.*—Land described as being bounded by a street or highway, conveys ownership to the center of the street or highway. Any variation from this interpretation must be explicitly stated in the description.

6. *Original Government Surveys Are Presumed Correct.*—Errors found in original surveys will in no way affect the location of the boundaries established under those surveys. In other words, no errors can be attributed to original government surveys, and the boundaries remain fixed as originally established.

350. Adverse Possession.—The many legal aspects of adverse possession can not be treated in this brief article, but it is desirable to direct the attention of the surveyor to the important fact that property lines are sometimes fixed by possession and use of the land as against original survey boundaries. The conditions and the period of time necessary to gain title are fixed by statute in the various states.

Adverse possession to become effective, must be plainly evident to the owner and hostile to his interests. Such possession may be evidenced by fencing, cultivation, erection of buildings, etc., and must be held without the permission of the owner.

Right to title by adverse possession may be acquired by individuals, corporations, and even by the state. But the statute does not run *against* the state; *i.e.*, property in a street or highway can not be acquired by adverse possession.

Under this principle, if a person should use the land up to a fence, and should recognize it as a boundary line, to the exclusion of the owner, for the statutory period, the fence then becomes the legal property line even though it may be shown later that it is not on the true and original line. However, if the possession of the land has not been held adversely, *i.e.*, to the exclusion of the owner, and the fence has merely served the convenience of the persons concerned, both parties recognizing that it was probably not on the true line, title can not be claimed.

It is therefore clear that the application of the principle of adverse possession is entirely a matter of intention and belief. If land is held openly and notoriously with the intent to acquire title, or with the belief that the occupation is proper and right, then title will be granted if and when the statutory requirements are fulfilled. But if by parol agreement or by actions it is manifest that the parties concerned had no intention to occupy beyond the true line, at the same time knowing that the location of the true line was uncertain, then title can not be gained adversely.

Adverse possession under "color of title" will "ripen into title" under the statute of limitations in some jurisdictions in half the time required without color of title; for example, if title may be gained without color of title in 20 years, it may be gained in 10 years with it.

351. Liability of Surveyors.—It has been held in court decisions that a surveyor is a member of a learned profession and may be held liable for incompetent services rendered. This principle applies to both a county surveyor and to any person engaged in private practice.

The general principle invoked in such cases is that the surveyor is bound to exhibit that degree of prudence, judgment, and skill which may reasonably be expected of a member of his profession.

CHAPTER XIX

CALCULATION OF AREAS OF LAND

352. Remarks.—As stated in the preceding chapter, one of the primary objects of a land survey is to determine the area of the tract or tracts with which the survey is concerned. During the progress of a survey of this character a closed traverse is run, the lines of the traverse being made to coincide with property lines when such procedure is possible. Where the boundaries are irregular or curved, or where they are occupied by objects which make direct measurement impossible, they are located with respect to the traverse line by appropriate angular and linear measurements. In the usual course of such a survey the lengths and bearings of all straight boundary lines are determined either directly or by computation, the irregular boundaries are located with respect to traverse lines by perpendicular offsets taken at appropriate intervals, and the radii and central angles of boundaries which form the arcs of circular curves are obtained.

In ordinary land surveying, such as described in the preceding chapter, the area of a tract of land is taken as its projection upon a horizontal plane, and is not the actual area of the surface of the land.

In the United States the common units of area are for rural lands, the acre; and for urban lands, the square foot. There are 640 acres in 1 sq. mile; 1 acre = 10 sq. Gunter's chains = 160 sq. rd. = 4,047 sq. m. = 4,840 sq. yd. = 43,560 sq. ft.

353. Methods of Determining Area.—The two most common methods of determining the area within a closed traverse are:

1. *The double meridian distance method.* This is the method which is almost universally employed for finding the area of lands for which the boundaries are straight lines whose lengths and bearings are known. Its advantage lies in the ease with which computations may be made and checked. The basis of computations are the latitudes and departures of the several courses forming the boundary traverse.

2. *The coordinate method.* Where the coordinates or total latitudes and departures of the corners of a tract have been found as described in Arts. 240 to 253, the area of the tract may be determined most expeditiously by computations based upon the coordinates.

These two methods will now be described in detail.

354. Principles of the Double Meridian Distance Method.—In computing area by the double meridian distance method, the latitudes and departures of all the courses are determined as described in Art. 241, and the survey is balanced, usually by one of the rules of Art. 243. A reference meridian is then assumed to pass through some corner of the tract, usually for convenience the most westerly point in the survey, and double the areas of the trapezoids or triangles formed by orthographically projecting the several traverse lines upon the meridian are computed. The algebraic sum of these double areas is double the area within the traverse.

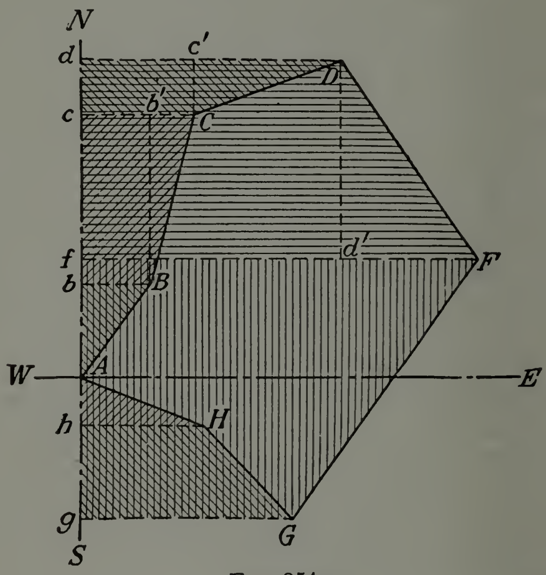


Fig. 354.

Thus, in Fig. 354, *ABCDFGH* represents a closed traverse and *SN* represents a meridian passing through the corner *A*, which is the most westerly point. The points *B*, *C*, *D*, etc., are projected upon the meridian to the points *b*, *c*, *d*, etc., the lines of projection *Bb*, *Cc*, *Dd*, etc., being perpendicular to the meridian.

The perpendicular distance from the meridian to any point in the traverse is called the *meridian distance* of the point. Thus the meridian distance of *B* is *Bb*. Meridian distances are considered as being positive if they lie east of the reference meridian and are considered as being negative if they lie west of the reference meridian.

The sum of the meridian distances of the two extremities of a line is called the *double meridian distance* of the line. Thus the double meridian distance of *BC* is *Bb* + *Cc*. It is clear that if the meridian passes through the most westerly corner of the traverse, the double meridian distance of all lines will be positive, which is a convenience, but not a necessity, in computing.

As explained in Art. 241, the length of the orthographic projection of a line upon the meridian is the latitude of the line, latitudes being considered as positive if the direction of the line has a northerly component and negative if the direction of the line has a southerly component. Thus the latitude of *BC* is *bc* and is positive, and that of *DF* is *df* and is negative.

From the figure it is seen that each trapezoid or triangle for which a course in the traverse is one side, is bounded on the north and south by meridian distances and on the west by the latitude of that course. Thus the projection trapezoid for *BC* is *BCcb*. It is therefore seen that the double area of any triangle or trapezoid formed by projecting a given course upon the meridian is the product of the double meridian distance of the course and its latitude.

$$\text{Double Area} = \text{D.M.D.} \times \text{Latitude}$$

In calculating double areas account is taken of signs. If the meridian extends through the most westerly point, all double meridian distances are positive, hence the sign of a double area is the same as that of the corresponding latitude. Thus in the figure the double areas of *AbB*, *BbcC*, *CcdD*, *GghH*, and *HhA* are positive, the latitudes *Ab*, *bc*, *cd*, *gh*, and *hA* being positive; while the double areas of *FfdD* and *GgfF* are negative, the latitudes *df* and *fg* being negative. Those projection trapezoids and triangles for which the double areas are positive have, in the figure, been cross-hatched with sloping lines and those trapezoids for which the double areas are negative have been cross-hatched with vertical or horizontal lines. Consulting the figure, it will be seen that the projected areas outside the traverse are considered once as positive and once as negative, being cross-hatched by both inclined and vertical or horizontal lines, hence the algebraic sum of the corresponding double areas is zero.

It is therefore clear that the algebraic sum of all double areas is equal to twice the area of the tract within the traverse. Whether this algebraic sum is a positive or negative quantity is determined solely by the order in which the lines of the traverse are considered. If the reference meridian passes through the most westerly corner, then a clockwise order of lines, as in the figure, results in a negative double area, and a counterclockwise order results in a positive double area.

355. When the reference meridian passes through a traverse point there is an intimate relation between the departures and double meridian distances. Thus, again referring to Fig. 354, it is seen that the D.M.D. of AB is bB, which is equal to the departure of the course both in magnitude and sign. And the D.M.D. of BC is equal to $bB + cb' + b'C$, which is equal to the D.M.D. of AB, plus the departure of AB, plus the departure of BC. Similar quantities make up the D.M.D.'s of CD and DF. For the last line HA the D.M.D. is hH, which is equal in magnitude but opposite in sign to the departure of HA.

Following are three rules for determining D.M.D.'s, so useful as to be worth memorizing, which are deduced from the above relations:

1. The D.M.D. of the first course (reckoned from the point through which the reference meridan passes) is equal to the departure of that course.
2. The D.M.D. of any other course is equal to the D.M.D. of the preceding course, plus the departure of the preceding course, plus the departure of the course itself.
3. The D.M.D. of the last course is numerically equal to the departure of the course but with opposite sign.

The first two of the above rules are employed in computing values. The third rule is useful as a check upon the accuracy of the computations. Assuming the departures as balanced, the D.M.D. of the last line, as found by computing the D.M.D.'s in succession around the traverse from the first line, must be numerically equal to the departure of the last line if no mistake in addition or subtraction has been made. When this condition is realized it may be concluded that the intermediate D.M.D.'s are correct, and recomputation is unnecessary. In calculating D.M.D.'s by the above rules, due regard must be given to signs.

356. Area within a Closed Traverse by the D.M.D Method.—Following is a summary of the steps employed in calculating by the D.M.D. method the area within a closed traverse when the lengths and bearings of the sides are known:

1. Compute the latitudes and departures of all courses as described in Art. 241.
2. Find the error of closure in latitude and in departure as described in Art. 242.
3. Balance the latitudes and departures in accordance with one of the rules of Art. 245.
4. Assume that the reference meridian passes through the most westerly point in the survey, and calculate the D.M.D.'s by the rules of the preceding article, using the corrected departures.
5. Compute the double areas by multiplying each D.M.D. by the corresponding corrected latitude.

6. Find the algebraic sum of the double areas, and determine the area by dividing this sum by two.

Area of Balsam Park, Island Pond, Vermont
By D.M.D. Method.

47.

Field Notes
Book No. 3
Page 47.

Computations
Aug. 17, 1913.
Computed and
Checked by J. Q. M.

Line	Cal. Bear.	Dist. 66 ft. Ch.	Latitudes		Departures		Corrected		D.M.Ds.	Double Areas	
			N	S	E	W	Lats.	Deps.		+	−
A−B	S 80°29½′ W	34.464		5.694		33.991	− 5.693	−33.990	61.812		351.89
B−C	S 33 04 W	25.493		21.364		13.911	−21.361	−13.911	13.911		297.15
C−D	S 33 46¾ E	33.934		28.205	18.867		−28.201	+18.867	18.867		532.06
D−E	N 87 58¼ E	28.625	1.013		28.607		+ 1.013	+28.608	66.342	67.21	
E−A	N 0 27 E	54.235	54.234		0.426		+54.242	+ 0.426	95.376	5173.51	
		176.751	55.247	55.263	47.900	47.902	ΣL=0	ΣD=0		5240.72	1181.10
				55.247		47.900				1181.10	
				.016		.002				2 | 4059.62	
										2029.81 Sq. Ch.	
										or 202.981 Ac.	

$E.\ of\ C. = \frac{.016}{176.751} = \frac{1}{11{,}000}$.

$E = \sqrt{.016^2 + .002^2} = \underline{0.016}$ Chains.

Line	A−B	B−C	C−D	D−E	E−A
Lat.	5.694	21.364	28.205	1.013	54.234
Log. Lat.	0.75542	1.32968	1.45032	0.00584	1.73427
Log. Cos.	9.21805	9.92326	9.91969	8.54899	9.99999
Log. Dist.	1.53737	1.40642	1.53063	1.45674	1.73428
Log. Sin.	9.99399	9.73689	9.74509	9.99973	7.89535
Log. Dep.	1.53136	1.14331	2.27572	1.45647	9.62964
Dep.	33.991	13.911	18.867	28.607	0.426
Log. Cor. Lat.	0.75534	1.32962	1.45026	0.00584	1.73434
Log. D.M.D.	1.79107	1.14336	1.27570	1.82179	1.97944
Log. D.A.	2.54641	2.47298	2.72596	1.82763	3.71378
D'ble Area	351.89	297.15	532.06	67.21	5173.51

Note:
Survey Balanced by Transit Rule.

Fig. 356.—Computations for area.

The above steps are illustrated in the computations shown by Fig. 356 in which the area of a tract within a transit traverse is determined. It is seen that distances are in 66-ft. chains, and computations are made by the use of logarithms. The survey is balanced by the transit rule, and the corrected latitudes and departures are recorded.

The point *C* is the most westerly point in the survey, and the double meridian distances are computed by beginning with the line *CD* for which the D.M.D. and corrected departure are of the same magnitude and sign. The D.M.D.'s for lines *DE*, *EA*, *AB*, and *BC* are calculated by the second of the rules given in Art. 355. The D.M.D. of *BC*, the last line in the traverse, is seen to be numerically equal to the corrected departure of this line but with opposite sign, hence the D.M.D. computations are correct.

Below the calculations of latitudes and departures are given the logarithmic computations for double areas, each D.M.D. being multiplied by the corresponding corrected latitude.

In the last two columns of the upper tabulation are recorded the positive and negative double areas. These are algebraically added, and the result is divided by two as shown, the result being the area in square chains.

The correctness of the computations for latitudes, departures, and double areas can not be readily checked except by recomputations of similar character. The corrections applied to latitudes and departures are checked if the algebraic sums of the corrected latitudes and departures are zero. The application of the third of the rules for D.M.D.'s as already explained, serves to verify the calculations for D.M.D.'s.

While in Fig. 356 computations by logarithms have been shown, where a computing machine is available it would ordinarily be used in preference to logarithms.

357. Area by Coordinates.—The coordinates of the corners of the tract are useful in finding not only the lengths and bearings of the boundary lines, but also in computing the area. The coordinate axes are usually a true meridian and parallel at right angles thereto, but may be an assumed meridian and parallel bearing no particular relation to the points of the compass. The intersection of the reference meridian and parallel, or the origin of coordinates, may or may not be one of the corners of the tract.

In Fig. 357, *ABCDF* represents a tract the area of which is to be determined, *SN* being the reference meridian, *WE* being the reference parallel, and the coordinates of *A*, *B*, etc., being known. For any point these coordinates are the perpendicular distance from the reference meridian, previously defined as the *meridian distance*, and the perpendicular distance from the reference parallel, previously defined as the *parallel distance*. Thus, for *A* the meridian distance is $aA = m_1$, and the parallel distance is $a'A = p_1$; for *B* the meridian distance is $bB = m_2$, and the parallel distance is $b'B = p_2$; and so on, around the traverse. Meridian distances are regarded as positive or negative according to whether they lie to the east or to the west of the reference meridian; parallel distances are regarded as positive or negative according to whether they lie to the north or south of the reference parallel. In the figure all meridian and parallel distances are positive, since all points lie in the north-east quadrant.

For reasons explained in the discussions of the D.M.D. method, the algebraic sum of the areas of the projection trapezoids either upon the reference meridian or reference parallel is equal to the area of the tract. Then, considering the trapezoids with bases on the meridian,

$$2 \text{ area } ABCDF = (m_1 + m_2)(p_1 - p_2) + (m_2 + m_3)(p_2 - p_3) + (m_3 + m_4)(p_3 - p_4) + (m_4 + m_5)(p_4 - p_5) + (m_5 + m_1)(p_5 - p_1) \quad . \qquad (1)$$

which form is practically identical with that employed in calculating area by the D.M.D. method.

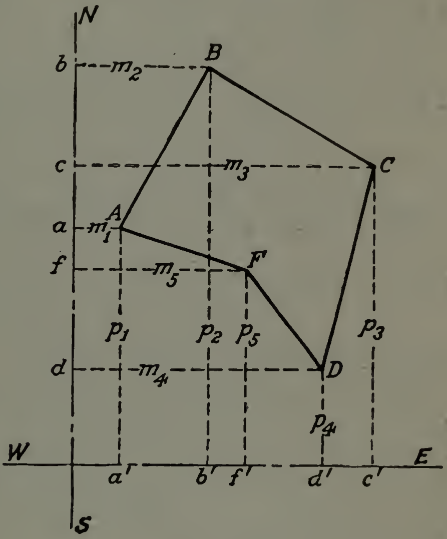


Fig. 357.

By multiplication and a rearrangement of terms in the above equation there is obtained:

$$2 \text{ area} = m_2p_1 - m_1p_2 + m_3p_2 - m_2p_3 + m_4p_3 - m_3p_4 + m_5p_4 - m_4p_5 + m_1p_5 - m_5p_1 \qquad (2)$$

When Eq. (2) is employed, computations may be conveniently made by tabulating in order around the tract each parallel distance below the corresponding meridian distance as follows:

$$\frac{m_1}{p_1} \times \frac{m_2}{p_2} \times \frac{m_3}{p_3} \times \frac{m_4}{p_4} \times \frac{m_5}{p_5} \times \frac{m_1}{p_1} \qquad (3)$$

Comparing the two expressions it will be seen that the difference between the sum of the products of the coordinates joined by full lines in Eq. (3) and the sum of the products of coordinates joined by dotted lines in Eq. (3) is equal to twice the area of the tract.

Example 1: By the use of Eq. (3) calculate the area of the tract whose corners are defined by coordinates as shown below:

Corner	1	2	3	4	5
Meridian distance, ft	300	400	600	1,000	1,200
Parallel distance, ft	300	800	1,200	1,000	400

Writing the coordinates in fractional form, as in Eq. (3), we have

$$2\text{ area} = \frac{300}{300}\times\frac{400}{800}\times\frac{600}{1{,}200}\times\frac{1{,}000}{1{,}000}\times\frac{1{,}200}{400}\times\frac{300}{300}$$

$$2\text{ area} = 120{,}000 + 480{,}000 + 1{,}200{,}000 + 1{,}200{,}000 + 120{,}000 - 240{,}000 - 480{,}000 - 600{,}000 - 400{,}000 - 360{,}000.$$

$$2\text{ area} = 3{,}120{,}000 - 2{,}080{,}000 = 1{,}040{,}000\text{ sq. ft.}$$

$$\text{Area} = \frac{1{,}040{,}000}{2} = 520{,}000\text{ sq. ft.}\quad \textit{Ans.}$$

358. Equation (1) may also be written in the form

$$2\text{ area} = p_1(m_2 - m_5) + p_2(m_3 - m_1) + p_3(m_4 - m_2) + p_4(m_5 - m_3) + p_5(m_1 - m_4) \qquad (4)$$

which form leads to the following rule:

Rule.—*To determine the area of a tract of land when the coordinates of its corners are known, multiply the parallel distance of each corner by the difference between the meridian distances of the following and the preceding corners, always algebraically subtracting the preceding from the following. One-half the algebraic sum of the resulting products is the required area.*

359. Areas of Tracts with Irregular or Curved Boundaries.—As before stated, it frequently happens that the boundary of a tract of land follows some irregular or curved line, as, for example, a stream or road. In such cases it is customary to run a traverse in some convenient location near the boundary and to locate the boundary by offsets from the traverse line. The determination of the area of the entire tract involves computing the area within the closed traverse, by methods which have already been described, and adding to this the area of the irregular figure between the traverse line and the curved boundary. The data which are available for computing the irregular area consist of offset distances, and the corresponding distances along the traverse line . Where the boundary is irregular, it is necessary to take offsets at points of change, hence the offsets for a boundary of this character are, in general, taken at irregular intervals. Where a segment of the boundary is straight, offsets are taken only at the ends. Where the boundary is a gradual curve, offsets are ordinarily taken at regular intervals.

Obviously an irregular boundary is not usually a succession of straight lines between offsets, but often the boundary is not a sharply defined line and if the offsets are taken sufficiently close together, the error involved in considering the boundary as straight between offsets is small as compared with the inaccuracies of the measured offsets. Irregular areas computed on such an assumption are said to be calculated by the *trapezoidal method* or by the *trapezoidal rule.*

When the curved boundaries are of such definite character as to make it justifiable, a somewhat higher degree of accuracy in the computed area may be obtained by assuming that the boundary is made up of segments of parabolas. When the area within a curved boundary is found by making this assumption, it is said to be computed by *Simpson's method* or by *Simpson's rule.*

360. Trapezoidal Method: Offsets at Regular Intervals.—Let Fig. 360 represent a tract lying between a traverse line AB and an irregular boundary CD, offsets h_1, h_2, . . . h_n having been taken at the regular intervals d. Summing up the areas of the trapezoids, the total area is

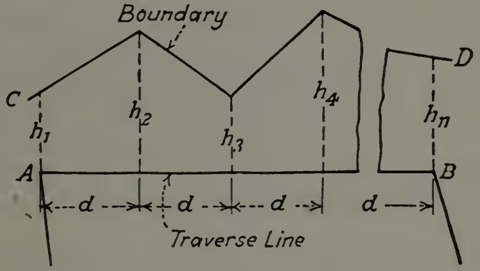


FIG. 360.

$$\text{Area} = \frac{h_1 + h_2}{2}\cdot d + \frac{h_2 + h_3}{2}\cdot d + \frac{h_3 + h_4}{2}\cdot d \cdot \cdot \cdot + \frac{h_{(n-1)} + h_n}{2}\cdot d$$

$$\text{Area} = d\left(\frac{h_1 + h_n}{2} + h_2 + h_3 + h_4 \cdot \cdot \cdot + h_{(n-1)}\right) \qquad (1)$$

which is the expression commonly used for finding an irregular area when offsets are taken at regular intervals.

Equation (1) may be conveniently expressed in the form of the following rule:

Trapezoidal Rule.—Add to the sum of the intermediate offsets, the average of the end offsets. The product of the quantity thus determined and the common distance between offsets is the required area.

Example 1: By the trapezoidal method find the area between a traverse line and curved boundary, rectangular offsets being taken at intervals of 20 ft. and the values of the offsets in feet being $h_1 = 3.2$, $h_2 = 10.4$, $h_3 = 12.8$, $h_4 = 11.2$, $h_5 = 4.4$. Applying the above rule, there results

$$\text{Area} = 20\left(\frac{3.2 + 4.4}{2} + 10.4 + 12.8 + 11.2\right) = 20(3.8 + 34.4)$$

$$= 20 \times 38.2 = 764 \text{ sq. ft.} \quad \textit{Ans.}$$

361. Simpson's Method: Offsets at Regular Intervals.—In Fig. 361 let AB be a portion of a traverse line, let DFC be a portion of the curved boundary assumed to be the arc of a parabola, and let h_1, h_2, and h_3 be any three consecutive rectangular offsets from traverse line to boundary taken at the regular interval d.

The area between curve and traverse line may be considered as composed of the trapezoid $ABCD$ plus the area between the parabolic arc DFC and the corresponding chord DC. It is one of the properties of the parabola that this latter area is equal to two-thirds the area of enclosing parallelogram $CDEFG$. This being recognized, the area between the traverse line and curved boundary within the length of $2d$ is given by the expression

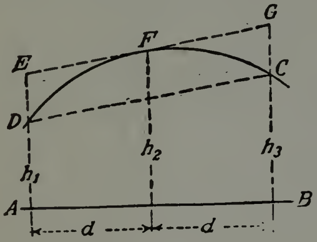


FIG. 361.

$$\Delta A_{1,2} = \frac{(h_1 + h_3)}{2}2d + \left(h_2 - \frac{h_1 + h_3}{2}\right)2d \cdot \frac{2}{3}$$

$$\Delta A_{1,2} = \tfrac{4}{3}h_2 d + \tfrac{1}{3}(h_1 + h_3)d$$

$$\Delta A_{1,2} = \frac{d}{3}(h_1 + 4h_2 + h_3)$$

Similarly for the next two intervals

$$\Delta A_{3,4} = \frac{d}{3}(h_3 + 4h_4 + h_5)$$

Summing up these partial areas for $(n - 1)$ intervals, n being an odd number and representing the number of offsets, the total area is

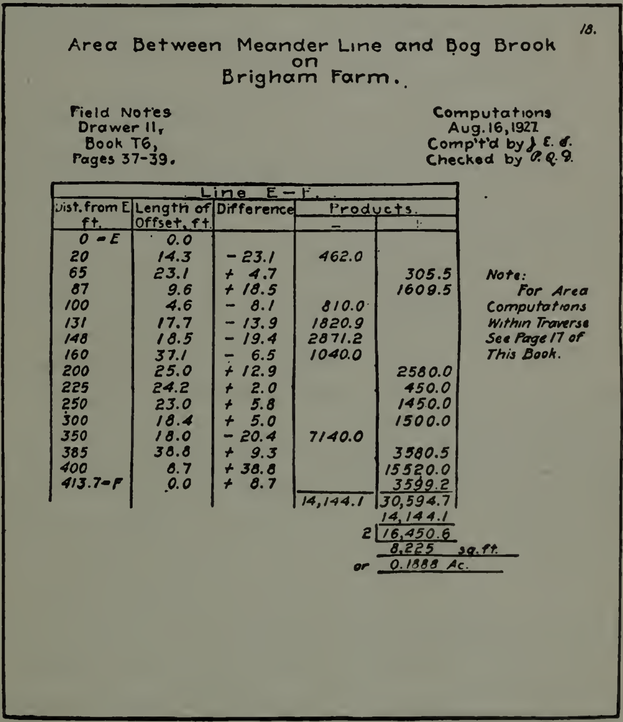
$$\text{Area} = \frac{d}{3}[h_1 + h_n + 2(h_3 + h_5 + \cdots h_{(n-2)}) + 4(h_2 + h_4 + \cdots h_{(n-1)})] \quad (2)$$

which is conveniently expressed in the following rule which is applicable to any case where the number of offsets is odd and the interval between the offsets is uniform.

Simpson's One-third Rule.—Find the sum of the end offsets, plus twice the sum of the odd intermediate offsets, plus four times the sum of the even intermediate offsets. Multiply the quantity thus determined by one-third the interval between offsets, and the result is the required area.

Example 2: By Simpson's one-third rule find the area between a traverse line and a curved boundary, rectangular offsets being taken at intervals of 20 ft. and the values of the offsets in feet being $h_1 = 3.2$, $h_2 = 10.4$, $h_3 = 12.8$, $h_4 = 11.2$, $h_5 = 4.4$. (The statement of the problem is the same as for Example 1.)
Applying Simpson's rule, there results

$$\begin{aligned}\text{Area} &= \tfrac{20}{3}\,[3.2 + 4.4 + 2(12.8) + 4(10.4 + 11.2)]\\ &= \tfrac{20}{3}\,(7.6 + 25.6 + 86.4) = 798 \text{ sq. ft.}\end{aligned}$$



18.

Area Between Meander Line and Bog Brook
on
Brigham Farm.

Field Notes
Drawer II,
Book T6,
Pages 37-39.

Computations
Aug. 16, 1927
Comp't'd by J. E. G.
Checked by P. Q. G.

Line E—F.				
Dist. from E ft.	Length of Offset, ft.	Difference	Products.	
			−	+
0 = E	0.0			
20	14.3	− 23.1	462.0	
65	23.1	+ 4.7		305.5
87	9.6	+ 18.5		1609.5
100	4.6	− 8.1	810.0	
131	17.7	− 13.9	1820.9	
148	18.5	− 19.4	2871.2	
160	37.1	− 6.5	1040.0	
200	25.0	+ 12.9		2580.0
225	24.2	+ 2.0		450.0
250	23.0	+ 5.8		1450.0
300	18.4	+ 5.0		1500.0
350	18.0	− 20.4	7140.0	
385	38.8	+ 9.3		3580.5
400	8.7	+ 38.8		15520.0
413.7 = F	0.0	+ 8.7		3599.2
			14,144.1	30,594.7

14,144.1
2 | 16,450.6
8,225 sq. ft.
or 0.1888 Ac.

Note:
For Area Computations Within Traverse See Page 17 of This Book.

Fig. 362.—Computations for area using trapezoidal rule.

362. Trapezoidal Method: Offsets at Irregular Intervals.—The method of coordinates described in Art. 257 may be applied to this problem by assuming the origin as being on the traverse line and at the point where the first offset is taken. The coordinate axes are then the traverse line and a line at right angles thereto. The rule of Art. 358 may then be modified to the following:

Rule.—*Multiply the distance, along the course, of each intermediate offset from the first, by the difference between the two adjacent offsets, always subtracting the following from the preceding. Also multiply the distance of the last offset from the first by the sum of the last two offsets. The algebraic sum of these products, divided by two, is the required area.*

The application of the rule to a specific problem is illustrated by the computations of Fig. 362, where the area between a meander line and stream is determined.

363. Partition of Land.—The problems involved in the partition or division of lands are so numerous as to preclude the possibility of discussing each one, but four of the simpler cases frequently encountered in the subdivision of irregular tracts will be described in the succeeding articles. Methods of subdividing the U. S. Public Lands are given in the following chapter.

In general, when a given tract is to be divided into two or more parts, a resurvey is run, latitudes and departures are computed, the survey is balanced, and the area of the entire tract is determined. The corrected latitudes and departures are further employed in the computations of subdivision.

The cases presently to be discussed are: (*a*) to find the area cut off by a line running between two points in the boundary; (*b*) to find the area cut off by a line running in a given direction from a given point in the boundary; (*c*) to cut off a required area by a line passing through a given point in the boundary; and (*d*) to cut off a required area by a line running in a given direction.

364. Area Cut Off by a Line between Two Points.—In Fig. 364 let *ABCDEFG* represent a tract of land to be divided in two parts by a line extending from *A* to *D*. A survey of the tract has been made, the latitudes and departures have been balanced, and the area has been computed.

It is desired to determine the length and direction of the cut-off line *AD* without additional field measurements, and to calculate the areas of each of the two parts into which the tract is divided.

Either of the two parts may be considered as a closed traverse with the length and bearing of one side unknown, that being the side *DA*. Considering the part *ABCDA*, the latitudes and departures of *AB*, *BC*, and *CD* are given, hence the latitude, departure, length, and bearing of *DA* may be determined as described in Art. 247. The area of either part may now be found by the method of double meridian distances (Art. 356).

365. Area Cut Off by a Line Running in a Given Direction.—In Fig. 365, *ABCDEFG* represents a tract of known dimensions, for which the balanced latitudes and departures are given; and *DH* represents a line running in a given direction which passes through the point *D*, dividing the tract into two parts.

It is desired to calculate from the given data the lengths *DH* and *HA* and the area of each of the two parts into which the tract is divided.

Each of the two parts may be considered as a closed traverse for which the lengths of two sides are unknown; the values of the unknown lengths may be computed as described in Art. 251. Considering the part *ABCDHA*, the latitudes and departures of *AB*, *BC*, and *CD* are known, from which the length and bearing of *DA* is calculated. In the triangle *ADH* the lengths of the sides *DH* and *HA* are found, and their latitudes and departures are calculated. With the latitudes and departures of *AB*, *BC*, *CD*, *DH*, and *HA* known, the area of *ABCDHA* is computed by the D.M.D. method.

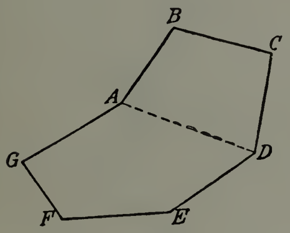


FIG. 364.

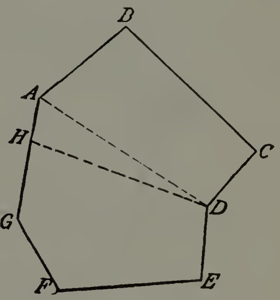


FIG. 365.

In the field the length and direction of the side *DH* is laid off from *D*, and a check on field work and calculations is obtained if the point *H* thus established lies on the line *GA* and if the calculated distance *HA* agrees with the observed distance *HA*.

The computations are further verified by seeing that the sums of the latitudes and departures of *AB*, *BC*, *CD*, *DH*, and *HA* are equal to zero. (This is on the assumption that the latitudes and departures of both *DH* and *HA* are based upon the lengths of these lines as computed from the triangle *ADH*.) The area computations may be checked by observing that the sum of the areas of the two parts, each computed independently of the other, is equal to the area of the entire tract.

366. To Cut Off a Required Area by a Line through a Given Point.—In Fig. 366, *ABCDEFA* represents a tract of land of known dimensions, the corrected latitudes and departures of the sides being given. And *G* represents a point in the boundary through which a line is to pass cutting off a required area from the tract. It is assumed that the area within the tract has been computed by the D.M.D. method and that a sketch of the tract has been prepared.

To find the length and direction of the dividing line the procedure is as follows:

Draw a line *GF* to that corner of the traverse which, from inspection of the sketch, will come nearest being on the required line of division. Calculate the latitude and departure of *CG*. Then in the traverse *ABCGFA* all sides are known except *GF*, which is unknown in both length and direction. By the methods of Art. 247, determine the latitude, departure, length, and bearing of *GF*. By the D.M.D. method compute the area of *FABCG*, the amount cut off by the line *FG*. Find the difference between this area and that required.

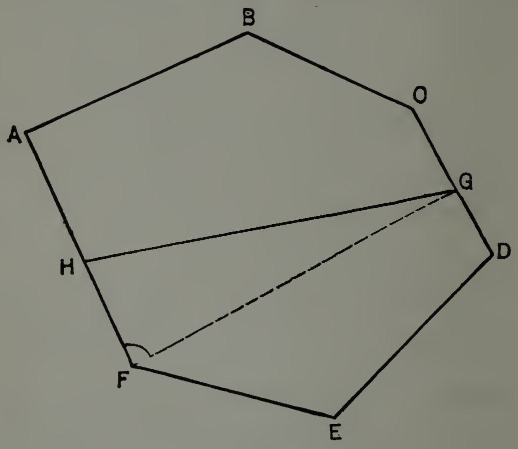


Fig. 366.

In the figure it is assumed that *FABCG* has an area greater than the desired amount, *GH* being the correct position of the dividing line. Then the triangle *GFH* represents this excess area, and since the angle *F* may be calculated from known bearings, there are given in this triangle, one side *FG*, one angle *F*, and the area. Calculate the length *HF* by the relation

$$HF = \frac{2 \times \text{area } GFH}{FG \sin F} \tag{1}$$

Solve the triangle for angle *G* and length *GH*.

By the known direction of *GF* and the angle *G*, calculate the bearing of *GH*. Compute the latitudes and departures of the lines *FH*, *GH*, and *HA*.

367. To Cut Off a Required Area by a Line Running in a Given Direction.—In Fig. 367, *ABCDEF* represents a tract of land of known dimensions and area, which is to be divided into two parts, each of a required area, by a line running in a given direction. The figure is assumed to be drawn at least roughly to scale, and the corrected latitudes and departures are known.

Through the corner that seems likely to be nearest the line cutting off the required area, a trial line *DG* is drawn in the given direction. In the figure *ABCDG*, the latitudes and departures of *AB*, *BC*, and *CD* and the bearings of *DG* and *GB* are known. *GBCDG* therefore is a closed traverse for which the length of two sides *GB* and *DG* are unknown. By the methods of Art. 251, these unknown quantities are found and the latitudes and departures of the courses are determined. By the D.M.D. method the area cut off by the trial line is computed. The difference between this area and that required is represented in the figure by the trapezoid *DGJH* in which the side *DG* is known. The angles at *D* and *G* may be calculated from the known bearings of adjacent sides, and in this way θ and ϕ are determined. Then

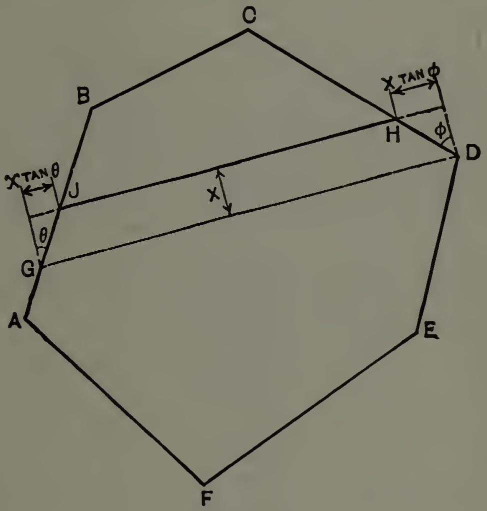


Fig. 367.

$$\textit{Area of trapezoid} = GD \cdot x + \frac{x^2}{2}(\tan\theta + \tan\phi),$$

in which $\tan\theta$ or $\tan\phi$ is positive or negative according as θ or ϕ lies within or without the trapezoid, and x is the altitude of the trapezoid. (In the figure both angles lie without the trapezoid and hence the tangents are negative.) The value of x is found by solving this expression. Then $GJ = x \sec \theta$; $DH = x \sec \phi$; and $JH = GD + x(\tan \theta + \tan \phi)$ in which the signs of $\tan \theta$ and $\tan \phi$ are as given above. In the field the points H and J are established on the lines CD and AB, and at the calculated distances from the adjacent corners. The side JH is then measured. If this measured value agrees with the computed value, the field work and portions of the computations are verified.

A further check on the computations is introduced by calculating the area of $BCHJ$ and comparing it with the required area of this figure.

368. Problems.

1. A square field contains 40 acres. What are its dimensions in chains, in rods, and in feet?

2. How many acres are there in a rectangular tract 50 × 100 ft.? in a tract 400 × 400 ft.? in a tract 2,640 × 2,640 ft.?

3. What is the area of a triangle when its three sides are 219.0, 317.2, and 301.6 ft.? When two sides are 1,167.1 and 392.7 ft. and the included angle 39°46′?

4. Find the error of closure of the following traverse, balance the survey by the compass rule, and compute the area in acres by the D.M.D. method using four-place tables of logarithms:

Course	Bearing	Length, feet
AB	S45¾°E	294.4
BC	N65½°E	262.4
CD	N35¼°E	313.6
DE	S64¼°W	392.0
EF	S59°W	196.0
FA	S26¼°W	240.0

5. Following are the notes for a transit traverse, distances being in Gunter's chains:

Course	Bearing	Length
AB	S58°08′E	10.24
BC	S67°07′E	9.32
CD	S9°39′W	24.00
DE	S84°22′W	24.92
EF	N6°21′E	18.92
FA	N29°52′E	18.80

Calculate the latitudes and departures, balance the survey by the transit rule, and calculate the area in acres by the D.M.D. method. Use four-place logarithms.

6. Given the following azimuth traverse:

Course	Azimuth	Length, feet
AB	126°30′	236.34
BC	195°43′	212.36
CD	174°47′	182.78
DE	301°10′	313.10
EF	42°04′	164.38
FA	2°05′	243.34

Calculate bearings, balance the survey by the compass rule, and compute the area in acres by the D.M.D. method. Use five-place logarithms.

7. Given the following azimuth traverse:

Course	Azimuth	Length, feet
AB	141°19′00″	422.23
BC	138°45′00″	302.91
CD	139°43′30″	437.24
DE	191°34′30″	254.61
EF	199°24′00″	903.95
FG	282°03′30″	1,583.09
GH	30°00′30″	1,496.70
HI	41°15′00″	505.13
IA	101°11′00″	73.07

Calculate bearings, compute latitudes and departures, balance the survey by the compass rule, and find the area in acres by the D.M.D. method, using six-place logarithms.

8. Given the following data resulting from the survey of a traverse:

Course	Bearing	Length, feet
AB	N48°10′E	264.4
BC	N87°31′E	295.6
CD	S8°09′E	281.3
DE	S82°02′W	376.2
EA	N48°23′W	213.6

Balance the survey by the compass rule and calculate the area by the D.M.D. method, employing natural functions and combining direct multiplication and the use of the slide rule.

9. Given the following closed traverse:

Course	Bearing	Distance, feet
AB	S86°24′E	321.2
BC	N74°19′E	354.5
CD	S3°55′E	195.6
DE	S65°02′W	323.6
EF	S27°42′W	266.8
FG	N39°12′W	301.9
GA	N14°33′W	267.5

Balance the survey by the compass rule. Calculate the area by the D.M.D. method.

10. Given the following notes for a closed traverse:

Course	Bearing	Distance, feet
AB	N6°37′W	932.4
BC	N16°43′E	500.0
CD	N2°38′E	1,018.0
DE	N78°53′E	664.9
EF	S25°57′E	900.6
FG	S21°28′W	468.1
GH	S10°22′E	938.7
HI	S45°23′W	595.4
IA	N87°57′W	715.3

Calculate the latitudes and departures, and balance the survey by the compass rule. Assume that the coordinates of *A* are 270.0N and 340.0E and compute the coordinates of all other corners. By the coordinate method calculate the area.

11. A traverse *ABCD* is established inside a four-sided field and the corners of the field are located by angular and linear measurements from the traverse stations, all as indicated by the following data:

Course	Bearing	Length, feet
AB	S89°58′E	296.4
AE	N20°00′W	34.2
BC	S43°20′W	333.9
BF	N35°20′E	16.9
CD	S80°21′W	215.6
CG	S73°00′E	27.6
DA	N27°24′E	314.2
DH	S36°30′W	15.7

Calculate the latitudes and departures and balance by compass rule. Calculate the coordinates of each transit point and of each property corner, using *D* as an origin of coordinates. Calculate the length and bearing of each side of the field *EFGH* and tabulate results. Calculate the area of the field by the coordinate method.

12. Given the following offsets from traverse line to irregular boundary measured at points 25 ft. apart:

Distance, feet	Offset, feet	Distance, feet	Offset, feet
0	0.0	175	81.7
25	57.5	200	95.2
50	87.3	225	97.4
75	90.8	250	85.6
100	93.2	275	73.4
125	79.2	300	24.1
150	60.0		

By the trapezoidal rule calculate the area between traverse line and boundary.

13. Given the data of problem 12, calculate the required area by Simpson's one-third rule.

14. Following are offsets from a traverse line to an irregular boundary taken at irregular intervals:

Distance, feet	Offset, feet	Distance, feet	Offset, feet
0	30.4	200	87.7
20	61.7	210	44.5
60	102.8	260	40.0
70	57.5	290	81.1
100	61.1	360	95.2
180	93.3	410	73.5

Compute the area between traverse line and boundry by means of the trapezoidal rule (Art. 362).

15. Given the data of problem 5, find the area north of a line running from *F* to a point *G* on the line *CD* and distant 10.00 chains from *C*. Calculate the length and bearing of *FG*. Prepare a description of each subdivision.

16. Given the data of problem 6, find the area cut off by a line passing through *F* whose bearing is N80°E. Calculate the length of the line and the distance from *B* to the point of intersection of the line with *BC*. Prepare a description of each subdivision.

17. Given the data of problem 7, the tract to be divided by a line passing through *G*, and the subdivision south of this line to contain 20.00 acres. Find the length and direction of the dividing line and the distance from the nearest corner to the point of intersection of this line with the main traverse. Prepare a description of each subdivision.

CHAPTER XX

UNITED STATES PUBLIC LAND SURVEYS

369. Remarks.—This chapter deals with the methods of subdividing the Public Lands of the United States in accordance with regulations imposed by law and carried out by the Surveying Service of the General Land Office. The methods described are those now in force, but with minor modifications these methods have in principle been followed in all the surveys of the public domain since the rectangular system of subdivision was inaugurated. Under this system the Public Lands of the States of Alabama, Arizona, Arkansas, California, Colorado, Florida, Idaho, Illinois, Indiana, Iowa, Kansas, Louisiana, Michigan, Minnesota, Mississippi, Missouri, Montana, Nebraska, Nevada, New Mexico, North Dakota, Oklahoma, Ohio, Oregon, South Dakota, Utah, Washington, Wisconsin, and Wyoming and the Territory of Alaska have been or are in progress of being surveyed.

The laws regulating the subdivision of Public Lands and the surveying methods employed are fully described in *Manual of Instructions for the Survey of the Public Lands of the United States*, published by the General Land Office, from which the material for this chapter is freely drawn.

370. General Scheme of Subdivision.—The regulations for the subdivision of Public Lands have been altered from time to time in the years since 1785, when the rectangular system was inaugurated; hence, the methods employed in surveying various regions of the United States show marked differences, depending upon the dates when the surveys were made. In general principle, however, the system has remained unchanged, the primary unit being the township, bounded by meridional and latitudinal lines, as nearly as may be 6 miles square, divided into 36 secondary units, called *sections*, as nearly as may be 1 mile square. Obviously, since the meridians converge, it is impossible to lay out a square township by such lines, and having a township which is not square, it is impossible that each of the 36 sections contained therein should be 1 mile square even though all measurements are without error.

Since the earliest surveys, townships and sections have been located with respect to principal axes passing through an origin or an *initial point,* as it is called, the lines forming the axes consisting of a true meridian, called the *principal meridian,* and a true parallel of latitude, called the *base line.* Each principal meridian is given a name to which all subdivisions are referred.

At the present time, the practice is to establish secondary axes at intervals of 24 miles north or south of the base line and at intervals of 24 miles east or west of the principal meridian, thus dividing the tract being surveyed into quadrangles bounded by true meridians 24 miles long and by true parallels, the south boundary of each quadrangle being 24 miles long and the north boundary being 24 miles long, less the convergence of the meridians in that distance. These secondary parallels are called *standard parallels* and each is continuous throughout its length. The secondary meridians are called *guide meridians,* and each is broken at the base line and at each standard parallel.

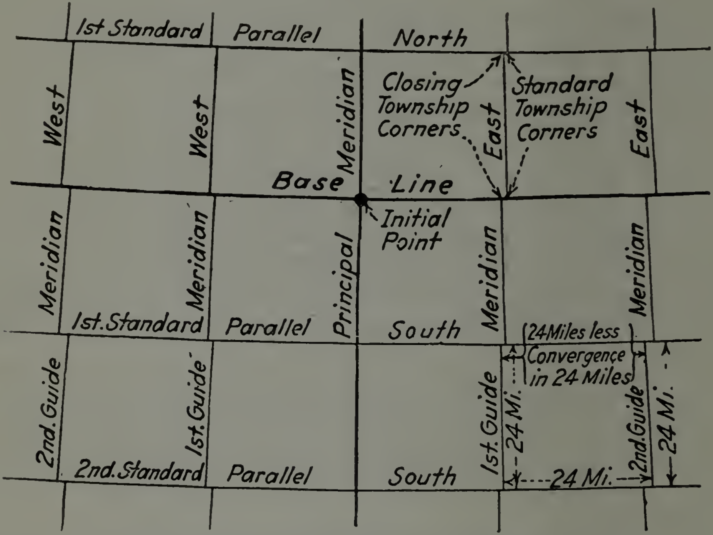

1st Standard
Parallel
North
West
Meridian
West
Meridian
Closing Township Corners
East
Meridian
Standard Township Corners
East
Meridian
Base
Line
Initial Point
Principal
Meridian
1st. Standard
Parallel
South
(24 Miles less Convergence in 24 Miles)
2nd. Guide
1st. Guide
1st. Guide
24 Mi.
2nd. Guide
24 Mi.
2nd. Standard
Parallel
South
24 Mi.

Fig. 370.—Guide meridians and standard parallels.

A typical system of principal and secondary axes is shown in Fig. 370. Both base line and standard parallels, being everywhere perpendicular to the direction of the meridian, are marked on the ground as curved lines, the rate of curvature depending upon the latitude. Principal meridian and guide meridians, being true north and south lines, are laid out as straight lines but converge toward the north, the rate of convergence depending upon the latitude.

Standard parallels are counted north or south of the base line. Thus the *second standard parallel south* indicates a parallel 48 miles south of the base line. Guide meridians are counted east or west of the principal meridian. Thus the *third guide meridian west* is 72 miles west of the principal meridian.

The extent of the surveys which are referred to a given Initial Point may be found by consulting a map published by the General Land Office entitled "United States, Showing Principal Meridians, Base Lines, and Areas Governed Thereby."

371. The division of the quadrangles created by the Standard Parallels and Guide Meridians into townships is accomplished by laying off true meridional lines called *range lines* at intervals of 6 miles along each standard parallel, the range line extending north 24 miles to the next standard parallel; and by joining the township corners established at intervals of 6 miles on the range lines, guide meridians, and principal meridian with latitudinal lines, called *township lines* (see Fig. 371).

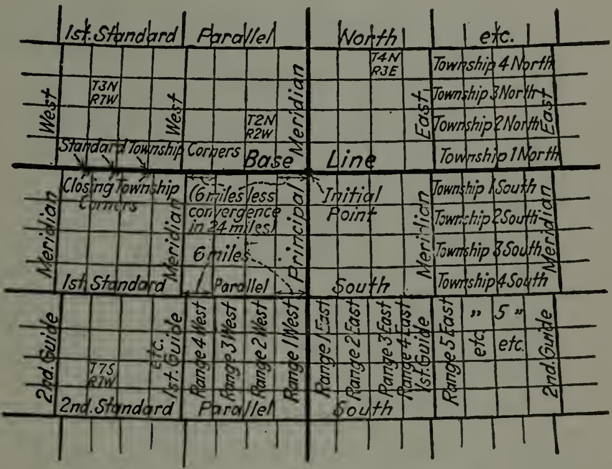
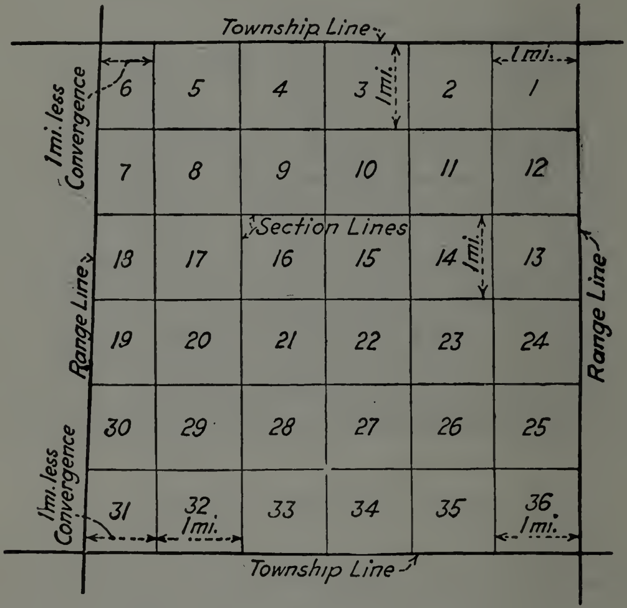

1st. Standard Parallel North etc.
T4N R3E
Township 4 North
T3N R7W
Township 3 North
T2N R2W
Township 2 North
West
West
Meridian
East
East
Standard Township Corners
Base Line
Township 1 North
Closing Township Corners
(6 miles less convergence in 24 miles)
Initial Point
Township 1 South
Township 2 South
Township 3 South
Township 4 South
Meridian
Meridian
Principal Meridian
Meridian
Meridian
6 miles
1st Standard Parallel South
2nd. Guide
etc. 1st. Guide
Range 4 West
Range 3 West
Range 2 West
Range 1 West
Range 1 East
Range 2 East
Range 3 East
Range 4 East
1st. Guide
Range 5 East
" etc.
5 " etc.
2nd. Guide
T7S R7W
2nd. Standard Parallel South

Fig. 371.—Township and range lines.

If a subdivision survey of this character is without error it is evident that the east and west boundaries will be 6 miles long, but that the north and south boundaries, except south boundaries of townships lying north of a base line or standard parallel, will be less than 6 miles long, due to the convergence of the meridians.

A row of townships extending north and south is called a *range;* and a row extending east and west is called a *tier.* Ranges are counted east or west of the principal meridian, and tiers are counted north or south of the base line. Usually for purposes of description the word "tier" is omitted and "township" is substituted therefor. The plan of subdivision is illustrated by Fig. 371. A township is designated by the number of its tier and range, and the name of the principal meridian. Thus, T7S, R7W is read *Township seven south, Range seven west,* and designates a township in the seventh tier south of the base line and the seventh range west of the principal meridian. If the township were among those referred to the initial point of the Third Principal Meridian, for example, the location of the township would be completely described by the designation T7S, R7W of the Third Principal Meridian.



Township Line
1 mi.
1 mi. less Convergence
6
5
4
3
1 mi.
2
1
7
8
9
10
11
12
Section Lines
Range Line
18
17
16
15
14
1 mi.
13
Range Line
19
20
21
22
23
24
30
29
28
27
26
25
1 mi. less Convergence
31
32
1 mi.
33
34
35
36
1 mi.
Township Line

Fig. 372.—Numbering of sections.

372. The division of townships into sections is performed by establishing, at intervals of 1 mile, meridional lines parallel with the east boundary of the township and by joining the section corners established at intervals of a mile with straight latitudinal lines. These lines, called *section lines,* divide each township into 36 sections. The plan of designating the sections in each township, illustrated by Fig. 372, consists of numbering them consecutively from east to west and from west to east, beginning with No. 1 in the northeast corner and ending with No. 36 in the southeast corner. Thus

Section 16 is a section whose center is 3½ miles north and 3½ miles west of the southeast corner of a township.

A section is legally described by giving its number, the tier and range of the township in which the section is located, and the name of the principal meridian to which the surveys in the given locality are referred. For example, the location of a section is defined by the description *Section* 16, T7S, R7W, *of the Third Principal Meridian.*

On account of the convergence of the range lines (true meridians) forming the east and west boundaries of townships, the north and south boundaries are in general less than 6 miles in length, the south boundaries of townships north of a standard parallel or correction line being exceptions. Since the meridional section lines are run parallel to the east boundary of the township, it follows that, if the surveys be without error, all sections except those adjacent to the west boundary will be 1 mile square, but that those adjacent to the west boundary will have a latitudinal dimension less than 1 mile by an amount equal to the convergence of the range lines within the distance from the section to the nearest correction line to the south.

373. The principal meridian, base line, standard parallels, and guide meridians are called *standard lines* and the corners placed during the process of establishing these lines are called *standard corners.* In addition, there are along the base line and each standard parallel, a series of corners established later during the process of subdivision into townships and sections, which are designated as *correction corners;* hence the term *correction line* is used to designate either a base line or standard parallel.

374. It should be understood that the plan of subdivision just described is the one which is carried out when conditions allow. There are, of course, always present the errors of measurement, so that the actual lengths and directions established in the field do not entirely agree with the theoretical values. But in addition, conditions met in the field often make it inexpedient or impossible to establish the lines of the survey in exact accordance with the specified plan. Thus there are numerous instances of standard parallels and guide meridians having been originally established at intervals of 30 and 36 miles, under old regulations, and regions partially surveyed. Later, under present regulations, meridians have been established between the old ones, and recent subdivisions are, hence, referred to standard lines many of which are less than 24 miles apart. Also the presence of large bodies of water, mountain ranges, Indian reservations, etc., may greatly modify the method of division, many townships and sections being made fractional.

375. Principal Meridian.—This line is established as a true meridian through the initial point, either north or south, or in both directions, as conditions require. Permanent quarter-section and section corners are established alternately at intervals of 40 chains (½ mile) and regular township corners are placed at intervals of 480 chains (6 miles)

376. Base Line.—From the initial point the base line is extended east and west on a true parallel of latitude, standard quarter-section and section corners being established alternately at intervals of 40 chains (½ mile) and standard township corners being placed at intervals of 480 chains (6 miles).

377. Standard Parallels.—At intervals of 24 miles north and south of the base line, true parallels of latitude, called *standard parallels* are run east and west from the principal meridian, these lines being established in a manner identical with that prescribed for the survey of the base line.

378. Guide Meridians.—These lines are extended north from the base line and standard parallels at intervals of 24 miles east and west of the principal meridian. Each meridian is established as a true meridian in a manner identical with that employed in laying off the principal meridian. The guide meridians terminate at the points of their intersection with the standard parallels, and hence are broken lines, each segment being theoretically 24 miles long. Errors of measurement are placed in the most northerly half mile of each 24-mile segment. At the true point of intersection of the guide meridian and standard parallel, a closing township corner is established.

379. Convergency of Meridians.—In Fig. 379*a*, let ACP and BDP represent two meridians, P being the north pole of the earth, and let CD be the arc of a parallel of latitude at a latitude $\phi = COA = DOB$ at which latitude it is desired to determine the angular and linear convergency of the meridians. Also let O be the center of the earth and AB be an arc of the equator intercepted by the two meridians. Consider the earth as a perfect sphere.

Then the difference in longitude between the two meridians is

$$\lambda = \frac{CD}{CO'} \text{ or } CD = CO' \cdot \lambda$$

The latitude of the arc CD is

$$\phi = DOB = DEO'$$

Then,

$$\sin \phi = \frac{DO'}{DE} \text{ or } DE = \frac{DO'}{\sin \phi}$$

With a negligible error the angle of convergence is

$$\theta = \frac{CD}{DE}$$

Substituting the values for CD and DE obtained above

$$\theta = \lambda \sin \phi \tag{1}$$

Let the distance between two meridians measured along a parallel be $d = CD$, and let the radius of the earth at the parallel be R. Then from the figure

$$\lambda = \frac{CD}{CO'} = \frac{d}{R \cos \phi}$$

Substituting this value in (1), there results

$$\theta = \frac{d \sin \phi}{R \cos \phi} = \frac{d \tan \phi}{R},$$

where θ is in radians. (2)

If d is in miles and $R =$ 20,890,000 ft., the approximate mean radius of the earth, then θ in seconds is from (2)

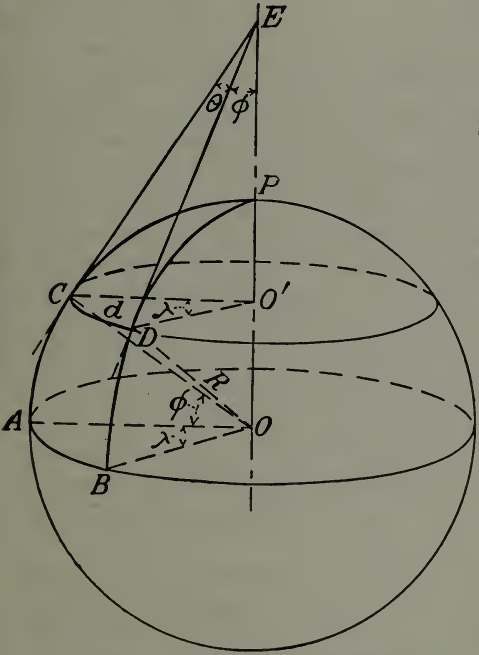


FIG. 379a.—Convergency of meridians.

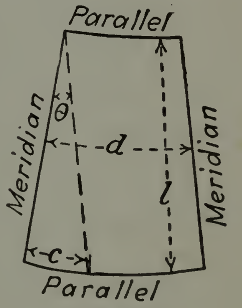


FIG. 379b.

$$\theta'' = 52.13d \tan \phi. \tag{3}$$

In Fig. 379b let l be the length of the meridian between two parallels and let θ be the mean angle of convergence of two meridians whose mean latitude is ϕ and whose mean distance apart measured on a parallel is d. Also let the linear convergence of the two meridians, measured along a parallel, be c. Then, with small approximation, $\theta = \frac{c}{l}$.

Substituting this value in (2) and solving for c, there results

$$c = \frac{dl \tan \phi}{R} \tag{4}$$

which gives with sufficient accuracy for land surveying the linear convergence between two meridians.

380. To Lay Off a Parallel of Latitude.—Since the base line and standard parallels, as well as the latitudinal township lines, are true parallels of latitude, they are curved lines when established on the surface of the earth, as is evident from the fact that meridians converge and that a parallel of latitude is a line whose direction at any point is perpendicular to the direction of the meridian at that point. Its projection on the surface of the earth is the base element of a cone whose vertex is at the earth's center, and the radius of whose base is $R \sin \phi$, in which R is the earth's radius and ϕ is the latitude. It is defined by a plane at right angles to the earth's axis cutting the earth's surface on a circle whose radius decreases as the latitude increases. It is true that the rate of curvature within the latitudes of the United States is so small that two points, say a quarter of a mile apart, on the same parallel of latitude will, for all practical purposes, define the direction of the curve at either point, but the projection of a line so defined in either direction would describe a great circle of the earth, gradually departing southerly from the true parallel. The great circle tangent to the parallel at any point along the parallel is called the *tangent to the parallel*, and it coincides with the true latitude curve only at the point of origin.

Though the tangent is a straight line, its bearing is not constant, but varies with the distance from the point of tangency, its deflection from true east or true west being equal to the angle of convergence of the meridians within the distance from the point of tangency to the given point.

By means of a sketch it can be readily shown that, within the limits of precision necessary in land surveying, the offset from tangent to parallel at any distance from the point of tangency is one-half the linear convergency of the meridians within an equal distance. With small error a parallel of latitude may, within the limits of distance here considered, be assumed to behave as a parabola. Hence, the offset from tangent to curve at any point, for all practical purposes, may be said to vary as the square of the distance from the point of tangency.

381. Township Exteriors.—The range lines or meridional boundaries of the townships are run from south to north on true meridians, quarter-section and section corners being established alternately at intervals of 40 chains.

Each township line forming the north or south boundary of a township is run as a random parallel, from the old towards the new meridional boundary. On the true parallel are established quarter-section and section corners, alternately at intervals of 40 chains, measurements being made from the boundary last run. The fractional measurement is placed in the most westerly half mile.

A range line is terminated at its intersection with a standard parallel, the excess or deficiency in the measured distance between standard parallels being placed in the most northerly half mile. At the point of intersection between the range line and standard parallel, a closing township corner is established.

When a full 24-mile quadrangle is to be divided into townships, the survey is usually begun at the southeast corner of the southwest township of the quadrangle (see Fig. 371). The range line is run 6 miles north, and the latitudinal boundary connecting the regular township corner previously established on the guide meridian or principal meridian, with the 6-mile point on the range line, is established. The range line is then continued another 6 miles and a second latitudinal boundary is established in the same manner as the first, connecting the second regular township corner north of the standard parallel on the guide or principal meridian, with the 12-mile point on the range line. Again the process is repeated, and then the range line is extended north of the 18-mile point to the closing township corner on the standard parallel. The most westerly range of townships is thus surveyed.

In a similar manner the boundaries of the townships forming the next range to the east are established.

Finally, the third range line, started at the southwest corner of the southeast township, is laid off as the others, but at the 6-, 12-, and 18-mile points latitudinal lines are run to the west to connect with corresponding township corners, and also to the east to connect with the first, second, and third regular township corners north of the standard parallel on the guide meridian.

382. Subdivision of Townships.—Following the normal plan for subdividing townships with regular boundaries, the subdivisional survey is begun on the south boundary of the township at the section corner between sections 35 and 36 (see Fig. 382) and the line between sections 35 and 36 is run in a northerly direction parallel to the east boundary of the township, the quarter-section corner between 35 and 36 being set at 40 chains, and the section corner common to sections 25, 26, 35, and 36 being set at 80 chains. From this latter corner a random line is run eastward to connect with the section corner on the township boundary, the quarter-section corner common to sections 25 and 36 being placed midway between section corners.

The above process is repeated for the successive meridional and latitudinal lines, the order in which the lines are surveyed being as indicated by the numbers on the section lines in the figure. Each meridional section line is run north parallel to the east boundary, and except for the western range of sections, each random latitudinal section line is run east parallel to the south boundary of the section to the south.

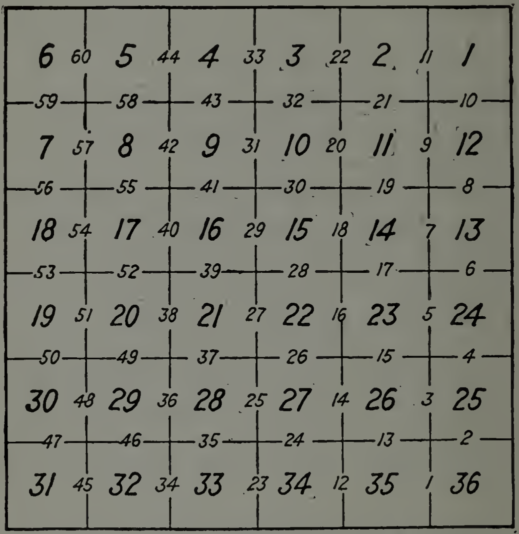

6 60 5 44 4 33 3 22 2 11 1
59 58 43 32 21 10
7 57 8 42 9 31 10 20 11 9 12
56 55 41 30 19 8
18 54 17 40 16 29 15 18 14 7 13
53 52 39 28 17 6
19 51 20 38 21 27 22 16 23 5 24
50 49 37 26 15 4
30 48 29 36 28 25 27 14 26 3 25
47 46 35 24 13 2
31 45 32 34 33 23 34 12 35 1 36

Fig. 382.—Order of establishing section lines.

When the northern boundary of the township is not a base line or standard parallel, the section lines to the south are run north as random lines to connect with section corners already established on the township lines, the quarter-section corners being placed a full 40 chains from the south boundary of the sections. In this way the excess or deficiency in linear measurement is, as before, placed in that portion of the line between the permanent quarter-section corner and the north boundary of the township.

When the north boundary of the township is a base line or standard parallel, the section lines to the south are run as true lines parallel to the east boundary of the township, permanent quarter-section corners being set at 40 chains, and closing section corners being established at the points of intersection of the section lines and base line or standard parallel.

When the most westerly range is reached, a random line parallel to the south boundary of the township is run west from the corner of sections 29, 30, 31, and 32, to the previously established section corner on the range line or guide meridian at a full 40 chains from the corner of sections 29, 30, 31, and 32. The deficiency due to convergency of the meridians and the excess or deficiency due to errors in linear measurements are thrown in the most westerly half-mile.

The survey of the other sections comprising the most westerly range is continued in similar manner. Lastly, the meridional section line extending north of the corner of sections 5, 6, 7, and 8 is established as were corresponding lines to the east, the procedure depending upon whether or not the north boundary is a correction line, as previously explained.

383. Subdivision of Sections.—While acts of Congress contain the fundamental provisions for the subdivision of sections into quarter-sections and quarter-quarter sections, the sections are not, except in rare instances, subdivided in the field by United States surveyors; but certain lines of subdivision are shown upon the official plats, and the surveyor in private practice is compelled to correlate conditions found on the ground with those shown on the official plat. In the work of subdividing sections into the parts shown on the official plat the local surveyor can not properly serve his client unless he is familiar with the land laws regarding the subdivision of sections, nor in the event of the loss of original monuments, can the surveyor expect legally to restore the same unless he understands the principles employed in the execution of the original survey.

384. Subdivision by Protraction.—Upon the official government township plats the interior boundaries of quarter-sections are shown as straight dotted lines connecting opposite quarter-section corners. The sections adjacent to the north and west boundaries of a normal township, except section 6, are further subdivided into parts containing two regular half-quarter sections and four lots, the latter containing the fractional areas resulting from the plan of subdivision of the normal township.

385. In addition to sections made fractional by reason of their being adjacent to the north and west boundaries of a township, they are also frequently made fractional on account of meanderable bodies of water, mining claims, and other segregated areas being within their limits. Such sections are subdivided by protraction into such regular and fractional parts as are necessary for the entry of the undisposed public lands and to describe these lands separately from the segregated areas.

386. Subdivisional Surveys.—The rules for the subdivision of sections given in the following paragraphs are based upon the General Land Laws. When an entryman has acquired title to a certain legal subdivision he becomes the owner of the identical ground area represented by the same subdivision on the official plat. Preliminary to subdivision it is necessary to identify the actual boundaries of the section, as it can not be legally subdivided until the section and exterior quarter-section corners have either been found or have been restored to their original positions, and the resulting courses and distances have been redetermined in the field. Having located the opposite quarter-section corners, the legal center of the section, or interior quarter-section corner may be placed. If the boundaries of quarter-quarter sections or fractional lots are to be established on the ground, it is necessary to measure the lengths of the boundaries of the quarter section and to fix thereon the quarter-quarter section corners in accordance with the proportional distances given upon the official plat, when the legal center of the quarter section may be located.

387. Subdivision of Sections into Quarter-sections.—According to law, the procedure to be followed in the subdivision of a section into quarter-sections is to run straight lines between the established opposite quarter-section corners. The point of intersection of lines thus run is the quarter-section corner common to each of the four quarter-sections into which the section is divided. It is called the *interior quarter-section corner* and is the legal center of the section.

388. Subdivision of Fractional Sections.—Where opposite corresponding quarter-section corners of a section have not been or can not be fixed, as is frequently the case when sections are made fractional by streams, lakes, etc., the lines of sectional subdivisions are run on courses mean between those of adjoining established section lines, or are run on courses parallel to the east, south, west, or north boundary of the section where there is no opposite section line.

389. Subdivision of Regular Quarter-section into Quarter-quarter Sections.—Preliminary to the subdivision of regular quarter-sections, the quarter-quarter-section corners are established at points midway between the section and exterior quarter-section corners and between the exterior quarter-section corners and the center of the section. The quarter-quarter-section corners having thus been established, the center lines of the quarter-section are run as straight lines between opposite corresponding quarter-quarter-section corners on the boundaries of the quarter-section. The intersection of these lines is common to the four quarter-quarter sections into which the quarter-section is divided. It is called the *interior quarter-quarter-section corner*, and it marks the legal center of the quarter-section.

390. Subdivision of an Irregular Quarter-section into Quarter-quarter Sections.—This case arises when the quarter-section is adjacent to the north or east boundary of a regular township and also when the quarter-section adjoins any irregular boundary of an irregular township. The procedure is the same as that outlined in the preceding article, except that the quarter-quarter-section corners on the boundaries of the quarter-section which are normal to the township exterior are placed at 20 chains, proportionate measurement, counting from the regular quarter-section corner.

391. Meandering.—In the process of surveying the public lands, all navigable bodies of water and important rivers and lakes below the line of mean high water are segregated from the lands which are open to private ownership. In the process of subdivision, the regular section lines are run to an intersection with the mean high-water mark of such a body of water where corners, called *meander corners*, are established. The traverse which is run between meander corners, approximately following the margin of a permanent body of water, is called a *meander line*, and the process of establishing such lines is called *meandering.*

Meander lines are not in any sense boundaries but are lines which are run for the purpose of approximately locating the water boundaries, and while the official plats show fractional lots as bounded in part by meander lines, it is an established principle that ownership does not stop at such boundaries.

In running a meander line, the surveyor begins at a meander corner and follows the bank or shore line, as closely as convenience permits, to the next meander corner, the traverse being a succession of straight lines.

392. Marking Lines between Corners.—As a final step in the survey of the public lands, it is the aim permanently to fix the position of the legal lines of subdivision with reference to objects on the surface of the earth. This is accomplished (1) by setting monuments at the regular corners, (2) by finding the location of the officially surveyed lines with respect to natural features of the terrain, and (3) by indicating the position of the regular lines through living timber by *blazing* and by *hack marks*.

393. Notes.—The field notes taken in connection with the survey of public lands are required to be in narrative form and are designed to furnish not only a record of the exact surveying procedure followed in the field, but also to provide a report showing the character of the land, soil, and timber traversed by the lines of subdivision, and to give a detailed schedule of the topographic features adjacent to the lines, together with reference measurements showing the position of the lines with respect to natural objects, to improvements, and to the lines of other surveys. In this way, the notes serve three purposes: (1) The field procedure is made a matter of official record; (2) the general characteristics of the territory served by the subdivision surveys are secured; and (3) the reference measurements to objects along the surveyed lines furnish evidence by which the established points and lines become practically unchangeable.

394. Corners.—In the subdivision of the public lands as described in the preceding articles, it is required that the United States surveyors shall permanently mark the position of the township, section, exterior quarter-section, and meander corners, and also such quarter-quarter-section corners as it is necessary to establish in connection with the subdivision of fractional sections. For this purpose monuments of a character, and marked in a manner specified by regulations of the General Land Office, are employed.

The location of every such corner monument is, in accordance with definite rule, referred to such nearby objects as are available and suitable for this purpose; and when the location of the true corner can not be marked in the ordinary manner an appropriate witness corner is established.

395. Corner Material.—At the present time the regulations of the General Land Office specify as the official corner monument, which is to be used when circumstances will permit, a post made from commercial wrought-iron pipe, filled with concrete, 1 to 3 in. in diameter and 36 in. long, to the top of which is riveted a brass cap on which the corner markings are stamped with steel dies.

Where the procedure is duly authorized, durable native stone may be substituted for the model iron post described above. The required corner markings are cut with a chisel, and the stone is ordinarily set with about three-quarters of its length in the ground.

396. Witness Corners.—Where a true corner point falls within an unmeandered stream or lake or within a marsh or in an inaccessible place, a witness corner is established in a convenient location nearby, preferably upon one of the surveyed lines leading to the position of the regular corner. Also where the true point falls within the traveled limits of a road, a cross-marked stone is deposited below the road surface, and a witness corner is placed in a suitable location outside the roadway.

397. Marking Corners.—All classes of monuments are marked in accordance with a system which has been designed to provide a ready identification of the position and character of the monument on which the markings appear. Iron posts and tree corners are marked with capital letters which are themselves keys to the character of the monument and with Arabic figures giving the section and township and range numbers of the adjacent subdivisions and the year in which the survey was made. Certain additional marks in the shape of *notches* and *grooves* are placed on the vertical edges or faces of stone monuments, the numbers of marks in the case of an exterior corner being equal to the distance in miles along the township or range line from the monument to the adjoining township corner, and in the case of interior corners, being equal to the distance in miles along section lines from the monument to the adjoining township boundary, thus furnishing a means of determining the numbers of the adjoining sections.

Following is an index of the ordinary markings common to all classes of corners:

Mark	Meaning	Mark	Meaning
AMC	Auxiliary meander corner	S	South
BO	Bearing object	SMC	Special meander corner
BT	Bearing tree	T	Township
C	Center	W	West
CC	Closing corner	WC	Witness corner
E	East	WP	Witness point
MC	Meander corner	¼	Quarter-section
N	North	1/16	Quarter-quarter, or sixteenth-section
R	Range		
S	Section		

All standard township, section, and quarter-section corners on base line and standard parallels are marked "SC." All closing township and section corners on these lines are marked "CC."

398. Restoration of Lost Corners.—It is one of the important duties of the local or county surveyor, in the relocation of property lines, to examine such evidence as is available and to identify the official corners if they exist. Should a search of this kind result in failure, then it is the duty of the surveyor to employ a process of field measurements which will result in the lost corner being restored to its most probable original position. As here employed the term *corner* is used to designate a point established by a survey, while the term *monument* is used to indicate the object placed to mark the corner point upon the surface of the earth.

399. When the original position of a corner can not be determined beyond reasonable doubt, either from traces of the original monument or from other reliable evidence relating to the position of the original monument, the corner is said to be *lost;* and it is restored to its original position, as nearly as possible, by processes of surveying, which involve the retracement of lines leading to the corner.

In connection with all resurveys of this character, since the purpose is to restore corners to their original location, it is essential that the laying off of a given distance at the time of the resurvey should render the same absolute distance between two points on the ground as was the case during the original survey when the corresponding measurement was effected. For reasons which have been discussed in earlier chapters, the measurement of a given known line at the time of a resurvey will not in general agree with the length of the line as recorded in the original survey. Thus, where linear measurements are necessary to the restoration of a lost corner, the principle of proportionate measurements must be employed; this consists first of comparing the resurvey measurement with the original measurement between two existing corners between which lies a lost corner, and then of laying off a distance from one of the existing corners to the lost corner which bears the same proportion to the corresponding original measurement, that the resurvey distance between the two existing monuments bears to the original recorded length between these points. In this way, there is established by resurvey the same relation between the several parts of a line that originally existed. The ratio of the resurvey measurement to the original measurement of the whole distance having been applied to the original measurement of each part to secure the resurvey measurement of the corresponding part, then the sum of the measurements thus calculated for the several parts must equal the resurvey measurement of the whole distance.

400. Problems.

1. Find the angle of convergence between two meridians 6 miles apart whose mean latitude is 32°20′. Calculate the linear convergency, measured along a parallel of latitude, in a distance of 6 miles.

2. Find the angle of convergence between two meridians whose distance apart is 24 miles and whose mean latitude is 45°. Compute the linear convergency in a distance of 24 miles.

3. Find the length of 1° longitude at a latitude of 40°06′20″.

4. Calculate the offsets between the tangent and the parallel at intervals of ½ mile over a distance of 6 miles when the latitude is 40°06′20″.

5. Show the dimensions and areas of the protracted subdivisions of Section 2, as required by law to be shown by official plat, when the north, east, south, and west boundaries are respectively 80.24, 80.16, 79.92, and 80.20 chains.

6. Show the dimensions and areas of the protracted subdivisions of Section 7, as required by law to be shown on the official plat, when the north, east, south, and west boundaries are respectively 76.84, 80.00, 76.64, and 80.00 chains.

7. Show the dimensions and areas of the protracted subdivisions of Section 6, as required by law to be shown on the official plat, when the north, east, south, and west boundaries are respectively 76.36. 80.44 76.60, and 80.00 chains.

CHAPTER XXI

ROUTE SURVEYING

401. Remarks.—Surveys made for the purpose of locating and building railways, highways, canals, power transmission lines, and other utilities which are constructed across country for purposes of transportation or communication are called *route surveys*, and the work of conducting such surveys is called *route surveying*. Surveys of this character are necessary, first, for the purpose of selecting the general route to be followed, and second, to fix the grades, alignment, and other details of the selected route in order that the *project* may be faithfully constructed in accordance with a definite plan.

Obviously the character of the enterprise has its influence upon the route selected. It is clear that the location of any route involves a study to determine the manner in which certain definite requirements of the enterprise may economically be met.

The details of the surveying methods employed naturally vary somewhat with the character of the project, but certain field methods are generally applicable.

402. General Field Procedure.—Ordinarily, it is impossible to complete in one operation the work of surveying for the location of a route and for the construction of a utility between two given terminals. First, a study is made to determine the route to be selected. Any such study, whether made by means of some sort of a survey in the field, or by means of maps, profiles, or other records in the office, may be called a *reconnaissance*.

Following the reconnaisance, a transit line is run over the general route selected. This may mark exactly the line upon which construction is to be undertaken, in which case it is called the *location* or the *located line*, or it may mark the approximate route, in which case it is called *the preliminary line*, or sometimes *the preliminary location*. In the latter case, the *preliminary* is followed by the *location*.

After the location has been run, so-called construction surveys may be necessary for staking out structures, or to determine the amount of work performed during construction.

RAILROAD SURVEYS

403. Reconnaissance.—The purpose of the reconnaissance is the selection of the general route. It requires an extensive study of the whole area that might possibly be used for the location, in order that no possible route may be overlooked or disregarded. As a result of the reconnaissance, only one or two narrow strips of territory will be subjected to the more detailed and accurate study which is to follow. Consequently no possible route that is missed during the reconnaissance will be discovered by the later work.

There must be secured a mental picture of the topography of the whole area under investigation. Maps are of the greatest assistance and all available maps should be secured and studied. If contour maps of the region can be obtained the problem of the reconnaissance becomes a relatively simple one. If such data are not available, it may be found desirable roughly to map to small scale all or a part of the area under consideration.

In the field, a study is first made of the drainage channels, their position, size, direction, and slope. Then the divides or watershed-limit lines between the different drainage basins are examined and the elevations and positions of low points on the ridges, known as passes or saddles, are determined.

In this way are fixed certain definite points through which the line will run.

Although made by rough methods, the reconnaissance is probably the most important part of the work of location, because of the fact that it determines the general route to be used.

404. Preliminary Survey.—After the reconnaissance has fixed one or two general routes for further study, a narrow strip of country along each possible route is carefully surveyed and mapped, the strip being of sufficient width to ensure the final location falling within the limits of the survey. The area outside the strip thus surveyed is disregarded as offering no route as favorable to economic construction as the route or routes selected for preliminary survey.

The preliminary survey may be run by use of (1) transit, tape, and level; (2) transit and stadia; and (3) the plane table.

405. Transit-tape-level Method.—Formerly this method was employed practically to the exclusion of all others. It is especially adapted to lines through wooded country, but for lines through open country it has been largely supplanted by the transit-stadia method described in the succeeding article.

The survey corps is divided into three parties, the transit party, the level party, and the topography party. The transit party is organized in the usual manner. The chief of party usually runs the transit and, following the instructions of the locating engineer, he runs a continuous traverse at random approximately along the middle of the strip through which it appears that the final line will lie, observing angles in the traverse line, directing the chainmen, and recording angular and linear measurements. Usually the traverse is run by the method of deflection angles. The chainage is carried forward from the point of beginning, stakes, each marked with the station number, being set at all full 100-ft. stations and at any plus stations that are established. Roads, streams, land lines, etc., intersected by the traverse line are shown by sketch, and the plus to such features is determined.

The level party, consisting of levelman and rodman, follows the transit party, taking profile levels along the traverse in the usual manner, ground elevations being determined at all stakes set by the transit party, at changes in slope, and at roads and streams. At the same time that the profile levels are run, bench marks of more or less permanent character are established along the line at intervals of a mile or less.

The topography party, consisting of the topographer, the topographer's rodman, and frequently a tapeman, follows the level party, the elevations of traverse stations having been previously obtained from the levelman. By use of the hand level or clinometer, sometimes in flat country by use of the engineers' level, elevations are determined along transverse lines at right angles to the traverse, for a distance on either side of the traverse sufficient to construct a map wide enough to include the position of the located line. The transverse lines are normally taken at each 100-ft. station along the traverse, but in very irregular country may be as close together as 25 ft. and in very smooth country may be as far apart as 500 ft. In addition to the determination of elevations at critical points along the transverse lines, the topography party also locates such details as buildings, roads, property lines, streams, etc.

406. Transit-stadia Method.—This method is particularly adapted to railroad preliminary surveys through open country where clear sights may be obtained without cutting. Where the topography is not badly broken and the country is open, the transit-stadia method is, under normal conditions, the most satisfactory method in use today. As compared with the method of the preceding article, the transit-stadia survey as ordinarily performed requires a lesser number of men and is considerably more rapid in the field.

The usual procedure is to run the traverse and to take side shots at the same time. Thus horizontal and vertical control are established and details are observed in one operation. The details of conducting a survey of this character are more fully described in Art. 481.

407. Plane-table Method.—This method is occasionally employed for preliminary surveys, but is not adapted to use in wooded country, nor is it conveniently employed in extremes of weather. For very short lines the field procedure is much the same as with the transit-stadia method, except that the map of strip is constructed in the field as the work progresses. The use of the plane table is described in Arts. 435, 436, 483. On extensive surveys the plane table is employed as an auxiliary for the mapping of topographic details, the main traverse being run with transit and tape, and elevations of traverse stations being obtained by direct leveling, as described in Art. 405.

408. Preliminary Profile and Map.—A profile of the ground along the traverse line is prepared as described in Art. 148, the usual horizontal scale being 400 ft. = 1 in. and the most common vertical scale being 20 ft. = 1 in.

From the notes of the preliminary survey there is also prepared a map showing the topographic features and other details of the strip of country along the selected route. Contours are normally shown at 5-ft. intervals, but the interval may be 2 ft. or even 1 ft. in level country and may be 10 ft. or greater in rough country. (For methods of platting see Chap. XIV and for contour map construction see Art. 454.

Both the map and profile are employed by the locating engineer as a guide during the progress of the preliminary survey, and for this reason each day's work is ordinarily plotted before the next day's work in the field is begun.

409. Location Survey.—Upon the basis of the preliminary-line map and further detailed study of the ground surface, the position of the located center line is determined. Generally a so-called paper location is drawn upon the map at this stage of the work, before the line is staked out in the field; but sometimes the line may be run on the ground directly, the map being used in the field to assist in placing the line.

If a complete paper location is made, it is possible to construct from the contours a profile of the paper location, to fix a grade line on that profile, and to make an estimate of cost of construction.

In running the location survey, curves are introduced at the places where the direction of the line changes; such curves are fully staked out, with stakes not more than 100 ft. apart and sometimes with stakes 25 or 50 ft. apart.

410. Circular Curves.—Most of the curves are arcs of circles connected by straight lines tangent to them, called *tangents*. The point where the curve begins is commonly called the point of curve, written P.C.; similarly the point where the curve ends is called the point of tangent, written P.T.; and the point where two tangents produced intersect is called the point of intersection or the vertex, written P.I., or V.

Stationing progresses from the point of beginning, measurements being made from point to point around the curves. The 100-ft. distances from station to station on a curve are normally measured as chord distances of 100 ft. each, but these 100-ft. distances are sometimes considered as being distances along the arcs, the proper corresponding chord lengths being calculated for measurement in the field. Such will be the case along curved property boundaries and where curves of very short radius are used, say anything less than 100 or 200 ft.; also in highway surveys, discussed later in this chapter. In railroad practice it is the custom to express the lengths of the curves in terms of distances along a series of chords, each of length 100 feet or less.

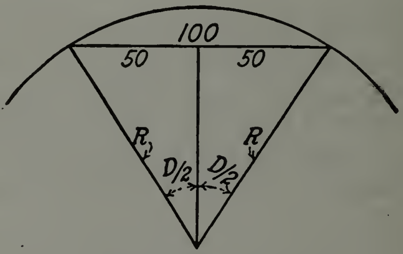


Fig. 410.—Degree of curve.

The sharpness of the curvature may be expressed in either of two ways:

1. By stating the length of the radius.

2. By stating the angle subtended at the center by a chord of 100 ft. This angle is called the "degree of curve," usually denoted by the letter *D*. Then, as illustrated by Fig. 410, if the radius be denoted by the letter *R*

$$\frac{50}{R} = \sin \tfrac{1}{2}D; \text{ or } R = \frac{50}{\sin \tfrac{1}{2}D}$$

From this relation the radius may be found if the degree of curve is known, or *vice versa.*

411. Geometry of the Circular Curve.—In discussing circular curves the following geometrical facts are employed:

1. An inscribed angle is measured by one-half its intercepted arc, and inscribed angles having the same or equal intercepted arcs are equal. Thus, in Fig. 411, the angle *ACB*, at any point *C* on the circumference, subtended by an arc *AB*, is one-half the angle *AOB* at

the center O, subtended by the same arc AB; and the angles at the points C and C' are equal.

2. An angle formed by a tangent and a chord is measured by one-half its intercepted arc. Thus, in Fig. 411, the angle at the point A

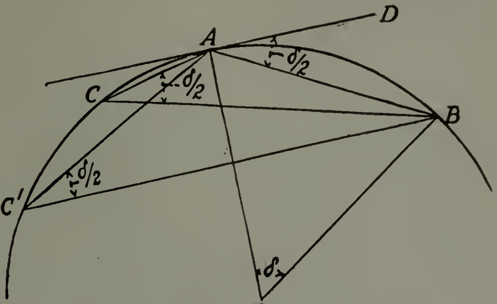


Fig. 411.

between AD, the tangent to the curve at that point, and the chord AB, is one-half the angle at the center O subtended by the same arc AB. This is a special case of the proposition above, when the point C moves to A.

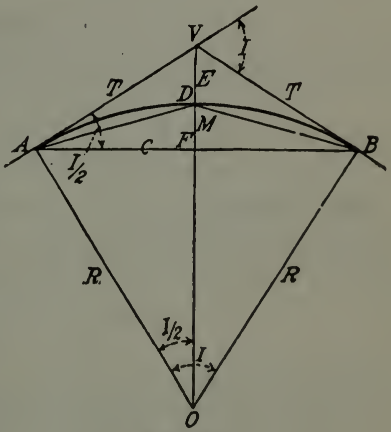


Fig. 412.

3. The two tangent distances to a circular curve, from the point of intersection of the tangents to the points of tangency, are equal in length.

4. Two angles are equal if their sides are perpendicular each to each, in the same order.

412. Curve Formulas.—Figure 412 represents a circular curve joining two straight lines or tangents. In the field the intersection angle I between the two tangents is measured. In any particular case the radius of the curve is selected to fit the topography and the operating conditions on the line when constructed. The line OV bisects the angles at V and at O, bisects the chord AB and the arc ADB, and is perpendicular to the chord AB at F. From the figure $\angle AOB = I$ and $\angle AOV = \angle VOB = \frac{1}{2}I$.

The chord $AB = C$ from the beginning to the end of the curve is called the *long chord*. The distance $AV = BV = T$ from vertex to P.C. or P.T. is called the *tangent distance.* The distance $DF = M$ from the mid-point of the arc to the mid-point of the chord is called the *middle ordinate.* The distance $DV = E$ from the mid-point of the arc to the vertex is called the *external distance.*

Given the radius of the curve $OA = OB = R$, and the intersection angle I, then in the triangle OAV,

$$\frac{T}{R} = \tan \tfrac{1}{2}I$$

$$T = R \tan \tfrac{1}{2}I = \text{tangent distance}$$

$$E = R \sec \tfrac{1}{2}I - R = R(\sec \tfrac{1}{2}I - 1) = R \operatorname{ex\,sec} \tfrac{1}{2}I = \text{external distance.}$$

In the triangle AOF,

$$\frac{C}{2} = R \sin \tfrac{1}{2}I$$

$$C = 2R \sin \tfrac{1}{2}I = \text{long chord}$$

$$M = R - R \cos \tfrac{1}{2}I = R(1 - \cos \tfrac{1}{2}I) = R \operatorname{vers} \tfrac{1}{2}I = \text{middle ordinate.}$$

In the triangle AVF, in which $\angle VAF = \tfrac{1}{2}I$,

$$\frac{C}{2} = AV \cos \tfrac{1}{2}I = T \cos \tfrac{1}{2}I$$

$$C = 2T \cos \tfrac{1}{2}I.$$

In the triangle ADF, in which $\angle DAF = \tfrac{1}{4}I$

$$\frac{M}{\tfrac{1}{2}C} = \tan \tfrac{1}{4}I$$

$$M = \tfrac{1}{2}C \tan \tfrac{1}{4}I.$$

413. Length of Curve. *Arc Lengths.*—The length of the circumference of a circle is $2\pi R$. This is the arc length for a full circle. The length of arc for any central angle I is

$$\text{Arc length} = \left(\frac{I^\circ}{360^\circ}\right)2\pi R,$$

in which the angle I° is expressed in degrees.

Chords.—As the distances measured in the field with the tape must of necessity be measured along straight lines, and as the 100-ft. tape is most commonly used, the *length of curve* in railroad practice is considered to be the sum of the lengths of a number of chords, normally each 100 ft. long. Thus if the central angle is I and the degree of curve is D,

$$\text{Length} = 100\frac{I}{D}$$

This is less than the corresponding arc length.

414. Curves by Deflection Angles.—Curves are usually staked out by the use of deflection angles turned from the tangent to the curve at the transit station, and by chords measured from station to station along the curve. The method is illustrated in Fig. 414, where ABC represents the curve, AX is a tangent to the curve at the point A, and angles XAB and XAC are the deflection angles from the tangent to the chords AB and AC.

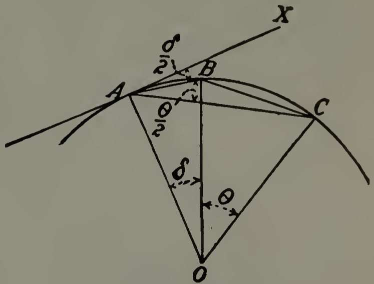


FIG. 414.

Assume the transit to be set up at the point A. Given: R, δ. θ. Required to locate the points B and C.

$$\angle XAB = \frac{\delta}{2}$$

$$AB = 2\,R \sin\frac{\delta}{2}$$

Turning the angle $\frac{\delta}{2}$ at A and measuring the distance AB, the point B is located in the field.

$$BC = 2\,R \sin\frac{\theta}{2}$$

$$\angle BAC = \frac{\theta}{2}$$

$$\angle XAC = \frac{\delta + \theta}{2}$$

At the point A the deflection angle XAC is set off, the distance BC is measured from B, and the forward end of the tape at C is lined in with the transit at A, the line of sight of which has been pointed along the line AC. Thus the point C is located.

If B is at a full station and the distance BC is the 100-ft. distance to the next full station at C, then $\theta = D =$ degree of curve, and $\angle BAC = \frac{D}{2}$.

415. Laying Off a Curve by Deflection Angles.—The first step in the field location of a curve is to mark on the ground the P.C. and the P.T. The deflection angle for each full station on the curve is calculated. If intermediate or plus stations are to be located also, deflection angles are calculated for these points. The transit is placed at the P.C., a sight is taken along the tangent, and each point on the curve is located by deflection angle and distance, the angle being turned from the tangent at the P.C. (except as noted below), and the distance being measured from each full station to points not more than 100 ft. ahead. If full stations only are to be located, the distance to each of such stations is measured from the previous station.

416. Transit Set-ups on the Curve.—Often it is impracticable or impossible, on account of obstacles, great length of curve, etc., to run all of a given curve with the transit at the P.C. and one or more set-ups are required on the curve between P.C. and P.T.

Figure 416 illustrates the case where the transit is set up at some intermediate point A. The curve is begun at P.C. and located as far as A, where a hub is set. The transit is then set at A. A backsight[1] is taken upon the last previous station at which the transit was set up, in this case the P.C., and the angle $\frac{\alpha}{2}$ (half the central angle subtended by the chord sighted over) is turned off as shown. The line of sight is then directed along the tangent at A. Deflections to points beyond A are turned as in the case previously explained. It is to be observed that the angle $\frac{\alpha}{2}$ between the tangent at A and the chord A-P.C. is equal to the deflection angle at the P.C. for the point A. Therefore, the vernier setting to locate point B from the transit station A, is the same as would have been used had the transit remained at the P.C. According to this method the following procedure may be used to orient the transit at any point on the curve.

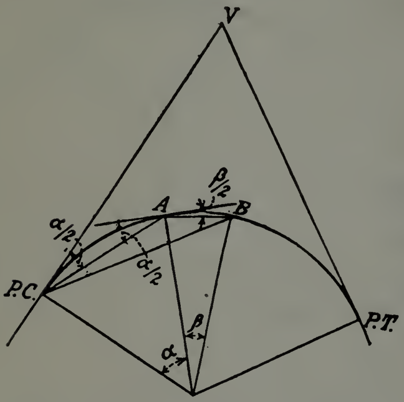


Fig. 416.

(a) Calculate deflections as for use at the P.C.; (b) when set up at any point on the curve, backsight at the last previous transit station with the vernier reading the deflection for the point sighted (as calculated under (a) above); to locate other points, use the deflection angles previously calculated for these points. When the point used as a backsight is the P.C., the backsight vernier reading is zero.

417. Location Field Work.—After the paper location has been made, the work of field location consists of laying off the line in the field so that it bears approximately the same relation to the preliminary traverse on the ground as does the paper location to the preliminary traverse on the map. This relation is accomplished generally by scaling from the map the distances from stations on the preliminary traverse to points of intersection between the preliminary traverse and the tangents of the paper location, and by measuring with a protractor the angles between tangents and intersecting traverse lines. In the field these governing points are established by chaining the scaled distance from the nearest station on the preliminary traverse, and the directions of the corresponding tangents are fixed by setting up the transit at the governing points and laying off the corresponding protracted angles. In this way a close agreement is always maintained between the paper and field location of tangents. Adjoining tangents are then run to an intersection, the intersection angles are measured, and the curve notes are figured as explained in the preceding articles, the degree of curve for a given curve being the same as that assumed in the paper location.

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[1] In turning deflection angles at points on a curve the backsights are taken with the telescope reversed, and the foresights with the telescope normal.

Beginning at some initial point for which the chainage is taken as zero, the field location is extended in the manner described above, stakes being set at full stations and frequently at multiples of 50 ft. or even 25 ft., and hubs being set at all P.I.'s, P.C.'s, P.T.'s, and at all intermediate points that are occupied by the transit. Transit notes are kept up the page in the manner illustrated by the following notes, and all important features such as roads, railroads, streams, property lines, etc., are shown in their proper relation to the located line by means of sketches on the right-hand page, the center line of the page being considered as the located line.

Station	Point	Deflection	Curve or bearing
100			
99	....	......	N73°10′W
98 + 56.7	P.T. ⊙	52°18′	
98	....	48°54′	
97	....	42°54′	D = 12°
96	....	36°54′	I = 104°36′
95	....	30°54′	T = 618.90
94	○	24°54′	R = 478.34
93	....	18°54′	L = 871.7
92	....	12°54′	
91	....	6°54′	
90	....	0°54′	
89 + 85.0	P.C. ⊙	0°00′	12°L
89			
88			

Profile levels are then run over the located line in the same manner as for the preliminary line, and from the data thus obtained a location profile is prepared showing the ground and grade lines and an alignment diagram.

In the light of a study of this profile and the preliminary map, grades are fixed. If minor modifications in alignment appear desirable, parts of the location are revised in this field. The line finally located is plotted in both plan and profile, when it is termed the final location. On the final location map are shown all features of importance in the immediate vicinity of the line, including the position and character of objects to which hubs are referenced and the position of location bench marks.

418. Railroad Construction Surveys.—The construction surveys consist essentially of (1) staking out earthwork and structures preparatory to and during the process of grading and construction, and (2) making the measurements necessary to determine the volume of work actually performed up to a given date, as a basis for payment of the contractor.

Generally just prior to the beginning of construction, the located line is rerun, missing stakes are replaced, hubs are referenced, etc. Also right-of-way boundaries are established and right-of-way maps are prepared.

Prior to grading, the line is cross-sectioned and slope stakes are set, as described in Art. 140. Also, if necessary, borrow pits are staked out and cross-sectioned as described in Art. 136. Cross-sections are plotted and volumes of earthwork are calculated as described in Chap. X. Lines and grades are also given for culverts, bridges, and other structures.

The general practice is to prepare a monthly estimate of work completed. In order to make such an estimate, it is necessary to make a quantity survey near the close of each month.

Before trimming the slopes, stakes, called *finishing stakes*, are often set to grade at the outer edge of the roadbed on either side of the center line.

During the progress of laying track, alignment is accurately established by setting tacked stakes along the center line at full stations on tangents and usually at fractional stations on curves. Also stakes, usually to one side of the track, are set to the elevation of the grade of top of rail.

HIGHWAY SURVEYS

419. Remarks.—Most highway surveys are made along established roads as a basis for improvements of such roads. Only small changes in alignment are possible, as the work is usually limited to existing right-of-way lands. Purchase of small parcels of adjoining land may sometimes be possible, permitting some improvement of alignment, as for example, at sharp curves. But the general route is fixed beforehand and no reconnaissance is necessary. Frequently no preliminary survey is required, and the location survey may be run at once, subject to small changes and adjustments after further study.

Occasionally the problem of the location of a new highway is presented, and in such cases reconnaissance and preliminary surveys will be run as for railroad location, using practically the same field methods.

420. Location Survey.—A transit line is run either on the road center line or offset to one edge of the proposed roadway. This line is stationed in the usual manner, stakes being set every 100 ft., and sometimes every 50 ft. or less. If a topographic map was not made from a preliminary survey, this work is done in connection with the location, the necessary field measurements being made and the whole plotted on the map of the route. In the field the transit line is fitted to the ground and the necessary curves are located.

421. Curves.—In highway practice, the degree of curve is defined as the central angle subtended by an arc 100 ft. in length, hence $R = \left(\frac{360°}{D°}\right)\left(\frac{100}{2\pi}\right)$. This gives radii for stated degrees of curve which are somewhat less than those obtained from the definition of degree of curve used in railroad work, previously defined. The radius of a 1° curve on this arc basis is 5,729.58 ft. and the radius varies inversely as the degree of curve, which means, for example, that the radius of a 10° curve is 572.96 ft. to the nearest hundredth of a foot. Field measurements with the tape must, of course, be made along the chords and not along the arc, and unless a correction is made for the difference between chord length and arc length, an error of greater or less magnitude will result. Common practice is to make no correction but to use chords so short as to reduce this error to a negligible amount. One authority recommends that 100-ft. chords be used for curves up to 5°, 50-ft. chords from 5° to 15°, 25-ft. chords from 15° to 30°, and 10-ft. chords for curves sharper than 30°. It may be assumed that such practice will give a sufficiently close approximation. Curves sharper than 20° should be used only in special cases and should be avoided whenever possible.

Curves are located by the deflection-angle method in most cases, but a short curve may be staked in sufficient detail by measuring from the vertex to each end of the curve along the tangents and from the vertex to the mid-point of the arc along the line from vertex toward center of curve.

422. Grades.—Main roads are designed, where possible, with grades flat enough to be climbed by automobiles without shifting gears, and this means that the grades should not exceed 6 or 7 per cent. Should such a rate of grade prove too expensive, steeper grades are used to reduce construction cost, the exact rate of grade selected depending upon the topography and the density of traffic to be handled.

In improving an existing road it is important to balance the earthwork quantities so that the excavated material will make all the fills, with no excess or deficiency, by reason of the fact that frequently there is no opportunity for waste or borrow of material along the line.

423. Problems.

1. Given: $I = 34°30'$, $D = 3°00'$, and P.C. = station 74 + 30.0. Required: R, L, T, and E; also deflection angles, arranged in notebook form, for staking out this curve, using 100-ft. stations.

2. Given: $I = 92°30'$, $T = 425.000$ ft., and P.C. = Station 25 + 10.0. Required: R, D, C, E, M, L; also deflection angles arranged in notebook form, for staking out this curve using 50-ft. stations.

3. If the curve of problem 2 represents the center line of a highway curve, suppose it is desired to set alignment stakes along two curves, one of which is to be 10 ft. outside, and the other 12 ft. inside of the center line; required: L_1, L_2, D_1, D_2, E_1, E_2, and the deflection angles arranged in notebook form, for staking out these curves.

4. Given: I = $60°40'$, $E = 125.5$ ft. Required: R, D, C, T, E, and L.

5. Two tangents AV and BV have an intersection angle of $45°00'$. A point C is located by the coordinates $VH = 270.2$ ft. and $HC = 157.4$ ft.; VH being measured along the tangent VA, and HC being measured perpendicular thereto. It is desired to connect the two tangents with a curve passing through the point C. Required: R, D, T, L, and E.

6. Having calculated the values in problem 5, change the value of D to that value which is a multiple of $10'$ and which is nearest to the calculated value. Change all other elements of the curve to agree with the new value of D and compute the deflection angles.

CHAPTER XXII

THE USE AND ADJUSTMENTS OF THE PLANE TABLE

424. General Remarks.—The essential features of the plane-table apparatus, shown in Fig. 428, are: (1) a drawing board supported by a tripod and capable of being rotated about a vertical axis (called the plane table), and (2) a line of sight having a fixed relation to a straightedge (called the alidade). It is possible to draw, on a sheet of paper placed on the board, a line to represent the direction of the line of sight, the map-distance to any point may be laid off, and the observed position of an object may be plotted.

In this book, as in common practice, unless otherwise indicated, the term *plane table* will refer to both the table and its accompanying alidade.

By means of the plane table, points on the ground to which observations are made immediately plotted in their correct relative positions on the drawing, all angles being plotted graphically. The plane-table method is more especially adapted to securing the details of the map, and on extensive surveys the primary points of horizontal and vertical control are established by other more precise methods. With its accessories, the plane table (often employed in conjunction with other surveying instruments) is adapted to mapping, regardless of scale, precision required, or magnitude of the survey.

425. Relation between the Transit and Plane Table.—It is helpful to a clear understanding of the principles which underlie the use of the plane table, to consider the similarity between angular measurements made with the transit, together with the office procedure of plotting the notes, and the corresponding operations with the plane table.

The plane-table board may be said to take the place of the graduated circle of the transit, and orientation consists in turning the table until some line on the paper becomes parallel with a corresponding line on the ground, clamping the table in this position. After orientation, the direction of any line is observed by turning the alidade until the line of sight coincides with the line (just as with the transit, the upper motion was rotated) and graphically the direction of the line is given by the position of the straightedge on the paper (just as with the transit, the numerical value of the azimuth was given by the vernier reading). If the table is oriented at any point A, Fig. 435, whose corresponding location on the paper is a the direction of the line AB on the ground is determined by pivoting the straightedge about the point a on the paper until B is sighted. The corresponding line ab on the paper is then established by drawing a line along the straightedge and by laying off to scale the measured distance AB. Thus it is seen that with the plane table there is employed what is virtually a combination of transit and drafting-room methods, but no record of numerical values is secured. The plane table is therefore suitable for mapping only.

426. Instruments; Types of Tables.—Three distinct types of tables are in common use, namely; the *Coast-survey*, the *Johnson*, and the *traverse* types.

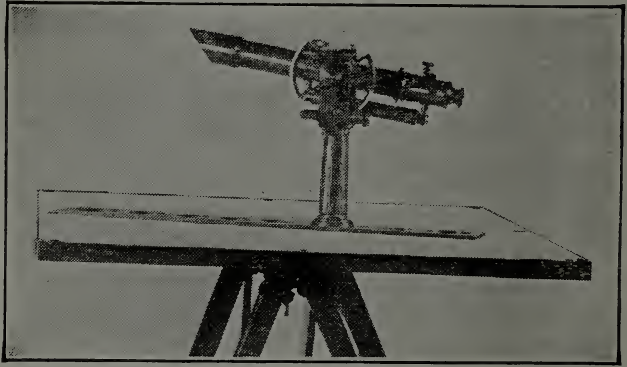

Fig. 428.—Plane table, Johnson type.

427. The Coast-survey Table.—This is the most stable of the three types. It is suitable for the relatively high degree of accuracy required by the U. S. Coast and Geodetic Survey on its shoreline charts, or for large-scale city work. The board, usually 24 by 30 in. is securely attached to a metal casting below, so arranged that the table can be accurately leveled by three leveling screws. By means of a clamp and tangent screw the board may be fixed in any position in azimuth. The paper on which the map is made is held in position by metal spring clamps. The tripod is of heavy and rigid construction.

428. The Johnson Table.—This table shown in Fig. 428. It has proved to be well adapted to the requirements of most plane-table work. The table consists of a board usually 24 by 30 in., gained into the under side of which is a circular brass plate into which a threaded opening has been cut. By this means the table can be screwed to the upper casting of the tripod head. This head consists of two castings which form a ball-and-socket joint held by two clamp screws. By loosening one clamp the ball is free to rotate in its socket, and thus the table may be brought to a level position. By tightening the clamp the table is fixed in this plane. By loosening the other clamp the table can be rotated about the vertical axis. The table is sufficiently stable and precise in its controls to be suitable for perhaps 90 per cent of topographic mapping requirements.

429. The Traverse Table.—The table, usually 15 by 15 in. in size, is attached to a simple tripod movement such that it can be rotated about the spindle and clamped in position. There is no leveling device, but by estimation the table is approximately leveled by adjusting the tripod legs. A box-trough compass, with needle generally 4 in. long, is gained into the board near one edge. The traverse table is suitable for: (*a*) military reconnaissance sketches; (*b*) traverses for small-scale maps, and for mapping roads to be used as control data on larger plane-table sheets; and (*c*) mapping ridges, valleys, and relatively inaccessible areas to fill in a topographic map being drawn on a larger sheet.

430. Alidades.—The plane-table alidade, is a combined sight and straightedge ruler.

431. The Peep-sight Alidade.—One type of alidade used in small-scale plane-table work consists of a peep sight, mounted on a ruler. The peep sight is formed by two sight vanes, either fixed or folding, similar to those employed on the surveyors' compass. The ruler generally consists of a brass plate 6 to 10 in. long, one edge of which is beveled and graduated to a suitable scale. It is often employed by topographers as an auxiliary to the telescopic alidade and is quite indispensable for sighting details while sketching.

432. The Telescopic Alidade.—The telescopic alidade (Fig. 428) is designed to afford greater accuracy in the control of the table, and especially to make possible the stadia method of measuring distances. The complete instrument usually consists of a brass ruler or straightedge approximately 18 by 3 in. beveled on one edge. Upon this ruler sometimes is mounted at one end a circular level, and at the other end a box compass with a 4-in. needle. In the center of the ruler is mounted a column which supports the telescope, vertical arc, and either a striding or an attached level.

The vernier of the vertical arc may be fixed or movable, or there may be an auxiliary level tube attached to the vernier arm as with the transit. It should be remarked that, because the plane table is relatively unstable as compared with the transit, a control level mounted on the vernier arm greatly facilitates the measurement of vertical angles, since it renders unnecessary an initial reading and index correction.

433. Setting Up and Orienting the Table.—The tripod legs should be well spread and firmly planted in the ground, and the table should be approximately waist high, so that the topographer may bend over the board without resting against it. The board is leveled, but since few tables are sufficiently rigid to remain level as the alidade is shifted about, no special attempt is made to see that the board is perfectly level each time an observation is made.

For plotted angles to be theoretically correct, the plotted position of the station at which the plane table is set should be exactly over the corresponding point on the ground. Practically, the degree of care exercised in bringing the plotted point over the ground point depends on the scale of the map. On small-scale work the plane table is set over the station without any attempt being made to place the plotted point vertically above the ground point; on very large-scale mapping the table is roughly set up, approximately oriented, and then the table is shifted bodily until, by plumbing, the point on the paper is practically over the station point. In any case the aim is to set up with sufficient care so that the plotted position of lines drawn from the station will be correctly shown within the scale of the map.

Orientation may be accomplished by any of the following methods:

1. By the magnetic needle. For rough mapping at small scale, orientation by the magnetic compass is often sufficiently exact. This method is susceptible to the same errors as those encountered when using the surveyors' compass, but possesses the advantage over other methods in that an error in the plotted direction of one line introduces no systematic errors in the lines plotted from succeeding stations. The table is oriented by rotating it until the fixed bearing (usually magnetic north) is observed. The table is then clamped, and all mapping at the station is carried on without disturbing the board.

2. By backsighting along an established line the position of which has previously been plotted. The method is equivalent to that employed on the azimuth traverse with the transit. Its advantage over the use of the magnetic needle lies in the increased precision obtainable; a disadvantage is that an error in direction of one line is transferred to succeeding lines. It is the method generally employed on intermediate- and large-scale mapping. If the plane table is set up at B (Fig. 435) on the line AB which has previously been plotted on the paper as ab, the straightedge of the alidade is placed along the line ba and the board is oriented by rotating it until the line of sight falls at A. The table is clamped in this position and is not again disturbed.

3. By the three-point and two-point problems, which will presently be explained (Arts. 440 to 444).

434. Method of Radiation.—When the table has been oriented, the direction to any object on the landscape may be drawn on the map by pivoting the alidade around the plotted position of the plane-table station, pointing the alidade toward the distant object, and drawing a line along the edge of the ruler. Thus in Fig. 434, the plane table is shown in position over station *O* in the center of the field. The plotted position of the plane-table station is indicated at *o* on the plane-table sheet. The alidade is pivoted about this point, and as sights are taken to the points *A*, *B*, *C*, etc., rays are drawn along the edge of the ruler. The distances are measured and are then plotted to scale along the corresponding rays. Thus the positions of the objects sighted are located on the map. This procedure is termed *radiation.*

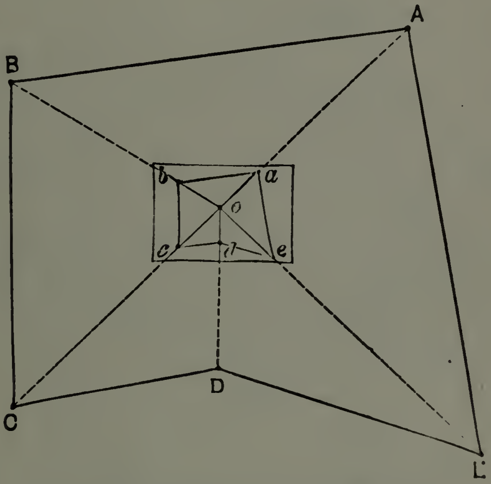

Fig. 434.—Radiation with plane table.

435. Method of Traversing.—In the traverse method the same principles are involved as in traversing with the transit. As each station is occupied the table is oriented, sometimes with the compass needle, but more generally by taking a backsight to the preceding station, the straightedge of the alidade being placed along the position of the preceding traverse line and the board being turned until the line of sight cuts the preceding station. With the board clamped in this position, the straightedge is pivoted about the plotted position of the station occupied, and a sight is taken to the following station and its position is plotted as in the radiation method just described.

Thus in Fig. 435, a series of traverse stations A, B, C, D, etc., is represented. Let the plane table be set up first at station A, and let a representing A, be plotted in such position that the other stations will fall within the limits of the sheet. With the straightedge passing through a, a sight is taken to station B; and b, its position on the map, is located by radiation as described above. The instrument is next set up at station B, the alidade is placed along the ray ba, the table is oriented by rotating the board until the line of sight points at station A, and is clamped in this position. A foresight is

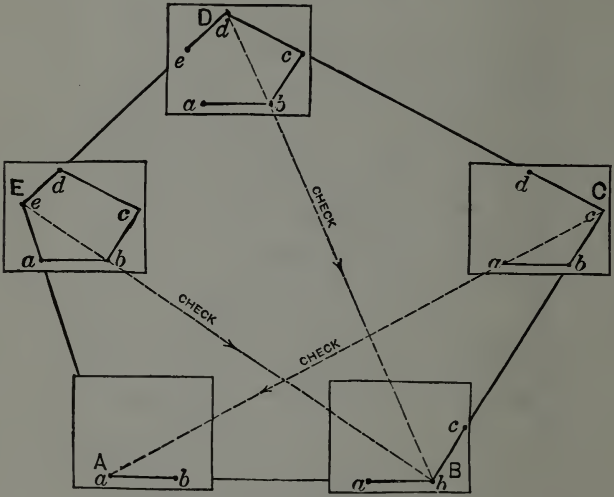

Fig. 435.—Traverse with plane table.

taken to station C, and its position is plotted at c. By a similar procedure the positions of the stations D and E are plotted at d and e.

436. Method of Intersection.—This method is similar to that employed with the transit in locating an object by angles taken from each end of a line of known length, and makes use of the principle that if the angles and one side of a triangle are known, the remaining sides may be determined. It is a very useful method for locating objects when distances to them are not conveniently obtainable. The position of an object is determined by setting up and orienting the table at each end of a line whose position has been plotted and at each of the two set-ups sighting at the object in question and drawing lines towards it, as in the radiation method. The intersection of the two lines thus drawn marks its plotted position.

Thus the plotted positions of the objects *A*, *B*, and *C* (Fig. 436) may be determined as follows: The table is set up and oriented over station *M*, the point *m* on the sheet marking its plotted position. The alidade is sighted on station *N*, by pivoting about the point *m*, a ray is drawn, and the distance from *M* to *N* is scaled along the line, to locate the point *n* on the sheet. The alidade is centered about the point *m* and rays of indefinite length are drawn toward the objects *A*, *B*, and *C*. The table is then moved to station *N*, set up, and oriented by backsighting to station *M*. The alidade is pivoted about *n* and, as before, rays are drawn toward the objects *A*, *B*, and *C*.

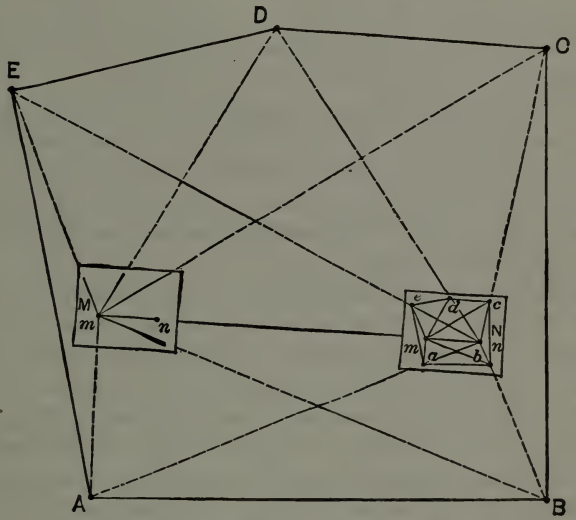

FIG. 436.—Intersection with plane table.

The intersections of these rays with the rays to corresponding objects drawn from *m* mark the plotted positions of the objects as at *a*, *b*, and *c*. Distances to the objects are not measured.

When the area being mapped is large and the scale permits, the method of intersection provides a very useful method of locating control stations for subsequent detail work.

This method is most advantageous where the terrain offers unobstructed sights, considerable relief, and many well-defined objects. It is more especially employed in intermediate- or small-scale mapping.

437. Principle of Resection.—It has already been explained in Art. 436 that by the method of intersection, the plotted position of an object is located on the plane-table sheet by rays drawn to it from two or more stations. Now let it be supposed that the plane table itself is the object to be located; that is, the position of the ground point over which the table has been set up is to be plotted on the map. This may be accomplished *after the board has been oriented* by drawing rays from two or more visible objects through the corresponding plotted positions of the objects on the map; and the plotted position of the plane table will be defined by the point where these two or more rays cross one another. This procedure is termed *resection*, and the rays are said to *resect* at the plotted position of the plane-table station on the map. The fact should be emphasized that the principle of resection is used only after the table has been oriented. Four methods of orientation are commonly used: (1) by the magnetic needle, (2) by backsighting along a line the direction of which has been plotted but the length of which is unknown, (3) by the three-point problem, and (4) by the two-point problem.

In the two latter methods, orientation and resection are accomplished simultaneously, as will be shown in later paragraphs. These methods of locating the position of a station which has not previously been plotted are often of the greatest convenience, for they enable the topographer to select advantageous stations for his work which otherwise could not be used.

438. Resection where Orientation Is Effected by Use of the Magnetic Needle.—The method of orientation by use of the magnetic needle has been explained in Art. 433. When this method of orientation is used, the method of resection is as follows: If P be the station over which the plane table is set up but the plotted position of which is unknown, and if A and B be two visible stations the plotted positions of which are a and b, then p, the plotted position of P, is determined by resecting through a in the direction of A and through b in the direction of B. The method, of course, would be utilized only for small-scale or rough mapping for which the relatively large errors due to orienting with the compass needle would not impair the usefulness of the map.

439. Resection with Orientation by a Backsight.—If the table is oriented by backsighting along a line (Art. 433), the method of resection is as follows: Suppose that the topographer wishes to occupy station C (Fig. 439), a convenient summit, toward which a single ray has been drawn from station B, perhaps during a previous day's work. Also, suppose that stations A and B are visible from station C, but that the latter station has not been located on the map. Under these conditions, the topographer occupies station C, and he orients the table by placing the alidade rule along the ray drawn from station B and turning the board until station B is sighted. Then he pivots the alidade about the point a on the map, sights at station A, and draws a ray to cross, *i.e.*, to resect, the ray drawn from *B*. The point *c* where these two rays cross marks the plotted position of the occupied station. As in the method of intersection, so in the method of resection, the position of the plotted point will be indefinite if the angle between the intersecting or resecting rays is small.

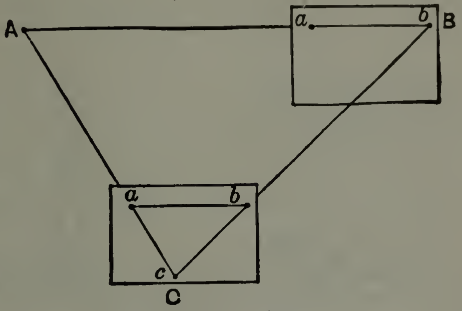

Fig. 439.—Resection.

440. The Three-point Problem.—Frequently the topographer wishes to avail himself of an advantageous position in the field, which has not been located on the map and towards which no ray from located stations has been drawn, and at the same time orientation by use of the compass needle is not sufficiently accurate. When three located stations are visible the three-point problem, of which there are several solutions, offers a convenient method of orienting and resecting simultaneously. In the United States, Lehmann's solution is the one most commonly used by experienced topographers. The mechanical or tracing-cloth solution, while somewhat simpler to understand, is not as satisfactory nor as expeditious under the usual field conditions.

441. Lehmann's or the Coast-survey Method.—The following paragraphs taken from "A Plane-table Manual" by D. B. Wainwright, describes the application of this method.

When the table is imperfectly oriented, the lines drawn from the three projected points, when sighting on the corresponding actual points, will not intersect at one point unless all four are on the circumference of a circle (see Fig. 441*c*, indeterminate position). Except in this case, two of the lines will be parallel, intersected by a third (see Fig. 441*d*, station on range between two fixed points, and Fig. 441*b*, station on prolongation of range line), or they will form a small triangle called the *triangle of error* (Figs. 441*a*, 441*c*, 441*e*, and 441*f*). The solution of the three-point problem determines the location of the station occupied and orients the table simultaneously.

The relative positions of the three fixed points with reference to the new station have an important bearing on the strength of its determination.

In the following statement in regard to the different groupings met with in practice, for the sake of brevity the term "fixed points" will be understood to mean points already determined and plotted on the sheet; the "great triangle" referred to is one formed by the three fixed points, and the "great circle" is the circle passing through them.

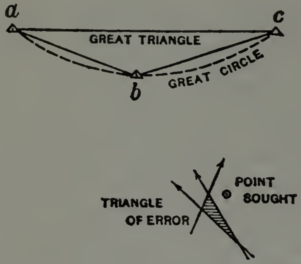
a
c
GREAT TRIANGLE
b
GREAT CIRCLE
TRIANGLE OF ERROR
POINT SOUGHT

Fig. 441*a*.

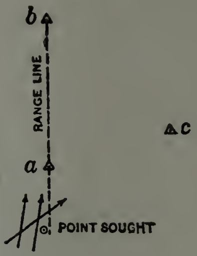
b
RANGE LINE
C
a
POINT SOUGHT

Fig. 441*b*.

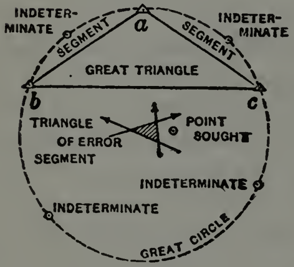
INDETERMINATE
a
INDETERMINATE
SEGMENT
SEGMENT
GREAT TRIANGLE
b
c
TRIANGLE OF ERROR
POINT SOUGHT
SEGMENT
INDETERMINATE
INDETERMINATE
GREAT CIRCLE

Fig. 441*c*.

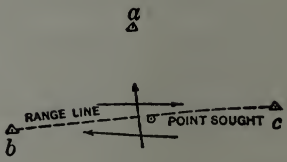
a
RANGE LINE
POINT SOUGHT
b
c

Fig. 441*d*.

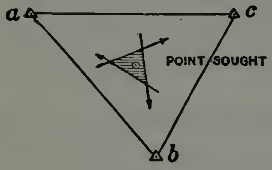
a
c
POINT SOUGHT
b

Fig. 441*e*.

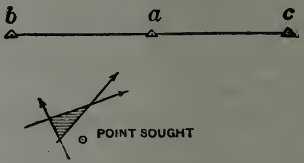
b
a
c
POINT SOUGHT

Fig. 441*f*.

When the new station is outside the great circle, the strength for determination of a position will be weak when the middle point, as seen from the new station, is the farthest of the three and the angles are small (see Fig. 441*c*).

If the new station is located outside the circle, and some distance below it, the angles are small and the determination correspondingly weak.

The determination increases in strength for given angles as the middle point approaches the new station (Fig. 441*a*).

When one angle is small or 0° (points in range), the determination will be strong, provided the two points making the small angle or range are not too near each other when compared to the distances to the new station and to the third point; provided also the angle to the third point is not too small (Fig. 441*b*).

When the new station lies on or near the great circle, its position is indeterminate (Fig. 441*c*).

When the new station is within the great circle, the strength of its determination increases as it approaches the center of gravity of the great triangle (Figs. 441*c*, 441*d*, and 441*e*).

The method following solves by estimation instead of by actual construction. It utilizes certain geometrical relations of the lines and points to serve as guides in making an estimate. It is most rapid in the hands of an experienced topographer, but for those having only occasional use for a graphic solution the tracing-cloth method is recommended.

442. Rules for Lehmann's Method.—The directions are stated in the form of rules.

The term "point sought" will be understood to mean the true position on the sheet of the projected point of the station occupied. The surveyor is assumed to be facing the signals, and the directions right and left are given accordingly.

Rule 1.—*The point sought is always distant from each of the three lines drawn from the three fixed points in proportion to the distances of the corresponding actual points from the station occupied, and it will aways be found on the corresponding side of each of the lines drawn from the fixed points.*[1]

The simplest case for the application of this rule occurs when the station to be determined is within the triangle formed by the three fixed points; the point sought must then be within the triangle of error to satisfy the conditions (Fig. 441*e*).

Although Rule 1 is sufficient in itself for the solution of the problem, there are two subordinate rules which materially assist the topographer in reaching a decision as to the proper location of the point sought with reference to the lines from the fixed points.

Rule 2.—*When the point sought is without the great circle, it is always on the same side of the line from the most distant point as the intersection of the other two lines* (Fig. 441*a*).

Rule 3.—*When the point sought falls within either of the three segments of the great circle formed by the sides of the great triangle, the line drawn from the middle point lies between the point sought and the intersection of the other two lines* (Figs. 441*c*, 441*d*, and 441*f*).

Application of Rules.—In practice the topographer first decides the relation of the new station with reference to the fixed points, whether it is within the great triangle or in one of the segments or outside the great circle. He then determines the position of the point sought with reference to one line (if within one of the segments or without the great circle by Rule 2 or 3); it then follows from Rule 1 that it must be on the corresponding side of the other two lines. Finally, he estimates the relative distances of the three actual points from him and marks the position of the point sought a proportionate distance from each of the three lines.

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[1] That is, if it is on the right side of one line it will be on the right of the other two.

443. The Tracing-cloth Method.—A very simple procedure, known as the tracing-cloth method, is as follows: Fasten a piece of tracing cloth or tracing paper on the plane table over the map. Choose any convenient point on the tracing cloth to represent the unknown point over which the plane table is set, and from it draw rays to the three distant and known points. Then loosen the cloth and shift it over the map until the three rays pass through the plotted positions of the corresponding known points. The intersection of the rays marks the point sought. It is pricked through onto the plane-table map, and the table is oriented in the usual manner.

444. The Two-point Problem.—The purpose of the two-point problem is to orient the table and to locate the point sought when

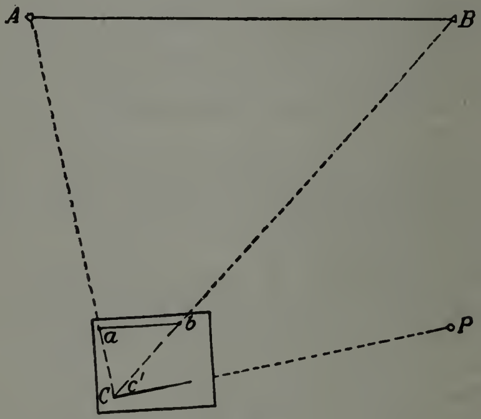

Fig. 444*a*.

two stations only are visible, and when it is impossible or undesirable to occupy either of them, as when the signals are inaccessible or at a considerable distance.

The positions of the two known stations *A* and *B* having been plotted on the sheet at *a* and *b* (Fig. 444*a*), the procedure is as follows: The table is set up at some point *C*, by estimation is oriented as nearly as possible, and the point *c′* is located by resection through *a* and *b*. A sight is taken toward a fourth point *P* whose plotted position is desired, and a ray is drawn toward it. The table is taken to station *P*, and oriented by sighting along the ray *p′c′* toward *C*

(Fig. 444*b*). By resection through *a* the point *p′* is marked. Through this point the alidade is sighted toward station *B*, and a ray is drawn intersecting the line *c′b* at the point *b′*. The direction of the line *ab′* is now parallel to the direction of the line joining the two

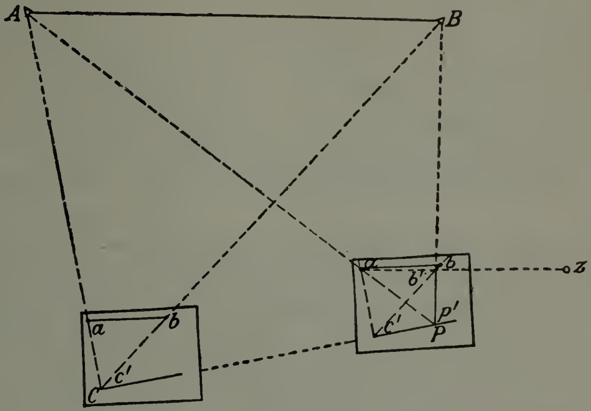

Fig. 444*b*.

stations *A* and *B*, therefore the angle *bab′* is a measure of the error in the attempted orientation of the table at station *C*. A convenient method of turning the table through this small angle into its true

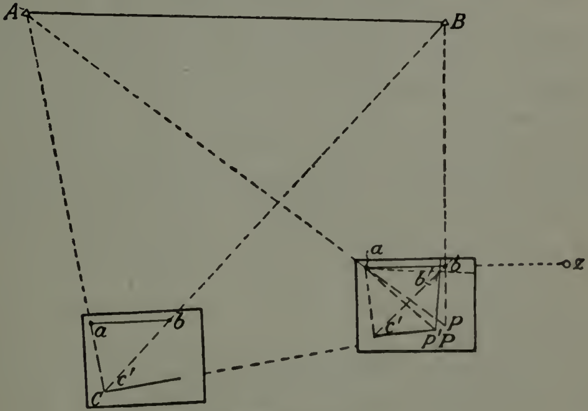

Fig. 444*c*.

position, is as follows: Place the alidade along the line *ab′* and set a signal on the line or select some definite point as *Z*, on the line of sight. Next, place the alidade along the line *ab* and turn the table until the signal or selected point is sighted. The table is now oriented, and by resection through *a* and *b* the true plotted position of the table is located at *p*, as shown in Fig. 444*c*.

445. Measurement of Difference in Elevation.—In the preceding articles there have been described the elementary methods employed in fixing on the plane-table sheet the *horizontal projection* of points on the ground. Since the principal use of the plane-table instrument is in the work of topographic mapping, much use is made of the attached or the striding level, whichever is provided, in determining the *elevations* of ground points. The methods are exactly similar to those employed in the case of the transit alidade, when used in leveling work; namely, direct leveling, stadia leveling, and trigonometric leveling.

The elevation of distant points whose locations have been determined by the method of intersection, may be secured readily by the method of trigonometric leveling (Art. 80), in which the horizontal distance is scaled directly from the map. The conditions as to distance and to the accuracy required will determine whether or not the corrections for refraction and the earth's curvature should be applied.

446. Adjustments of the Plane-table Alidade; Remarks.—The adjustments of the telescopic alidade introduce no principles of adjustment that have not been explained as they apply to the transit and the wye level. Observations made with the plane-table are not required to be as precise as those made with the engineers' transit; and accordingly, the adjustments in general need not be as refined, and in one or two cases are omitted entirely. The telescope of the alidade is not reversed in altitude as is that of the transit, and hence no appreciable error is introduced through any lack of perpendicularity between the line of sight and the horizontal axis, nor through any lack of parallelism between the line of sight and the edge of the ruler. Also, it may be assumed without appreciable resultant errors that the edges of the ruler are straight and parallel, and that the horizontal axis is parallel to the plane of the ruler.

447. Adjustments.—(1) *To make the axes of the control levels parallel to the plane of the ruler.*

Test and Correction.—Same as for the plate levels of the transit (Art. 203, Adjustment 2), except that reversal of the plate or ruler is effected by a carefully marked guide line on the plane-table sheet. The plane table itself is not reversed.

(2) *To make the vertical cross-hair lie in a plane perpendicular to the horizontal axis.*

Test and Correction.—Same as for the corresponding adjustment of the transit (Art. 203, Adjustment 1).

(3) *To make the axis of the telescope level parallel to the line of sight.*[1]

Test and Correction.—Same as for the telescope level of the transit (Art. 203, Adjustment 6).

For the alidade having a telescope which may be rotated about its axis in a sleeve, *i.e.*, *tube-in-sleeve type*, the following two adjustments take the place of that described in Adjustment (3).

(3*a*) *To make the line of sight coincide with the axis of the telescope.*[2]

Test.—Sight the intersection of the cross-hairs on some well-defined point. Rotate the telescope carefully in the sleeve through 180°. Usually the limits of rotation are fixed by a shoulder and a lug. If the intersection of the cross-hairs remains on the same point the line of sight is in adjustment.

Correction.—If the cross-hairs have apparently moved away from the point, bring each hair halfway back to its original position by means of the capstan screws holding the cross-hair ring, as in the corresponding adjustment of the wye level (Art. 112, Adjustment 4). By means of the tangent screw, set the intersection of the cross-hairs again on the point and repeat the test.

(3*b*) *To make the axis of the striding level parallel to the axis of the telescope* (*and hence parallel to the line of sight*).

Test.—Place the striding level on the telescope and bring the bubble to the center of the tube. Remove the level carefully, turn it end for end, and replace it on the telescope barrel. If the bubble returns to its central position, the level is in adjustment.

Correction.—If the bubble is off center, move it through one-half of the displacement by means of the adjusting screw at one end of the tube. Bring the bubble to the center by means of the tangent screw and repeat the test.

(4) *To make the vernier read zero when the line of sight is horizontal.*[3] Same as for corresponding adjustment of the transit (Art. 203, Adjustment 7).

(4*a*) *To adjust the auxiliary level on the vernier arm so that its axis is parallel to the axis of the telescope when the vernier reads zero.*[4]—Same as Adjustment 7*a*, Art. 203.

448. Sources of Error.—The sources of error which affect the plane-table method are, in the main, those which affect transit and plotting work, and the discussion relating to those subjects need not be repeated here. Three sources of error, however, should be considered; namely, (1) setting over a point, (2) drawing rays, and (3) instability of the table.

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[1] Applicable to an alidade with telescope rigidly fixed to the horizontal axis, *i.e.*, the fixed-tube type.

[2] Applicable to an alidade with telescope designed to rotate about its axis, *i.e.*, the tube-in-sleeve type.

[3] Not applicable to an alidade having a level attached to the vernier arm.

[4] Not applicable to an alidade having a fixed vernier.

Setting Over a Point.—Because plotted results only are required, it is not necessary to center the plane table over a ground point with any greater precision than is required by the scale of the map. For scales smaller than 1 in. = 50 ft., the table is usually set up by estimating with the eye when the plotted position of the station is over the ground point. For scales equal to or larger than 1 in. = 50 ft., the plotted position of the station on the map will be placed vertically over the ground point, by first setting the table up roughly, and then shifting it bodily until a plumb-line held from the under side of the board and just beneath the plotted point, comes over the ground point.

Drawing Rays.—The accuracy of plane-table mapping depends largely upon the precision with which the rays are drawn, consequently when sighting the alidade, either in orientation or in plotting the position of points, its direction should be controlled or indicated by rays of considerable length, for the same reason that accuracy in plotting angles is increased if the size of the protractor or if the base used in laying off tangents is increased. Only enough of each ray is drawn to insure the plotted position falling upon it, with one or two additional dashes drawn near the ends of the alidade ruler, to mark its direction. Fine, thin lines are desirable in plane-table mapping both for accuracy and legibility; hence, sharply pointed, hard (6H to 9H) pencils are used. A needle is employed for plotting station points.

Instability of the Table.—If it is manipulated with care, the plane table can be oriented with considerable precision; a principal source of error in its use, however, arises from the fact that its initial position is subject to continual disturbance by the topographer while he is working upon it. Errors from this source may be kept within reasonable limits: (1) by planting the tripod firmly in the ground, (2) by setting the table approximately waist high so the topographer can bend over it without leaning against it, (3) by using the alidade and scale with great care to prevent the exertion of undue pressure upon or against the table, and (4) by testing the orientation of the board occasionally and correcting its position if necessary. This latter test is always applied before plotting the position of a new instrument station.

449. Field Checks.—An important advantage of the plane-table method is that it provides many convenient opportunities for verifying the plotted positions of points on the map. This may be accomplished, after the table has been set up and oriented, by drawing a ray toward any visible object whose position has previously been plotted. If this ray passes through the plotted point, the verification is effected. The plotted position of the plane table itself may also be verified by resecting from distant visible objects whose plotted positions are known to be correct. When sketching, these checks should always be applied at each station to guard against faulty orientation, mistakes in observations, etc.

CHAPTER XXIII

TOPOGRAPHIC MAPPING

450. General Remarks.—A topographic map shows by the use of suitable symbols: (*a*) the configuration of the earth's surface, called the *relief;* (*b*) other natural features, such as timber, streams, ponds, lakes, etc.; and (*c*) the physical changes wrought upon the earth's surface by the works of man, such as buildings, highways, canals, etc. Many maps and plats of various sorts may show such objects as are listed under (*b*) and (*c*) above, but such a map may not properly be called a topographic map unless, in addition, it represents the shape of the ground surface. Therefore, the distinguishing characteristic of a topographic map, as compared with others, is the representation of the terrestrial relief.

Of the symbols employed to represent relief, the one having by far the widest use is the contour line, and in the following paragraphs the purpose will be to indicate the principles which underlie the interpretation, the construction, and the use made of contour lines on topographic maps.

451. Contour Lines.—A contour line on the ground may be defined as an imaginary line passing through all points of equal elevation. It may be thought of as the trace formed by the intersection of a level surface with the ground surface. The shore line of a still body of water is the best visible example of a contour line on the ground.

If on the drawing, there are plotted the positions of several ground points of equal elevation, say 720 ft. above sea level, the line on the map joining these points is called a *contour line.* A contour line may therefore be defined as a line, either on the map or on the ground, which joins all points of equal elevation. It will be a useful distinction, however, in the discussion which follows, if the ground line is called a *contour*, and the corresponding map line is called a *contour line.*

The use of the contour line has the advantage that it permits the representation of relief with much greater facility and with far greater accuracy, than do other symbols.

452. Characteristics of Contour Lines.—The principal characteristics of contour lines may be determined by examining a contour map and comparing it with the terrain which it represents.

These characteristics are as follows:

1. On steep slopes, the contour lines are spaced more closely than on gentle slopes. Thus the horizontal distance between contours is inversely proportional to the slope.

2. On the uniform slopes, the contour lines are spaced uniformly.

3. Along plane surfaces, contour lines are straight and parallel to each other.

4. Since contour lines represent level lines they are perpendicular to the lines of steepest slope and therefore are perpendicular to ridge and valley lines where they cross such lines.

5. Evidently since all land areas may be regarded as summits or islands above sea level, all contour lines must close upon themselves, either within or without the borders of the map. Therefore, a closed contour line on a map always indicates either a summit or a depression. The elevations of adjacent contour lines will usually indicate which condition is represented, but if a summit or a depression is represented by a single contour line, it would be impossible to tell which feature is represented. To prevent confusion, therefore, a depression is always indicated by a contour line hachured, which is called a *depression* contour.

6. Since contour lines represent contours of different elevation on the ground, they can not merge or cross each other on the map, save the rare exceptions, that in the first case they represent a vertical surface, as the abutments of a bridge, and in the second case, they indicate an overhanging cliff or cave.

8. A single contour line can not lie between two contour lines of higher or lower elevation.

453. The Contour Interval.—The vertical distance between two contour lines is called the *contour interval.* The choice of interval depends upon the purpose and scale of the map, and upon the character of the terrain represented. For large-scale maps of flat country the interval may be as small as ½ ft.; for small-scale maps of rough country the interval may be 50 ft., 100 ft., or even more. For maps of intermediate scale, such as are used for many engineering studies, the interval is usually 2 to 5 ft.

454. Contour Map Construction.—The construction of a topographic map normally consists of three operations, as follows: (*a*) the plotting of the horizontal control, or skeleton upon which the details of the map are hung, (*b*) the plotting of details, including the map location of points of known ground elevation, called ground points, by means of which the relief is to be indicated, and (*c*) the construction of contour lines at a given contour interval, the ground points being employed as guides in the proper location of the contour lines.

The common methods of plotting both horizontal control and details have been described in an earlier chapter and will not be considered further. Methods of plotting contours will now be discussed.

Regardless of the number of ground points whose plotted positions are known, it is evident that any contour line must be drawn in some degree by estimation. This condition requires that the draftsman use his skill and judgment to the end that the contour lines may best represent the actual configuration of the ground surface.

Contour lines are shown for elevations which are multiples of the contour interval, and usually each fifth contour line is made heavier than the rest. Contours ordinarily change direction most sharply where they cross ridge and valley lines, also the gradients of these lines are quite uniform, hence they become very important as aids to the correct drawing of the contour lines. Because of this fact, special care is taken in the field to locate the valley and ridge lines. Examples of such locations are shown plotted at points *a* (Fig. 456*a*). The stream lines are drawn through those points which represent valleys, and the contours are spaced along them before any attempt is made to interpolate or draw the contour lines. This procedure aids the draftsman in his interpretation of the data.

455. Interpolation.—Consider the two contour points *A*-2, *B*-2 (Fig. 456*a*), whose elevations are 848.0 and 852.0 ft. respectively. If the contour interval for this map is 2 ft. then the 848 and the 852-ft. contour lines pass through the corresponding points. Assuming that the slope is uniform between these points, the 850-ft. contour line passes through a point midway between the two points considered. If a 1-ft. interval were used, then the additional 849 and 851-ft. contour lines would be drawn through the quarter points of the line from *A*-2 to *B*-2. This process of spacing the contour lines proportionally between plotted points is called interpolation.

The procedure is usually not so simple as in the case cited above. It is noticed that the elevations at the corners of the other squares in the figure are mostly of such values that the contour lines do not pass through them. For this condition the interpolations may be made by estimation on the map, by arithmetical calculations, or by graphical means.

Interpolation by Estimation.—Since each contour map is the result of more or less interpretation by the draftsman, it is thought in many cases to be not inconsistent with the other methods of its construction if the interpolation is made by careful estimation accompanied occasionally by approximate mental calculations. This method is most commonly used on intermediate- and small-scale maps.

Interpolation by Calculation.—Where considerable accuracy is desired in the map, the errors of estimation may be eliminated by simple arithimetical calculations.

Interpolation by Graphical Means.—Various means of graphical interpolation are in use. One of these is shown in Fig. 455. A number of parallel lines are drawn at equal intervals on tracing cloth, each fifth or tenth line being made heavier or of a different color than the rest and numbered as shown. Now if it be desired to interpolate the position of 52- and 54-ft. contours between *a* with elevation of 50.7 and *b* with elevation of 55.1, the line corresponding to 0.7 ft. (left end) is placed over *a* and the tracing turned about *a* as a center until the line corresponding to 5.1 ft. (left end) covers *b*. The interpolated points are at the intersections of lines representing their elevations and the line *ab*, and may be pricked through the tracing cloth. Had the known points been much closer together, the figures at the right end of the tracing would have been used, or in other words, the value of each space would have been doubled; and if the scale were small, the contour interval large, and the topography rugged, each space might represent 1 ft. Thus it will be seen that, by assigning different values to the spaces, a single piece of tracing cloth prepared in this way can be made to suit a variety of conditions.

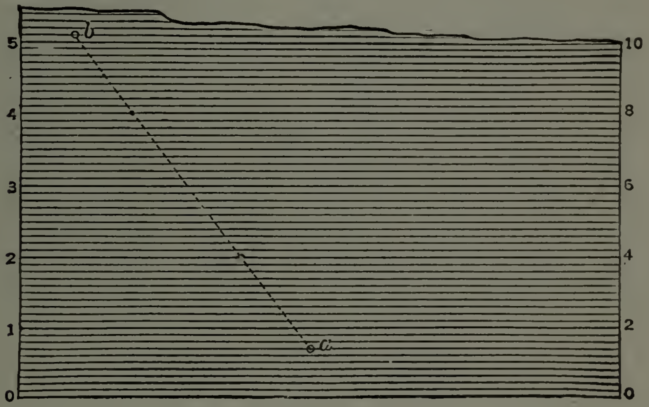


Fig. 455.

456. Systems of Ground Points.—Four typical systems of ground points are commonly used in drawing topographic maps. These may be called: (*a*) trace-point system, (*b*) coordinate-point system, (*c*) controlling-point system, and (*d*) cross-section-point system.

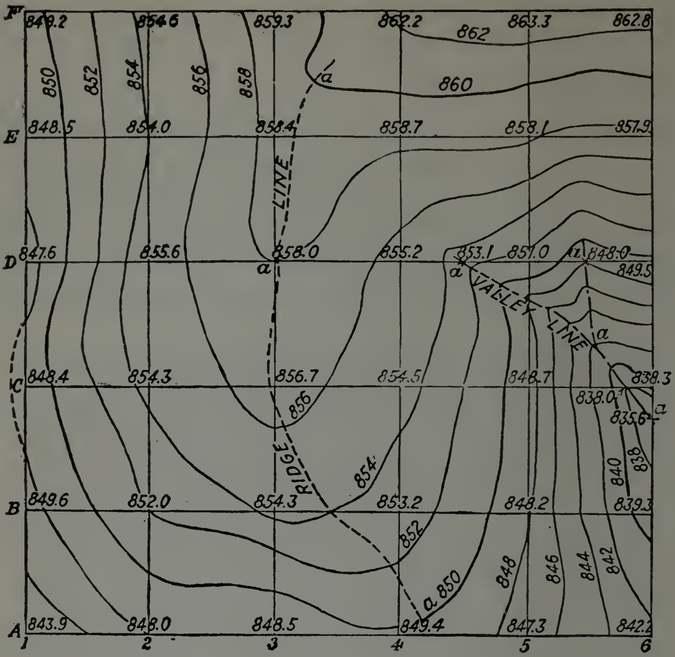

F
E
D
C
B
A
1
2
3
4
5
6
848.2
854.6
859.3
862.2
863.3
862.8
862
860
850
852
854
856
858
a
848.5
854.0
858.4
858.7
858.1
857.9
LINE
847.6
855.6
858.0
855.2
853.1
851.0
848.0
849.5
VALLEY
LINE
848.4
854.3
856.7
854.5
848.7
838.3
838.0
835.6
RIDGE
849.6
852.0
854.3
853.2
848.2
839.3
840
838
848
846
844
842
a 850
843.9
848.0
848.5
849.4
847.3
842.2

FIG. 456a.

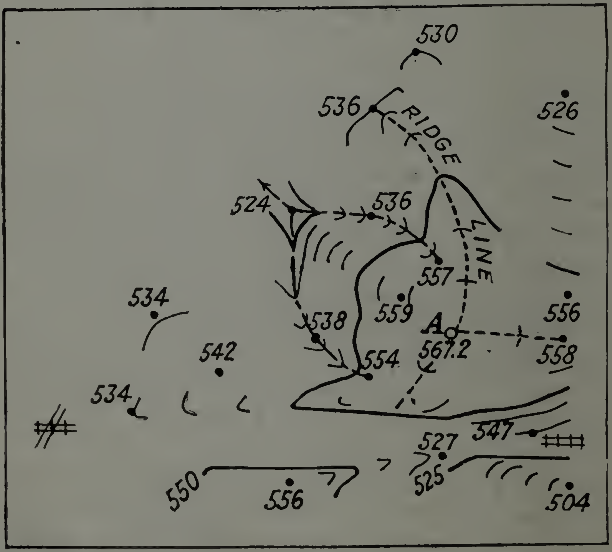

530
536
RIDGE
526
524
536
LINE
557
534
559
538
556
A
567.2
558
542
554
534
527
547
525
550
556
504

FIG. 456b.

The Trace-point System.—If a number of points on a given contour, called *contour points*, are located on the ground and their corresponding positions plotted on the map, the contour line may then be shown connecting these plotted points.

The Coordinate System.—By this method a system of squares or rectangles is plotted, and the elevations of the corners are recorded beside them. Also, the positions of valley and ridge lines are shown. Following the principles stated in Art. 452, the valley lines are sketched, the contour crossings are interpolated on the valley lines and on the sides of the squares, after which the contour lines are drawn. (See Fig. 456*a*.)

In this example the ridge line was of little aid, but under other conditions, a ridge line may be as important as a valley line.

The Controlling-point System.—The significance of valley and ridge lines has already been mentioned. It has also been noted that where a uniform slope exists between two contour points the intermediate contour lines on the map may be located by interpolation. Hence, if a system of points is provided which locates the summits, depressions, the valley and the ridge lines, and all important changes in slope, a contour map of the region may be drawn. Such a system is illustrated in Fig. 456*b*.

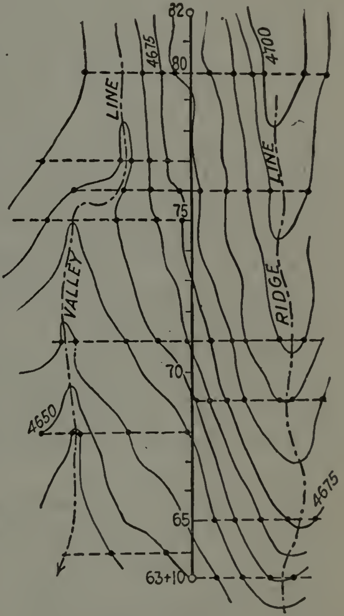


Fig. 456*c*.

In this sketch the ridge and valley lines have been drawn, the contours spaced along them and also spaced between other controlling points, and the fifth contour lines have been sketched.

The Cross-section-point System.—This system is most frequently used to draw the maps required in connection with route surveys. The field surveys locate the positions of contour points along lines normal to the transverse. In Fig. 456*c* the cross-section lines are dashed and the contour crossings are shown as dots. These dots obviously lie on the contour lines themselves, and hence the latter may be drawn as in the trace-point system.

457. Finishing the Map.—The subject of drafting in its general aspects has been discussed in Chap. IV, and the methods of plotting have been explained and map symbols have been given in Chap. XIV. The following paragraphs deal with these subjects only as they apply to the symbols on topographic maps.

458. Colors.—The standards most generally used for the hues, tints, and shades of colors are those employed by the cartographers of such organizations as the U. S. Geological Survey, and the U. S. Coast and Geodetic Survey.

The three primary colors are red, blue, and yellow. Any other color can be produced by mixtures, in varying amounts, of these primary colors. Thus a mixture of red and blue produces purple, blue and yellow yield green, and yellow and red yield orange. These different varieties of colors are called *hues*. If a color mixture having a given hue is thinned by adding water, the color is said to change in *tint*. Thus a hue may be given a light tint by adding water, or a darker tint by allowing the water to evaporate, or by adding more pigment. If black pigment is added to a mixture of a given hue, the color is said to change in *shade*. Thus various shades of a hue are secured by adding various amounts of black pigment. Any clean water may be used in mixing water colors, but distilled water should be used with all inks.

In addition to black, three colors most commonly used on topographic maps are as follows: burnt sienna (reddish brown) for all land forms, *i.e.*, contour lines, hachures, sand, etc.; prussian blue for all water features, *i.e.*, streams, lakes, marsh, etc.; and green for trees and growing crops.

If a considerable portion of a map is to be covered by symbols, it is sometimes best to use a flat tint. This is especially true in the cases of water and timber areas.

Flat tints may be applied to drawing papers but should not be used on tracing cloth because of the resultant distortion. Such tints should be very light and evenly applied, but the procedure of tinting is too complex to be fully described here. Colored tints are sometimes applied to tracings by the use of pencils.

459. Contour Lines.—Contour lines are fine, smooth, free-hand lines of uniform width. Each fifth line is weighted slightly heavier than the others to facilitate the reading of elevations on the map. If the contours are spaced closely on the map, only the fifth contour lines need be numbered; but on areas of low relief where the contour lines are spaced widely apart, each line may be numbered with its elevation. The line is broken to leave a space for the number. It is difficult to maintain an even weight of line unless a *contour pen* is used (see Art. 41).

CHAPTER XXIV

TOPOGRAPHIC SURVEYING

460. General Remarks.—The distinguishing feature of a topographic survey, as compared with other surveys, is the determination of the position, both in plan and in elevation, of the system of points of known elevation called ground points, necessary to the construction of the map.

Considering the range of uses which they serve and of the variation in character of the areas covered, it is plain that topographic surveys must vary widely. Some are simple in plan and execution, covering but a few acres; others are complex in plan, difficult in execution, and extend over hundreds of square miles.

461. Classes of Surveys.—Topographic surveys may be said to fall into three classes, namely, those for large-scale, intermediate-scale, and small-scale maps. The range of map scales within each designation is not definite, yet the classification is a useful one. The scale limits commonly accepted as belonging to each of these classes are as follows:

Large scale: $\frac{1}{120}$ to $\frac{1}{1,200}$; *i.e.*, 1 in. = 10 ft. to 1 in. = 100 ft.

Intermediate scale: $\frac{1}{1,200}$ to $\frac{1}{12,000}$; *i.e.*, 1 in. = 100 ft. to 1 in. = 1,000 ft.

Small scale: $\frac{1}{12,000}$ to $\frac{1}{240,000}$; *i.e.*, 1 in. = 1,000 ft. to 1 in. = 20,000 ft.

462. Control.—The system of measurements which locates the instrument stations in plan is termed the *horizontal control.* Triangulation or traversing provides the skeleton of the survey which is later clothed with the located positions of such objects as roads, houses, trees, streams, and ground points of known elevation which are employed in indicating the relief.

The system of measurements (lines of levels) which locates the instrument stations in elevation, together with the benchmarks established in the field, is called the *vertical control.*

In many cases the area is of such wide extent that the required number of instrument stations can not conveniently be included in a

TABLE 462.—TOPOGRAPHIC SURVEY CONTROL DATA (APPROXIMATE VALUES)

Scale of map	Kind of control	Triangulation					Traverse			Levels	
		Length of sight, miles	Average error of closure in triangles	Distance between bases, miles	Probable error in base measure	Maximum discrepancy between bases	Length of traverse, miles	Maximum error of angles	Maximum linear error of closure	Length of circuit, miles	Maximum error of closure
Small	Primary	10 to 200	1″	100 to 500	$\frac{1}{1,000,000}$ to $\frac{1}{4,000,000}$	$\frac{1}{80,000}$ to $\frac{1}{300,000}$	50 to 500	2″	$\frac{1}{50,000}$	50 to 500	4 mm. × $\sqrt{\text{kilometers}}$ or 0.017 ft. × $\sqrt{\text{miles}}$
	Secondary	5 to 20	3″ to 5″	50 to 200	$\frac{1}{1,000,000}$	$\frac{1}{100,000}$	25 to 200	10″	$\frac{1}{10,000}$		
	Tertiary	1 to 10	5″ to 15″	10 to 100	$\frac{1}{50,000}$ to $\frac{1}{20,000}$	$\frac{1}{5,000}$ to $\frac{1}{2,000}$	10 to 100	30″ to 1′	$\frac{1}{3,000}$	10 to 100	0.05 ft. × $\sqrt{\text{miles}}$
	Quartenary	½ to 2	1′ or graphical	2 to 10	$\frac{1}{5,000}$	$\frac{1}{500}$	1 to 2	3′ or compass	$\frac{1}{500}$ to $\frac{1}{1,000}$	1 to 10	0.1 to 0.5 ft. × $\sqrt{\text{miles}}$
Intermediate	Primary	1 to 5	10″ to 30″	5 to 50	$\frac{1}{10,000}$ to $\frac{1}{40,000}$	$\frac{1}{1,000}$ to $\frac{1}{4,000}$	1 to 20	30″ to 3′	$\frac{1}{1,000}$ to $\frac{1}{5,000}$	1 to 25	0.05 to 0.3 ft. × $\sqrt{\text{miles}}$
	Secondary	½ to 2	Graphical	1 to 5			1 to 5	1′ to 10′	$\frac{1}{500}$ to $\frac{1}{2,500}$	1 to 5	0.1 to 1.0 ft. × $\sqrt{\text{miles}}$

Large	Primary	1 to 5	2″ to 10″	2 to 20	$\frac{1}{20,000}$ to $\frac{1}{80,000}$	$\frac{1}{2,000}$ to $\frac{1}{5,000}$	1 to 5	30″ to 1′	$\frac{1}{5,000}$ to $\frac{1}{20,000}$	1 to 10	0.05 ft. to 0.1 ft. × $\sqrt{\text{miles}}$
	Secondary	¼ to 1	5″ to 20″	1 to 5			½ to 3	30″ to 2′	$\frac{1}{1,000}$ to $\frac{1}{5,000}$	1 to 3	0.05 ft. to 0.1 ft. × $\sqrt{\text{miles}}$

single traverse or a simple triangulation system, or else the measurements, which may be sufficiently accurate for surveys of small areas, will not be sufficiently accurate for the more extended survey because of the accumulative effect of the errors involved. In these cases the desired accuracy for the entire survey may be secured if a few stations evenly distributed over the tract are connected by a system of measurements of relatively high precision, called the *primary control.* Within this system then, subordinate systems of measurements to locate the instrument stations need not be made with such high precision because the distance in which the accumulative effects of the errors can take place will be not farther than from one primary control station to another. This system of measurements to locate instrument stations within and under the primary control is called the *secondary control.* The extensive surveys executed by the U. S. Coast and Geodetic Survey include four grades of control, called first order, second order, third order, and fourth order, corresponding respectively to primary, secondary, tertiary, and quartenary control, according to the more usual designation. It should be noted that the terms *primary* and *secondary* are purely relative, so that the degree of accuracy used on a secondary traverse for one survey might be sufficient for a primary traverse on another.

Table 462 gives approximate values of the limits of permissible errors for control measurements suitable to the different map scales.

The various classes of control apply either to triangulation or traverse methods; and often a combination of the two methods is used, as when the primary control consists of a triangulation system, and the secondary control consists of traverses connecting the triangulation stations.

463. Details.—For accurate mapping of the relief the ground points are of primary importance and the survey is planned with particular regard to them. Various systems of points are used in drawing the contour lines, and corresponding to each of these systems there are one or more methods designed to locate the points most expeditiously in the field. In addition to the ground points, the field measurements locate all objects necessary to the purpose of the survey.

464. Contour Interval.—The choice of a proper contour interval is based upon three principal considerations: (1) the desired accuracy of elevations read from the map, (2) the characteristic features of the landscape, and (3) the legibility of the map.

The contour interval may be thought of as the scale by which the vertical distances or elevations are measured on a map; the more refined the scale, that is, the smaller the interval, the more refined will be the measurements of the elevations of chosen points. Hence, disregarding the precision of the field measurements, the accuracy with which elevations may be taken from the map is determined by the contour interval.

Often field conditions exist where characteristic features require the use of a contour interval which would otherwise be inappropriate. Thus, if the shape of the terrain is such as to show much variation within a small area, or in other words, if the topography is of *fine texture*, then a smaller contour interval is required to show the greater complexity of configurations. On the other hand, if the landscape is composed of large, regular forms, or is of *coarse texture*, then a larger interval may be used.

TABLE 464.—RELATION BETWEEN SCALE OF MAP, SLOPE OF GROUND, AND CONTOUR INTERVAL

Range of scale	Slope of ground	Interval in feet
1 in. = 10 ft. to 100 ft.	Flat	0.5 or 1
	Rolling	1 or 2
	Hilly	2 or 5
1 in. = 100 ft. to 1,000 ft.	Flat	1, 2, or 5
	Rolling	2 or 5
	Hilly	5 or 10
1 in. = 1,000 ft. to 10,000 ft.	Flat	2, 5, or 10
	Rolling	10 or 20
	Hilly	20 or 50
	Mountainous	50, 100, or 200

In general, contour lines should not be spaced on the map more closely than 30 to the inch, although the legibility of the map depends largely on the fineness and precision with which the lines are drawn. Table 464 represents the best practice for usual conditions.

465. General Field Methods.—The manner of conducting a topographic survey is influenced by numerous conditions such as shape and size of tract, map scale and contour interval, the character of the terrain, and the purpose of the map.

The principal instruments used are the engineers' transit, the plane table, the engineers' level, and the hand level or clinometer.

Points of horizontal and vertical control having been established, the details which are to be mapped may be located by any of the several methods previously described in chapters dealing with chaining and with the use of transit and plane table. Ground points are either located directly upon the contours (when they are called contour points) or chosen without regard to the actual location of the contours. The elevations of these ground points are determined either by direct leveling or by trigonometric leveling, and their positions are determined with respect to points of horizontal control in the same manner as other details which are to be mapped.

In the location of topographic details, the stadia is used extensively for both large- and intermediate-scale maps. It is sufficiently precise except for maps of very large scale, say larger than 1 in. = 20 ft.

The use of the plane table makes it possible to portray the topography of open country more accurately, with the location of a smaller number of ground points, then by the use of the transit. However, the plane table is not a suitable instrument for intermediate- or large-scale mapping of heavily wooded country.

For intermediate- or large-scale mapping where contours are located directly with transit or plane table, the level is frequently used as an auxiliary instrument for directly determining the position of the contours on the ground.

For small-scale mapping a relatively small number of ground points are located by triangulation methods with the plane table, elevations being determined by scaled distances and vertical angles being measured with the telescopic alidade.

For large- and intermediate-scale mapping, ground points are commonly located by the method of radiation, and their elevations usually determined by trigonometric leveling, the transit-stadia or plane-table-stadia method being used.

Where the country is comparatively flat, elevations of ground points are determined by direct leveling.

Sometimes, where the tract to be mapped is wooded, where the topography is smooth, or for other reason, it is expedient to divide the tract into squares, in checker-board fashion, and by direct leveling to determine the elevations of the corners of the squares and of other critical points along the sides of the squares.

On route surveys, such as the preliminary surveys for highways, railroads, and canals, contour points are located along lines transverse to the main traverse, distances from traverse to contour points being measured with the tape, and elevations of contour points being determined usually with the hand level.

Where the system of ground points is located on the contours by direct leveling and the positions of these contour points are determined by radiation or intersection with plane table or transit, it is called the *trace-point* system.

Where the system of ground points is at the corners of squares and at critical points along their sides, all ground points located by linear measurement and their elevations being determined by direct leveling, it is called the *coordinate-point* system.

Where the ground points form an irregular system along ridge and valley lines and other critical features of the terrain and are located by radiation or intersection with transit or plane table, elevations being determined by trigonometric leveling or sometimes by direct leveling, it is called a *controlling-point* system.

Where the ground points form a system on lines transverse to a main traverse, the points being located by linear measurement and direct leveling, usually with the hand level, it is called a *cross-section-point* system.

466. Specifications for Topographic Maps.—The accuracy in horizontal dimensions may be required to be such that the average errors in scaled dimensions between definite points chosen at random shall not exceed a given value.

The accuracy of contour lines may be specified by assigning maximum values to: (1) the average error in elevations taken from the map; and (2) the maximum error indicated by random test profile lines.

INTERMEDIATE SCALE SURVEYS

467. General Remarks.—In the following paragraphs will be described the general field procedure that normally might be employed on an intermediate-scale topographic survey of a tract of considerable extent, where points of horizontal and vertical control are located in favorable positions for securing the map details which are obtained by the transit-stadia or plane-table methods. With modifications the same procedure might be employed on large-scale surveys. Obviously the character of the country, the extent and purpose of the survey, as well as the contour interval and scale of the map, have an important influence upon choice of methods and field procedure.

468. Horizontal Control.—The horizontal control may consist of a traverse system, a triangulation system, or a combination of the two. For an extensive survey there is first established a primary system, and this is extended by a secondary system. On surveys of less extent the primary system is all that is necessary.

469. Primary Traverse.—If primary control is to be established by a traverse for a survey for an intermediate-scale map, the transit-tape method is generally used.

The route is chosen with regard, first, to the position of the stations to be located, and, second, to ease of travel. Because of the accumulative effects of the errors in transit-tape traversing, it is desirable to provide checks at intervals not greater than 15 miles. Hence, it is best if possible to arrange for closed circuits whose lengths will not exceed 12 to 15 miles. If closed circuits can not conveniently be secured, checks for distance should be applied as the work proceeds and checks for azimuth by astronomical observations should be applied at intervals not exceeding 15 miles. If the area is such that a single closed circuit will suffice, the traverse is run near the perimeter of the tract. If several circuits are required, they are arranged, if possible, to divide the tract into areas approximately equal in size. The routes are taken along roads where these are suitably located, and elsewhere the traverse follows ridge or valley lines insofar as these natural control lines make possible a proper distribution of stations. Where they exist, advantage can sometimes be taken of the government land lines, because in flat or rolling country the roads usually follow these lines and the system provides an even and regular subdivision of the area.

It has been stated that the precision of the measurements for primary traverse vary with the purpose of the survey, with the scale of the map, and with the length of the traverse; hence, it will not be possible here to give anything more than a general statement regarding the accuracy required. If it is assumed that the length of the traverse will not exceed 15 miles, the requirements of most surveys will be met by ratios of precision of $\frac{1}{1{,}000}$ to $\frac{1}{10{,}000}$ which apply to map scales of 1 in. = 1,000 ft. (or more) to 1 in. = 100 ft., respectively.

The principles as regards the personnel, instruments, and methods stated in Chap. XIII apply to this work.

470. Secondary Traverse.—Wherever a secondary traverse is required to fix the positions of the instrument stations from which to locate the details with the transit, two general methods may be used: (1) the secondary traverse may be carried forward simultaneously and in connection with the survey for location of details; or (2) it may be run before and separate from the location of details. Because of economy in time the first method is frequently used.

If the details are to be located by the plane-table method, it will usually be desirable to run the secondary traverse with the transit before the location of details is begun because, in this case, there is no opportunity to adjust the secondary traverse to the primary stations if the details are mapped as the secondary traverse proceeds.

The route of the secondary traverse is selected with particular regard to the location of instrument stations that will be best situated for observing details. The route is frequently chosen along a ridge or a valley line, and in all cases the length is made such as to avoid an unduly large accumulation of errors.

The secondary traverse ordinarily is not more than three or four miles in length, and the needs of most surveys will be met if the ratios of error do not exceed $\frac{1}{3{,}000}$ and $\frac{1}{500}$ corresponding to the map scales of 1 in. = 100 ft., and 1 in. = 500 ft. (or more), respectively.

The personnel, instruments, and methods for traversing of this nature have been described in Chap. XIII.

It should be noted that the methods here described as being applicable to secondary traverse for surveys for intermediate-scale maps may, and usually do, apply to secondary traverse for large-scale, and to tertiary or quartenary control for small-scale maps.

Where the required precision is greater than $\frac{1}{1{,}000}$, distances are taped; for a precision less than $\frac{1}{1{,}000}$, distances are usually measured by stadia.

The plane table is frequently used for short secondary traverses where the required precision is not greater than $\frac{1}{1{,}000}$. The method of traversing is described in Art. 435. Where the required precision is low, ridge and valley lines are sometimes located by occupying alternate stations and orienting the table with the compass. For a precision greater than $\frac{1}{500}$, each traverse station is occupied and the table is oriented by backsighting to the preceding station.

Just as the error of closure of a primary traverse is adjusted, so must a secondary traverse between two primary points be adjusted.

471. Primary Triangulation.—The various field conditions favorable to the use of triangulation to establish the horizontal control are as follows: (1) a fairly extended area in an open, hilly region, (2) a city where accurate traversing is difficult because of street traffic, or (3) a rugged mountainous region where traverse work would be slow and laborious. Wooded regions seriously affect the usefulness of this method. Observation towers are there required to establish lines of vision between stations, and the necessary expense of time and money is not justified, except for surveys of considerable magnitude.

A general description of the personnel, instruments, and methods for this work is given in Chap. XXV.

472. Secondary Triangulation.—The primary triangulation has resulted in the location of a number of transit stations, hill-tops, chimneys, trees, and other prominent objects, whose positions have been plotted on the field sheets. The secondary control stations may now be located by one or another of the methods of traversing explained in Chap. XIII, or, where field conditions are suitable, by resection and intersection methods explained in Chap. XXII. If such a secondary triangulation system is to be established within and under the control of the primary triangulation, thus rendering a secondary traverse unnecessary, this work may be done with either the transit or the plane table. The advantages of the latter instrument are important, however, because it provides a ready means of solving the three-point and two-point problems. On the other hand, it is true that three-point determinations made with the transit can be conveniently plotted in the office by the use of a three-arm protractor; or, if this instrument is not available, a piece of tracing cloth on which the angles have been laid off can be used instead, as explained in Art. 443.

This method of secondary triangulation has the advantage that instrument stations can be chosen at strategic points, unaffected by the cumulative errors inherent in traversing.

473. Small Areas.—In the case of topographic surveys for intermediate-scale maps of small areas, *i.e.*, from a few acres up to a hundred or more, the horizontal control can be established by a single traverse, or by a simple triangulation system.

The instruments and methods are those which apply to secondary control, as described in the preceding articles.

474. Vertical Control.—The purpose of the vertical control for a topographic survey is to establish bench marks at convenient intervals over the area to serve as points of departure and closure for the leveling operations of the topographic parties when locating ground points.

Primary and secondary level routes are required in very much the same amount, and bear the same relation to each other, as do the primary and secondary traverses or triangulation systems. In fact, the level routes very often follow the traverse lines, the traverse stations being used as bench marks.

The datum may be assumed for a given survey; but the results of governmental precise levels, referred to sea-level datum, are now available for all but the most isolated regions of the United States.

475. Primary Leveling.—Four degrees of accuracy are commonly used in establishing the primary vertical control for topographic surveys for intermediate-scale maps: (1) a maximum error expressed by the coefficient of 0.05 ft. $\sqrt{\text{distance in miles}}$; (2) 0.1 ft. $\sqrt{\text{distance in miles}}$; (3) 0.3 ft. $\sqrt{\text{distance in miles}}$; and (4) 0.5 ft. $\sqrt{\text{distance in miles}}$. The first applies to very flat regions where a contour interval of 1 ft. or less is used. The second, third, and fourth coefficients apply to surveys in which no more exact use is made of the results than to determine the elevation of contour points for contours having 2-, 5-, and 10-ft. intervals respectively.

The leveling methods applicable to the first two degrees of accuracy stated above have been fully explained in Chap. VIII. For the last two degrees of accuracy mentioned, the larger permissible errors allow long sights and greater speed in manipulation of the level. The last degree of accuracy listed can be reached by careful stadia leveling (Art. 322), which method has important advantages in hilly country.

476. Secondary Leveling.—The lengths of the secondary level circuits are, in general, roughly one-fourth of the lengths of the primary circuits, and since the errors vary approximately, as the square root of the distance, the coefficients of permissible errors are about twice the corresponding coefficients used on the primary circuits.

477. Trigonometric Leveling.—The height of the instrument, either plane table or transit, which has been oriented by the two- or three-point problem, is usually determined by trigonometric leveling, *i.e.*, by a vertical angle and by a horizontal distance, either scaled from the map or measured directly by stadia or tape (see Art. 80). Usually two or more stations are observed, to increase the accuracy of the determination. The method is also applicable to field conditions where the horizontal control is established by triangulation and where a high degree of accuracy in the measured elevations is not required.

For very long sights, corrections for curvature of the earth and refraction should be made; or, preferably, reciprocal measurements to eliminate natural and instrumental errors, should be taken (see Arts. 80, 81).

478. Location of Details.—It is assumed in the remarks which follow that the necessary horizontal and vertical control measurements have been made and that the field party is concerned with the location of details only. If the plane-table method is to be used, the horizontal control is plotted on the plane-table sheet.

The adequacy with which the resultant map meets the purpose of the survey depends very largely upon the work of locating details, and the topographer should be completely informed as to the uses to be made of the map, to the end that he may give the right emphasis to each part of the work.

The four typical systems of ground points used in map construction have been described. The choice of the system best adapted to a particular survey depends upon the field conditions and upon the purpose of the survey. The trace-point and coordinate-point systems yield accurate results if a sufficient number of points are observed, hence, if the purpose of the map requires a comparatively high degree of accuracy, the expense of observing many points is warranted and one or the other of these systems is chosen. But, if the purpose of the survey does not warrant the location of as many points as are required by the two systems mentioned, then better results are secured by the use of controlling points, for the survey of wide areas, or of cross-section points for route surveys. Of course, these different systems are sometimes combined in the same survey, to meet the various field conditions.

In general, since the requirements for accuracy are not as high in the case of surveys for intermediate-scale maps as for large-scale maps, the controlling-point system is most commonly used in the former case, and the coordinate- or trace-point systems in the latter case. Most route surveys are drawn to an intermediate scale and the cross-section system is especially well adapted to that purpose and scale of map.

The field conditions more favorable to the use of the transit as compared with the plane table are many definite points to be located, or ground covered with brush, timber, or vegetation such that at a given station the area of visibility is small. The conditions more favorable to the plane table are open country and many irregular lines and areas to be mapped. Conditions are sometimes such that the advantages of both instruments can be secured by their use in combination. These conditions more often occur in the case of surveys for large-scale maps.

479. Accuracy Required in the Field Measurements to Objects Other than Contours.—The accuracy required in locating details depends not only upon the scale of the map but also upon the definiteness of the objects sighted. Such definite objects as buildings, bridges, boundary lines, etc., should be located with an accuracy consistent with that with which they can be plotted on the map, which may be assumed to be a map distance of about $1/50$ of an inch. Such less definite objects as shore lines, streams, edges of woods, etc., are located with an accuracy corresponding to a map distance of perhaps $1/20$ or $1/30$ of an inch.

480. Accuracy Required in the Field Measurements for the Location of Contours.—The veracity with which contour lines represent the terrain, depends upon: (*a*) the accuracy of the observations, (*b*) the number of observations, and (*c*) the distribution of the points located.

Ground points are definite, but since the contour lines must necessarily be generalized to a large extent, it would be inappropriate to locate the points with refined measurements. The error in position should be consistent with the error in elevation which, in general, should not exceed one-fifth of a contour interval, *i.e.*, the error in position should not exceed one-fifth of the map distance between contours, and the error in elevation should not exceed one-fifth of the vertical distance between contours. The purpose of a topographic survey will be better served by locating a greater number of points with less accuracy, within reasonable limits, than by locating fewer points with greater accuracy.

481. Details by Transit-Stadia.—The personnel of the topography party using the transit usually consists of the transitman, recorder, and one or two rodmen. In wooded country one or more axemen are usually needed to clear the lines of vision. The organization should be such as to allow not only the instrumentman but all other members of the party to work as rapidly as possible.

In locating ground points, the vertical angles are usually observed more precisely than are the horizontal angles. Accordingly, the vertical circle of the instrument assumes greater importance than the horizontal circle; but because all vertical angles are measured with respect to a horizontal plane, it is important that the horizontal plate should be truly horizontal and remain so without the need of constant releveling. If vertical angles are to be measured to the nearest minute of arc, a level tube of about 30″ sensitiveness should be attached to the vernier arm of the vertical arc. The plate levels, the vernier of the vertical arc, and the attached level should be carefully adjusted, and the value of the stadia constant should be carefully determined.

The transit is set up at a station on either the primary or the secondary control, the elevation of the station being known. The transitman then takes side shots to ground points and other features, observing stadia distances, vertical angles, and azimuths as described in Art. 324. Where possible the elevations of points are determined by direct leveling.

The recorder keeps the record of all values given him by the instrumentman, and describes all points by remarks or sketches. Notes are kept in form shown by Fig. 325.

The rodmen choose ground points at summits, depressions, along ridge and valley lines, and at important changes in slope. Objects other than contour points are also located, as the purpose of the survey may require. The matter of selecting points is important, and the rodmen should be carefully instructed and properly trained for their work. They must follow a systematic arrangement of routes such that the entire area is covered and no important objects are overlooked. The rodmen are instructed to report important, features which cannot be well discerned from the transit station.

The procedure is suitable for most surveys for intermediate-scale and for many small-scale maps where the terrain is rolling, hilly, or rough. But if the ground is flat, such that direct leveling over large areas is possible, a more expeditious method may be that of tracing out the contours on the ground. The method is explained in Art. 490, in connection with surveys for large-scale maps, where it is more often used.

482. Field Sketches.—Field sketches are valuable aids to supplement the observed data, especially where the ground exhibits many irregular features and where many details are to be mapped. They vary in character from free-hand sketches and notes entered in a cross-ruled book, to field drawings. Where details are numerous, a drawing board is set up near the transit and as the salient points are located by the transit, they are plotted on the drawing. The more complex and detailed topographic features are then sketched while the terrain is in view.

483. Details with Plane Table.—The personnel of the plane-table party for mapping details consists of the plane-table man, computer, and one or more rodmen.

The equipment usually includes a plane table, telescopic alidade with a control bubble on the vernier arm, a peep-sight alidade, scale, small triangles, 8H pencil, and a stadia slide rule or stadia tables. The alidade should be in good adjustment and the stadia constant should be carefully determined.

Previous to going into the field horizontal control has been adjusted and plotted on the plane-table sheet. The elevations of all bench marks are a matter of record in the hands of the computer, or sometimes are recorded on the sheet itself. A cover sheet of tough paper is placed over the plane-table sheet to prevent its becoming soiled during the fieldwork. A small portion of the cover is torn away to expose the sheet as the work progresses.

The plane-table man sets up his table at a convenient station, and orients by backsighting on the nearest adjacent station. He then directs the rodmen to the controlling points of the terrain. When the rodman presents his rod, the instrumentman pivots the peep-sight alidade about the plotted position of the station and draws a short portion of the ray near the end of the alidade farthest from the station point; directs the telescopic alidade upon the rod, reads the stadia intercept, sets the crosshair on the H.I. point, and motions the rodman forward; levels the control bubble on the vernier arm and reads the vertical angle; plots the point by scaling the horizontal distance (corrected for slope, if necessary, by the computer). The computer has now calculated the elevation of the point and the instrumentman records it on the map near the plotted point. As rapidly as sufficient data are secured, the plane-table man sketches the contour lines. Other objects of the terrain are located and either drawn in their finished form, or with sufficient detail so that they may be completed in the office.

The utmost skill of the topographer is used in judging the features of the terrain and in representing these on the map with the required precision and with the least expenditure of time.

Many objects are located by the method of intersections, the elevations being determined by trigonometric leveling.

484. Because the plane table permits a ready solution of the three-point and two-point problems, use is made of these methods to enable the topographer to utilize advantageous locations for the instrument which may not have been included in the control surveys. This will be true especially where the control has been established by triangulation.

485. Preliminary Route Surveys.—Preliminary surveys for the purpose of locating railroads, highways, canals, etc., are topographic surveys made for the purpose of securing data for an intermediate-scale topographic map of the belt of country through which the route will finally pass.

Usually the horizontal control is a continuous transit traverse with distances measured with the tape and stakes set every 100 ft.; and the vertical control is established by a line of profile levels following the transit, the elevations of the ground at the 100-ft. stations being determined, and bench marks being established. The topographic details on either side of the traverse are then obtained by the cross-section method, contours being directly located usually with the hand level, but sometimes with the engineers' level.

Where the country is open, the transit-stadia method is well adapted to preliminary route surveys. For short lines, traverse distances are often determined by stadia, and elevations of traverse stations by vertical angle and stadia distances. Often under these conditions, details are observed at each transit station as the traverse is run.

Where the country is open, there are many details, and the topography is complex, the plane-table method is advantageous, side shots being taken to controlling points and the map being drawn in the field.

486. Details by Cross-section Method.—The personnel of the topography party using the cross-section method consists of the topographer, one or two chainmen, and one or more axemen if needed.

The equipment consists of either a cross-ruled, wide-paged notebook, or sketch sheets mounted on a small board; a topographers' rod marked in feet; tape; hand level or clinometer; and a 5-ft. Jacob's staff on which the hand level or clinometer is mounted. The elevations of stations on the traverse are in the hands of the topographer.

At each traverse station a line is established, usually by estimation, at right angles to the traverse. The points where contours, at the given contour interval, intercept this cross line are then determined by use of the hand level and topographers' rod, and distances from the main traverse to these contour points, along the given cross line, are measured with the tape. (Thus if the contour interval were 5 ft. and the elevation of the ground at the traverse station were 492 ft., the 495-ft. contour would be located by trial when the topographer observed a rod reading 3 ft. less than that for the traverse station.) In this manner the contours are located over a strip of sufficient width, the party proceeding from cross line to cross line along the traverse. As fast as observations are made, the locations of contours are shown to scale in the notebook or on the sketch sheet, the center line of the cross-ruled page or sheet representing the traverse line, the locations of valley and ridge lines are sketched, and other features are plotted in their true position with respect to traverse and cross lines. Usually the directions or trends of the contours are shown at each cross line and along ridge and valley lines, but on the field sheets the contour lines are not sketched for their full length.

Sometimes, where the topography is regular, sketches are omitted entirely and distance from traverse to contour is recorded numerically.

Also for relatively small-scale maps, the purpose of the survey is sometimes met by measuring the distances to governing ground points (not on contours) by pacing, and the inclination of the ground slopes may be determined by the use of a clinometer. The topographer stations himself at a chaining station on the traverse line. The rodman moves out along a cross-section line, pacing the distance from the center line as he goes. When he reaches an important change in slope he halts and presents his rod. The topographer sights at a point on the rod at the same height above the ground as his eye and reads the angle of inclination. The rodman calls out the distance, and from a table of values, the topographer calculates the difference in elevation between the traverse station and the point indicated by the rod. By adding (or subtracting) this difference in elevation to the elevation of the center stake, the elevation of the point sighted is determined. The position of the point is then plotted and the elevation recorded in the sketch book. The topographer proceeds to the point thus located, and the rodman moves forward to the next important change in slope. This process is repeated until the limit of the strip for which the topography is being taken is reached.

Use of either the hand level or the clinometer has a definite advantage over other methods of locating points if the ground cover is dense, because along the cross-section lines the lengths of sight required are not great and the hand level or clinometer is suitable for sighting through very small openings in the undergrowth. The chainmen are thus able to penetrate rapidly thickets which would be prohibitive to stadia sights. Therefore, this method is sometimes used in forested areas of wide extent, the entire tract being surveyed by means of a series of strips having slightly overlapping edges.

LARGE-SCALE TOPOGRAPHIC SURVEYS

487. Remarks.—One of the principal factors which affects the accuracy of map construction is the number of located points per square inch of map area. Since the map area which represents a given unit of ground area varies as the square of the scale to which it is drawn, it is plain that as the scale increases, many more located points are required per acre of ground. Therefore, in the case of large-scale maps, the location of details becomes a much more important part of the survey as a whole than in the case of surveys for maps to smaller scales.

Methods of establishing horizontal and vertical control and of observing details are not unlike those employed on intermediate-scale surveys.

The degree of accuracy of the horizontal control, whether traverse or triangulation system, depends upon the area of the tract as well as upon the scale of the map. For small areas a precision of $\frac{1}{1{,}000}$ may be all that is required; on more extensive surveys the primary horizontal control often needs to be established with a precision of $\frac{1}{5{,}000}$ or higher. Traverses which form the horizontal control are usually run with transit and tape, and for most surveys of moderate area the traverse has definite advantages over the triangulation system. The triangulation system for city surveys overcomes the difficulty of executing precise transit-tape traverses in the traffic of busy streets. It has the additional advantages that the roofs of many structures provide suitable supports for the instrument stations, and that many objects, such as spires, flagstaffs, etc., provide excellent monuments for observations.

Vertical control is established by direct leveling. The contour interval for large-scale maps is usually 1 or 2 ft. Ordinarily for these intervals the vertical control is of sufficient precision if level circuits close within $0.05\text{ ft.}\sqrt{\text{distance in miles}}$ for extensive surveys, or $0.1\text{ ft.}\sqrt{\text{distance in miles}}$ for smaller areas.

The field work of locating ground points and other details for large-scale maps is accomplished by a variety of methods, instruments, and party organizations. A few typical methods will now be described.

488. The Coordinate-point Method Using the Transit and Tape. This method may be applied in the field in many ways, but the two most common methods of establishing the control and locating the interior points are illustrated in Fig. 488. An example of a finished map is shown in Fig. 456*a*.

By one method, a rectangular traverse, *A*, *B*, *C*, etc. (Fig. 488), is run near the perimeter of the tract. Stakes are set at every 100-ft. station as the traverse proceeds. In this case the error of closure is apparent in the field, for the point of ending should fall on the point of beginning. If this error is more than the permissible error, the hubs and stakes are reset in such manner as to reduce the error within allowable limits.

A line of profile levels is run around this traverse to close within a permissible error, thus establishing the elevation of each stake and hub. These readings, except on turning points, are taken on the ground, just in front of the numbered side of each stake. The elevations thus determined should be correct to the nearest 0.1 ft. which is sufficiently exact to control the lines of levels run within the tract to determine the elevations of the coordinate points.

The interior stakes are set by transit-tape lines beginning at a stake on one side and closing on the corresponding stake on the opposite side, as from *F*-18 to *B*-1, *F*-17 to *B*-2, etc.

The elevations of these interior stakes are then determined with the engineers' level, or with a hand level, depending on the lengths of the lines and the accuracy required.

Irregularities which can not be properly interpreted from observations on the corners of the 100-ft. squares are now observed by means of a hand level and tape and the features sketched in a cross-ruled book or on a sketch sheet. Other details, as fences, roads, buildings, etc., are located by measurements taken from adjacent coordinate points. With all this information recorded, the map is constructed in the office as explained in Art. 456.

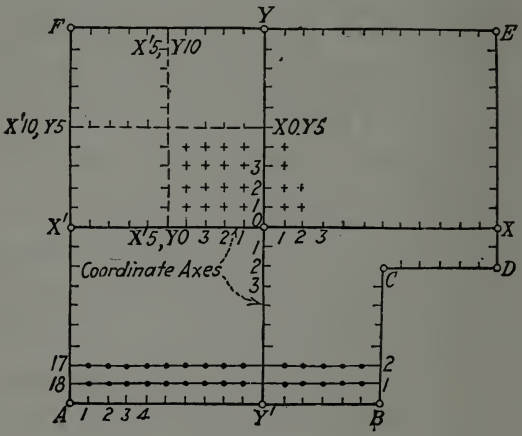


Fig. 488.

All bench marks and transit-line hubs are permanently referenced for future use, and in some cases, substantial stakes bearing identification marks are left at all 100-ft. station corners to serve as reference monuments during construction work.

By another method, coordinate axes are established by transit-tape lines as *OX*, *OX'*, *OY*, *OY'*, hubs being set at the extremities, and from these hubs coordinate lines are run parallel to the axes to intersect at the corners of the tract, thus affording checks on the work.

Stakes are set at each 100-ft. station along these lines, and levels are run to establish the vertical control. The interior points are set, their elevations are determined, other details are located, and the map is drawn as explained in the previous paragraphs.

489. The Coordinate-point Method Using the Plane Table.—The use of the plane table is a great aid to mapping by the coordinate method, if many irregular features are to be mapped. The methods of establishing the stations on the ground and of determining the elevations of these points are the same as when using the transit instrument, described in the previous article. The details are drawn, however, on a plane-table sheet in the field. The board is oriented by sighting along a traverse line on the ground, and details are sighted with a peep-sight alidade, the points being located by taped distances from adjacent coordinate points.

490. The Trace-point Method Using the Plane Table.—If visibility is good, greater accuracy and speed can sometimes be obtained by use of the trace-point system than by the coordinate system of ground points. The engineers' level may often be used to advantage with the plane table. It is readily moved from point to point, and direct rod readings are quickly made to place the rodman on a given contour.

The personnel of the party consists of a topographer at the plane table, a levelman, and one or more rodmen. A computer is sometimes added to the party.

The procedure is as follows: The levelman, having set up the level at a convenient station, directs the rodman about until he locates a point on a given contour. This point is immediately sighted by the plane-table man and plotted on the plane-table sheet. The rodman then moves to the next point usually on the same contour. The distances from the plane table to contour points are measured by the stadia method, but definite objects are usually located by taped distances.

491. The Controlling-point Method Using Plane Table.—The personnel and procedure for the transit-stadia method have been described in Art. 481, but in some cases where many details are to be sighted the transit may be used to advantage in combination with the plane table.

If this method is used, the transit is set up on the control station, the position of which is plotted on the plane-table sheet. The plane table is then set up near by and its position plotted on the map in its correct relation to the transit station. The plane table is oriented in the usual manner. When a rodman has selected a contour point, the transitman observes the distance and vertical angle to it; the plane-table man observes the direction with the peep-sight alidade, draws a ray towards it from the plotted position of the plane-table station, and plots its position at the correct distance scaled from the plotted position from the transit station. The transitman, or computer, if one is present, computes the elevation of the point, which is recorded for immediate use on the map.

This combination of instruments utilizes all the advantages of both and provides an extremely rapid and efficient method of mapping where conditions are favorable, *i.e.*, where many points are to be located, a large scale is used, and the visibility is good. As compared with the use of the plane table alone, the combined use of the two instruments will save time in the field, but may not reduce the total cost because of the expense incurred by the addition of a transitman to the field party.

CHAPTER XXV

TRIANGULATION

492. General Remarks.—A triangulation system is composed of a series of triangles, one, two, or all three sides of each triangle being also sides of adjacent triangles. The lines of a triangulation system form a network tying together the points or stations where the angles are measured. The vertices of the triangles are the triangulation stations.

By the use of the triangulation method, the necessity of measuring the length of every line is avoided. If it were possible to measure all the angles in a triangulation system and one side with absolute precision, then no further linear measurements would be necessary, for then the process of precisely calculating lengths could be carried through the series of connected triangles. Unavoidable errors in the field measurements, however, make it desirable that the lengths of two or more lines in each system be measured as a means of checking the calculated distances. The lines whose lengths are measured are called *base lines*.

The arrangement of the triangles in most systems affords many different geometrical figures for each of which the theoretical value of the sum of the included angles is known. Also, the sum of the angles about any station should equal 360° and, in any triangle, the lengths of the sides should be proportional to the sines of the angles opposite. These known conditions serve two purposes: first, as a measure of the precision of the angle measurements, and second, as a means of adjusting the errors in the measurements so as to secure the most probable values of the measured quantities.

The work of establishing a triangulation system may be divided as follows: (*a*) the reconnaissance, by means of which a proper selection of station locations can be made; (*b*) the erection of signals, and, in some cases, tripods or towers for elevating the instruments and the signals; (*c*) the measurement of the base lines; (*d*) the measurement of angles between the sides of triangles; (*e*) the astronomical observations from which to determine the true azimuths of the triangle sides; and (*f*) the computations, including the adjustment of the observations, and the calculation of the length of each triangle side and the coordinates of the stations.

There is a quality of triangulation corresponding to every degree of precision. Thus, triangulation may be used for a simple topographic survey covering but a few acres or it may be used to extend control of the highest order across the continent.

Where conditions are favorable, precise traversing may be comparable in accuracy with the most precise triangulation work. The relative merits of the triangulation method and the traverse method are based on the character of the terrain only, and not on the degree of precision to be attained. If favorable routes are available, the method of traversing is superior to the method of triangulation; but if the terrain offers many obstacles to traverse work, then triangulation is the superior method. Because of the character of the terrain near shore lines, the triangulation method is extensively used in surveys for hydrographic charts, and for maps of the coast line and of navigable rivers.

493. Classification of Triangulation Systems.—The bases on which triangulation systems are classified are: (*a*) the average closing angular error in the triangles of the system; and (*b*) the discrepancy between the measured length of a base line and its length as computed through the system from the one next adjacent.

The Board of Surveys and Maps (composed of representatives of the federal bodies engaged in surveying and mapping) has classified triangulation for the extensive surveys of the U. S. Government as follows:

	First order	Second order	Third order	Fourth order
Average triangle closure, seconds.	1	3	5	>5
Check on base	$\frac{1}{25,000}$	$\frac{1}{10,000}$	$\frac{1}{5,000}$	$>\frac{1}{5,000}$

This classification takes into account (*a*) the average closing angular error in the triangles of the system, and (*b*) the discrepancy between the measured length of a base line and its length as computed through the system from the one next adjacent.

First-order triangulation furnishes the primary horizontal control for small-scale mapping operations of the U. S. Coast and Geodetic Survey, the U. S. Geological Survey, etc., the triangle sides often being many miles in length. The system which extends across the continent and from Canada to Mexico is of this order.

Third- and fourth-order triangulation establishes points of horizontal control at short intervals in advantageous locations for detail mapping. For intermediate- and large-scale surveys of limited extent, triangulation of the third and fourth orders would be employed to establish points of primary and secondary control.

First- and second-order triangulation calls for methods of high precision not often necessary except on very extensive surveys. Third- and fourth-order triangulation calls for methods of ordinary precision, sufficiently exact for the usual intermediate- and large-scale surveys of limited area. In this book special emphasis will be given to triangulation methods of ordinary precision for surveys of limited extent.

494. Reconnaissance.—The reconnaissance survey provides data by means of which the most advantageous stations can be selected. These data relate to such conditions as the shape of the resulting triangles, number of stations to be occupied, number of angles to be measured, usefulness of the stations in later work, possibility and convenience of base-line measurements, accessibility of stations, visibility of stations from each other, and cost of the necessary signals.

The chief of party examines the terrain, choosing the sites where the visibility between stations is good and where the shapes of the resulting triangles will be the best possible. At each station the angles to other stations are read usually with a prismatic compass and distances are estimated or measured roughly, en route, so that the suitability of the system as a whole can be examined before signals are erected.

In open, hilly regions, stations can often be located on summits such that the instrument can be set up on the ground. Under adverse conditions, however, the instrument must be elevated by means of a tower to a height sufficient to enable all adjacent stations to be observed.

For night observations an electric signal-lamp may be employed. For long distances heliotropes of various designs are used as daylight signals, when the sun is shining, if flags or poles are invisible to the eye. In principle, the heliotrope consists of a mirror so arranged as to flash sunlight to the distant station.

495. Instruments for Angular Measurement.—For surveys of ordinary precision the instruments employed in the measurement of horizontal angles are the engineers' transit in its usual form and the *repeating theodolite*, similar in design to, but larger in size than, the ordinary engineers' transit. Quite commonly the verniers of the repeating theodolite read to 10 sec.

On surveys of high precision the angles are measured with a *direction instrument*. The principal distinguishing features of the direction instrument are: (1) that the horizontal plate has but a single tangent motion, and (2) that instead of verniers, micrometer microscopes are used to read the subdivisions of the graduated circle.

TRIANGULATION OF ORDINARY PRECISION

496. Triangulation Figures.—Triangulation figures may be divided into three forms, namely: the chain of simple triangles, the chain of polygons, and the chain of quadrilaterals. In an extensive triangulation system ordinarily all three types of figure will be found irregularly overlapping and intermingling.

(*a*) *Chain of Triangles.*—The distinguishing feature of the chain of single triangles (Fig. 496*a*) is that there is but one possible route by which the calculation of distances can be carried through the chain. If AB be the base line whose length is measured, and all the angles of the triangles are observed, then the length of the triangle sides in the chain (as AC, BC, BD, etc.) may be calculated progressively along the chain from the measured base line to the triangle side farthest removed from the base line. In case two lines are measured as base lines, one at each end of the system, the calculations may be carried from each toward the other, to a triangle side somewhere between them.

The sum of the angles of each triangle should, of course, be 180°. As the sum of the observed angles will not normally equal this amount, the angles are adjusted to satisfy this requirement before the distances are calculated.

(*b*) *Chain of Polygons.*—A polygon is composed of a group of triangles bounded by three, four, five, or more sides and having, in addition to the stations on the boundary of the figure, a station somewhere inside the figure which is at a vertex common to all the triangles of the figure. A chain of such polygons is illustrated in Fig. 496*b*.

As in the case of the chain of single triangles, the sum of the three measured angles of each triangle should equal 180°. Also, the sum of the angles about the central point should equal 360°. But there is a further condition that the length of any side may be calculated by two routes and that these two calculated lengths should agree. If all the angles were known exactly, the calculated value of a side would be the same by one route as by the other. The angles in a given polygon are so adjusted that this condition exists, and further, that the sum of the three angles of each triangle will equal 180° and that the sum of the angles about the central point will equal 360°. A similar adjustment is made for the other polygons in the triangulation chain.

(*c*) *Chain of Quadrilaterals.*—A quadrilateral is a group of four overlapping triangles bounded by four sides. A chain of quadrilaterals is shown in Fig. 496*c*. In the individual quadrilateral there is no station at the intersection of the diagonals. Consider one of the quadrilaterals, as *ABDC*. The measurement of the angles gives the angles of four triangles, *ABD*, *ACD*, *ABC*, and *BCD*, in each of which the sum of the angles must equal 180°. In addition, as in the case of the polygons, the length of any line must be the same when calculated by one route as when calculated by another.

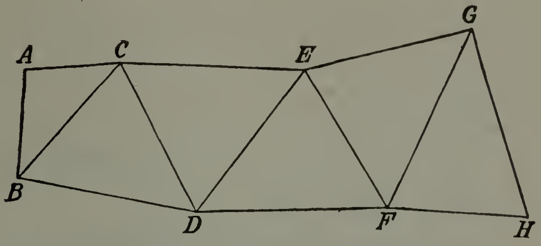


Fig. 496*a*.—Chain of single triangles.

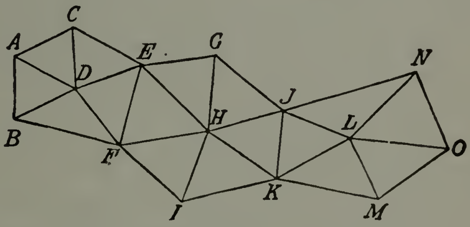


Fig. 496*b*.—Chain of polygons.

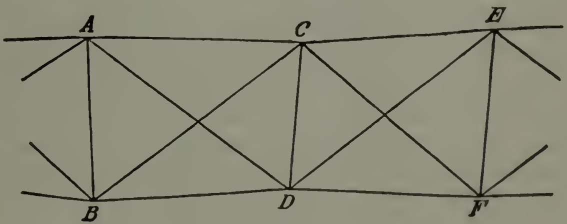


Fig. 496*c*.—Chain of quadrilaterals.

For example, consider *AB* as the side of known length and *CD* as the side whose length is required. There are four ways in which the required distance *CD* may be found. The four values of *CD* should agree, and will agree if the angles are precisely known. The adjustment of angles must be so carried out as to make their adjusted values satisfy this requirement as well as to make the sum of the three angles of each triangle equal 180°.

496a. Of these three forms of triangulation figures, the chain of single triangles is the simplest, requiring the measurement of fewer angles than does either of the other two. This type of figure, however, has the obvious weakness that, aside from the test of precision afforded by the measurement of more than one base line, the only check is in the sum of the angles of each triangle considered by itself. To reach the same precision in the determination of lengths, base lines would have to be placed closer together.

For more accurate work, quadrilaterals or polygons are used when possible in preference to single triangles, the quadrilaterals being best adapted for a relatively narrow chain between widely separated points, and the polygons being best adapted to surveys over wide areas.

497. Strength of Figure.—It has been shown in Fig. 13*a*, that values computed from the sine function of angles near 0° or 180° are subject to large ratios of error, and, since in triangulation computations the sine function is nearly always used, it follows that angles near 0° and 180° are undesirable. It has been found in practice that satisfactory results can be secured for most purposes if the angles used in the computations fall between the limits of 30° and 150°. Attention should be directed, however, to the fact that many angles which are measured in the field serve the purpose of providing adjustment conditions only, and are not used in computing the lengths of the sides in the system. Such angles may be very near 0° or 180° without impairing the excellence of the system as a whole.

498. Base Nets.—In a system of triangulation, long sides (within proper limits) are obviously more economical than short ones. It is difficult and expensive to measure long base lines; hence, in practice, the base lines are usually much shorter than the average length of the triangle sides. This condition necessitates the most careful attention to the location of the base lines and the immediately adjacent stations. The figure formed by this group of stations is called the *base net* and is formed so as to permit economical lengths of triangle sides to be used with a minimum loss in the precision of the base-line measurement.

The arrangements shown in Fig. 498 are examples of good base nets affording quick and accurate expansion of the base line to the longer sides of the system.

The number of base lines required will depend on the excellence of the shapes of the triangles in the system. In practice, the accuracy necessary for work of ordinary precision can be carried through a chain of from 10 to 30 triangles, depending upon the strength of the figures secured.

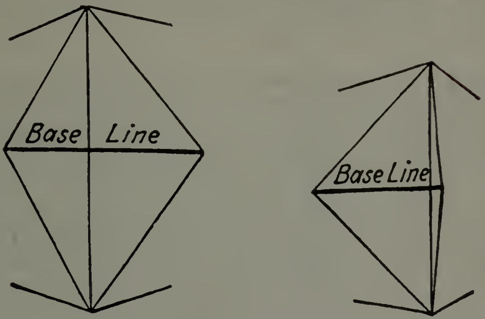

Base Line
Base Line

FIG. 498.—Base nets.

499. Triangulation System.—A triangulation system of a wide area is shown in Fig. 499. The system is, for the most part, composed of polygons. Two attached triangles are *WPZ* and *YST*. The computations for this system can be arranged to afford checks on the computed values of most of the sides except, of course, such ones as *WP* and *WZ*. The base nets are quadrilaterals.

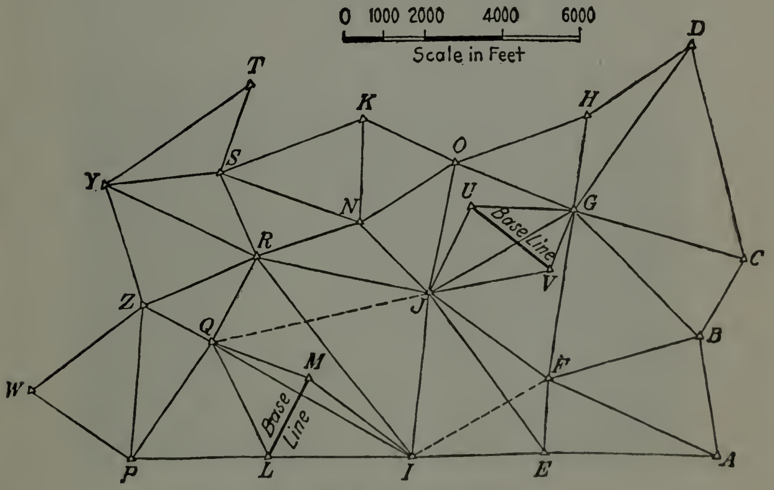

0
1000
2000
4000
6000
Scale in Feet
T
K
H
D
O
S
Y
U
N
G
Base Line
R
C
V
Z
J
Q
B
M
W
F
Base Line
P
L
I
E
A

FIG. 499.—Triangulation system.

In all systems, the base lines should be so placed that as many sides as possible can be included in the routes through which the computations are carried from one base line to the next. Sides not so included will not be subject to the positive checks afforded the other sides.

500. Signals and Instrument Supports.—Each triangulation station must be marked by some sort of signal visible from nearby stations from which it is to be sighted. If the station is one to which observations are taken, but which itself is not occupied for the measurement of angles, a relatively simple signal structure is used. This signal may be one constructed for the purpose, or may be an object already in place, such as a flagpole, chimney, or telegraph

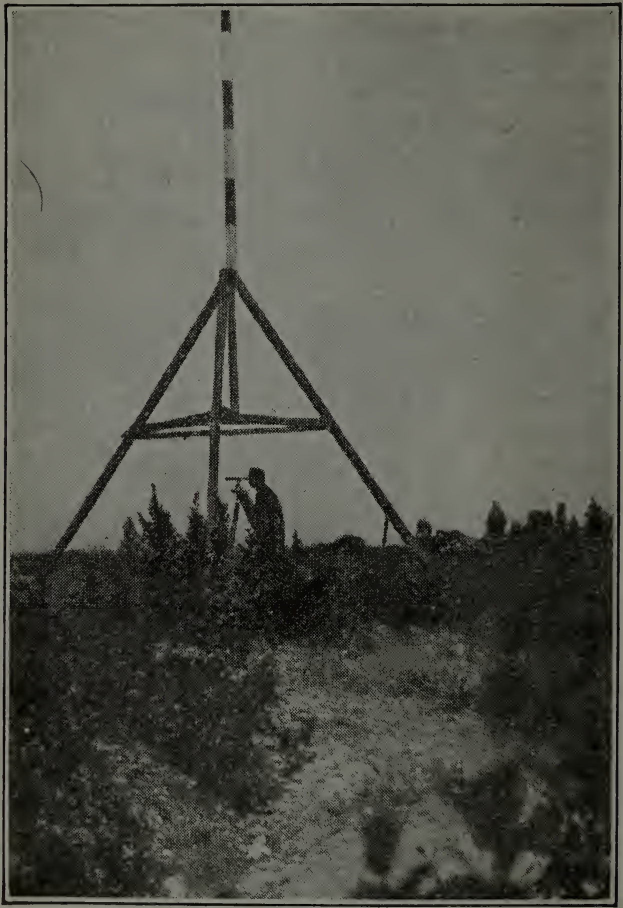


Fig. 500.—Tripod signal.

pole. To increase the visibility of a pole or tree signal, two square or rectangular targets are sometimes attached, being placed at right angles to each other.

At an instrument station it is desirable to have a signal so constructed that the instrument may be placed directly over the station when angular observations are to be taken. In a small triangulation system with triangle sides only a few hundred feet in length, and with but few angles to be measured, so that observations are soon completed, a temporary signal which is readily moved may be all that is necessary. A pole or a light tripod to which is attached a plumb-line may be centered over the station mark.

Where a larger and a more permanent signal is required that does not need to be moved to provide for setting up the instrument, a tripod like that shown in Fig. 500 may be used.

When the instrument must be elevated to secure visibility, a combined observing tower and signal may be built. A central structure, built in the tripod form, is designed to support the instrument. Around this, but entirely separate from it, is a four-sided structure supporting the platform upon which the observer stands.

501. Angle Measurements.—Thus far the term *triangulation station* has been used to designate instrument stations, *i.e.*, points where the instrument is set up to measure angles. In most triangulation systems secondary control is established by observations to stations in the vicinity of the primary or main stations, but these stations, called *minor stations*, are not used in the extension of the main system of triangles. Obviously, the angle measurements of such stations may be made with a lower degree of accuracy than is required in the main scheme.

The instrument is set up at each major station, and angles with vertex at the station are measured by the method of repetition described in Art. 200.

The exact procedure to be employed in measuring an angle by repetition depends somewhat upon the desired precision. When the method is employed primarily as a check, the angle is doubled usually without reversing the telescope between repetitions.

When it is desired to increase the precision a moderate amount, quite commonly the angle is multiplied four or six times, half of the observations being made with the telescope in the direct position, and half with it in the reversed position, and both verniers are read. Certain instrumental errors, such as those due to eccentricity and to nonadjustment of the horizontal axis, are eliminated in this manner.

When a high degree of precision is necessary, several sets of perhaps six repetitions are taken with the telescope in the direct position, and these sets are duplicated by others taken with the telescope in the reversed position. To eliminate errors of graduation, settings are so made that readings are distributed over various parts of the circle and verniers, and to eliminate eccentricity, both verniers are read. Furthermore, special care is taken so to manipulate the instrument that systematic errors due to lost motion in the clamps and to other causes will be eliminated.

If precise results are to be obtained, the instrument must be manipulated with care. The plate bubbles should be kept centered, but the leveling screws should not be disturbed except between repetitions. When turning on the lower motion, the hands should be in contact with the lower plate, and when turning on the upper motion, they should be in contact with the upper plate, and not the telescope. The last motion of the tangent screws should be clockwise or against the opposing spring. To eliminate the effect of twist in the tripod, after each repetition the instrument should be rotated on its lower motion in the same direction that it was turned on its upper motion. That is, the direction of movement is always either clockwise or counterclockwise. Due to the possibilities of unequal settlement of the tripod and to unequal expansion of the parts of the telescope, it is desirable that the observations be made as rapidly as is consistent with careful work. The instrument, so far as possible, should be protected from sun and wind.

502. Azimuth Determinations.—In calculating the coordinates of triangulation stations, a meridian of reference, either true or assumed, is used. For a small system of a few hundred acres or less, an assumed meridian will be satisfactory; but for an extensive system, it is desirable, and many times essential, that the true meridian be used. In the latter case, a determination of the azimuth of a triangle side is made at any convenient station, from which determination the azimuths of all other sides can be computed. If the system is many miles in extent, a determination is made at intervals of 20 to 30 triangles, as a check on the angle measurements.

BASE-LINE MEASUREMENTS

503. The Tape.—The steel tape is used for measurements of ordinary precision and the invar tape, for which the coefficient of expansion may be as low as 0.0000002 per 1°F. (about $\frac{1}{30}$ that of steel), is used for base-line measurements of the highest precision. The length of the tape should be accurately determined by comparison with a known standard of length. The U. S. Bureau of Standards at Washington, D. C., for a small fee, will compare the tape with a standard length, and will issue a certificate showing the true length of the tape under stated conditions as regards tension, temperature, and supports.

504. Measuring the Base Line.—Base-line measurements can be made satisfactorily over somewhat rough and uneven ground if provisions are made for properly supporting and stretching the tape. The top of the rail of a railroad track or the surface of a paved roadway of uniform grade may be used for base-line measurements, and these surfaces render unnecessary part of the special preparations required for measurements over uneven ground. These surfaces should not be used, however, when their temperatures are appreciably different from that of the surrounding air. Where the base line is over uneven ground, supports for the tape are provided by the use of 2- by 4-in. (sometimes 4- by 4-in.) posts driven firmly in the ground. These are placed on a transit line and at intervals of one tape length, as nearly as may be determined by careful preliminary measurements. A strip of copper or zinc is tacked to the top of each post to provide a suitable surface on which to mark the tape lengths. Profile levels are run over the tops of the posts to determine the gradient from post to post. The tape is usually supported at one, two, or three points between the end supports. These intermediate points must be placed accurately on the grade line between the tops of the two adjacent end posts.

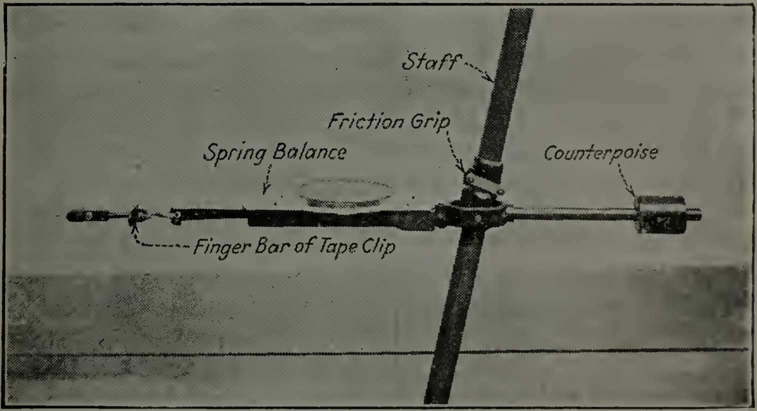


FIG. 504.—Tape stretcher and spring balance.

The party consists of from four to six men whose duties are indicated by the following description of the procedure. The tape is stretched and the proper tension is applied by means of a stretcher apparatus and a spring balance (sometimes a weight and pulley are used instead of the spring balance) attached to the forward end of the tape beyond the end support (Fig. 504). When the rear end of the tape is observed to coincide with the mark and when the proper tension is applied, the position of the forward end of the tape is marked by a fine line engraved on the metal strip on the top of the stake; thermometers held at the same height above the ground as the tape, one at each end and sometimes one at the mid-point of the length of tape, are read at the instant that the tape length is marked on the forward post. The tape is then carried forward (without allowing it to drag on ground) and the process is repeated. After a few measurements, the end of the tape will most likely fall either beyond or short of the limits of the metal strip of the next forward post. Accordingly, it will be necessary occasionally to use *set forwards* or *set backs* as may be necessary to keep the tape ends on top of the posts. These are measurements of small distances made by the use of a finely divided pocket scale and a pair of dividers.

505. Errors in Base-line Measurements.—The nature of the various sources of error in tape measurements as they affect work of ordinary precision is treated in Art. 68 but the special procedure used in base-line measurements effects changes in sources of error which should be noted.

1. *Tape Not Standard Length.*—If the tape is carefully handled the comparison made by the Bureau of Standards will be sufficiently precise to render the error from this source negligible for the duration of a field season. Experience has shown that a tape changes in length over a period of years even though it is not used. Hence, a tape should be standardized in the same season in which it is used.

2. *Variations in Temperature.*—It has been stated above that the effect of temperature is the most serious source of error in precise linear measurements. For example, an error of 0.5°F in the mean temperature of a steel tape introduces an error of $\frac{1}{300,000}$ in the measured length. This error alone is greater than the total error permitted in precise work.

Field conditions favorable to a small variation in the thermal expansion are those in which the air and the ground are at nearly the same temperature while the measurement is being made. These conditions are obtained on cloudy days or at night, and it is at such times that important base lines are usually measured. Measurements taken at such times have indicated probable errors as low as $\frac{1}{3{,}000{,}000}$.

It should be stated that at night a pavement may have a temperature considerably different from that of the air, hence before a base-line measurement is made along a road surface, observations should be taken to make sure that air and pavement temperatures agree closely.

3. *Tape Not Horizontal.*—This source of error is rendered negligible by determining the difference in elevation of adjacent posts by a line of profile levels run with a transit or an engineers' level. The gradient for each tape length is then readily computed and the corrections are calculated.

4. *Variations in Tension.*—This source of error is not important if only stretch in the tape is considered, as where the tape is supported throughout its length. Its effect on the amount of sag, if the tape is supported at intervals, is much more serious (see following paragraph).

5. *Sag in Tape.*—The effect of variable tension is more important the less the number of supports. Other things being equal, the shortening due to sag is directly dependent upon the tension.

A normal tension (see Art. 72) is used where conditions are favorable.

6. *Wind.*—If a strong wind is blowing normal to the base line and if the tape is not supported throughout its length, there will be a lateral displacement of the unsupported portions of the tape, thus producing an effect similar to that of sag. Wind also sets up vibrations which render the tape unsteady. It is quite impossible to calculate corrections for wind effects, and, accordingly, precise measurements should not be attempted if the tape is unsupported and if a strong cross wind is blowing.

7. *Marking the Tape Lengths.*—The magnitude of the errors resulting from this source depends upon the fineness of the tape graduation and of the line cut on the metal strip, and also upon the precision with which these lines are made to coincide when the tape lengths are being marked. The lines marking the ends of ordinary steel tapes are relatively coarse, but makers of invar tapes use lines not exceeding 0.002 in. in width. The Bureau of Standards is careful to state which edge of the tape is used in marking the comparison with the standard length. For a given tape, careful manipulation is the only means of reducing errors from this source.

506. Corrections.—The methods of computing and applying corrections to tape measurements are given in Arts. 66 to 74.

507. Reduction to Sea Level.—It is sometimes necessary to reduce the length of the base line to the equivalent length at mean sea level. The correction to be subtracted from the true length is given by the expression

$$C_l = \frac{LA}{R}$$

in which L is the length of the base line, A is the mean altitude of the base line above sea level and R is the radius of the earth (Mean R = 20,890,600 ft., Log R = 7.31995).

COMPUTATIONS

508. Adjustment of a Chain of Triangles.—In the case of the simple chain of triangles there are two kinds of adjustment to be made: (1) the station adjustment, to make the sum of the angles around each point equal to 360°, and (2) the figure adjustment, to make the sum of the three angles in each triangle equal 180°. In precise triangulation these two adjustments are made in one operation, by methods involving the principles of *least squares*, but the following approximate solution yields results that are sufficiently accurate for most cases.

To make the sum of the angles around each point equal to 360°, the angles are added together and the sum subtracted from 360°. This difference is divided by the number of angles around the point, and the quantity so found is algebraically added to each angle. To make the sum of the angles in each triangle equal to 180°, a similar plan is followed, using the values obtained by the first adjustment; that is, the three angles are added together and the sum subtracted from 180°. One-third of the difference is added to each of the three angles.

This method of adjustment assumes that all the angles were observed in the same way and with the same precision, and is only applicable when such is the case. If this is not the case, certain angles being measured with a higher precision than certain others, the method may be readily modified by weighting the observations of the several angles within the system.

Following is an example of the adjustment of the angles in a simple chain of three triangles, all of the observed angles being assumed to be of equal precision.

Example: Below are tabulated the observed angles in the chain of triangles shown in Fig. 508. The station and figure adjustments are to be made by dividing the errors proportionately among the angles.

The values after performing the station and figure adjustments are as shown.

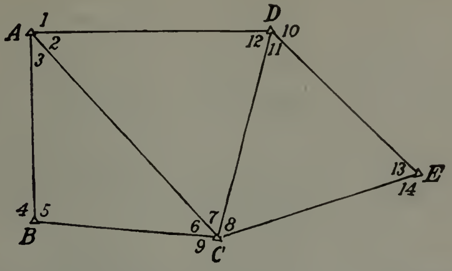


FIG. 508.—Adjustment of a chain of triangles.

STATION ADJUSTMENT

Angles	Observed values	Stations		
1	240°19′00″	*A*	Sum of observed angles	
2	73°31′10″		= 360°00′30″	
3	46°10′20″		Adjusted angles	
4	267°12′20″		1	240°18′50″
5	92°47′30″		2	73°31′00″
6	41°02′00″		3	46°10′10″
7	63°10′40″	*B*	Sum of observed angles	
8	74°43′10″		= 359°59′50″	
9	181°04′30″		Adjusted angles	
10	260°33′00″		4	267°12′25″
11	56°09′00″		5	92°47′35″
12	43°18′30″	*C*	Sum of observed angles	
13	49°07′50″		= 360°00′20″	
14	310°52′10″		Adjusted angles	
			6	41°01′55″
			7	63°10′35″
			8	74°43′05″
			9	181°04′25″
		D	Sum of observed angles	
			= 360°00′30″	
			Adjusted angles	
			10	260°32′50″
			11	56°08′50″
			12	43°18′20″
		E	Sum of observed angles	
			= 360°00′00″	
			Adjusted angles	
			13	49°07′50″
			14	310°52′10″

Figure Adjustment

Triangle *ABC*

Angles	Values from station adjustment	Values from figure adjustment
3	46°10′10″	46°10′16″
5	92°47′35″	92°47′42″
6	41°01′55″	41°02′02″
Sum....	179°59′40″	180°00′00″
	Triangle *ACD*	
2	73°31′00″	73°31′02″
7	63°10′35″	63°10′37″
12	43°18′20″	43°18′21″
Sum....	179°59′55″	180°00′00″
	Triangle *CDE*	
8	74°43′05″	74°43′10″
11	56°08′50″	56°08′55″
13	49°07′50″	49°07′55″
Sum....	179°59′45″	180°00′00″

509. Adjustment of a Quadrilateral.—In the adjustment of the angles of a quadrilateral, two conditions may be considered: The first is the geometric condition that the sum of the interior angles of a rectilinear figure is equal to $(n - 2)\ 180°$, in which n is the number of sides of the figure; the second is the trigonometric condition that in any triangle the sines of the angles are proportional to the lengths of the sides opposite. Each of these conditions will now be considered.

1. *Geometric Conditions.*—When all angles in a quadrilateral are measured, there are four overlapping triangles. These are shown as triangles *ABC*, *ACD*, *ABD*, and *BCD* in Fig. 509. In each of these triangles the sum of the three angles must be 180°. Hence from the figure,

$$b + c + d + e = 180° \tag{1}$$

$$a + f + g + h = 180° \tag{2}$$

$$a + b + c + h = 180° \tag{3}$$

$$d + e + f + g = 180° \tag{4}$$

Also the sum of the eight lettered angles in the figure must equal 360°, since they form the interior angles of a closed figure of four sides. This may be derived also by the addition of equations (1) and (2) or (3) and (4).

$$a + b + c + d + e + f + g + h = 360° \tag{5}$$

Further, since the vertically opposite angles at the intersection of the diagonals must be equal, it follows that

$$b + c = f + g \tag{6}$$
$$d + e = h + a \tag{7}$$

If any three of these seven equations are satisfied, the other four must, of necessity, be satisfied also. Equations (5), (6), and (7) are the ones most convenient to use.

The following procedure is suggested:

A. Adjust the angles around each point to make their sum equal to 360°.

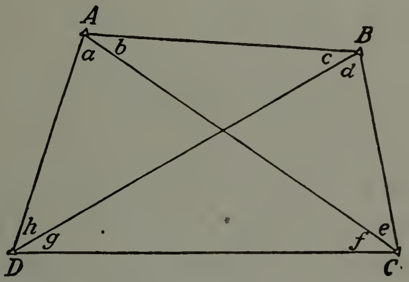


FIG. 509.—Adjustment of a quadrilateral.

B. Using the values resulting from adjustment 1, add the eight angles a, b, c, d, e, f, g, and h, and subtract their sum from 360°. Divide the difference by 8, and algebraically add the result to each of the eight angles, thus satisfying the conditions of equation (5).

C. Using the adjusted values from 2, find the difference between the sums $(b + c)$ and $(f + g)$ and divide that difference by four. Apply the result as a correction to each of the four angles, increasing each of the two whose sum is the smaller and decreasing each of the two whose sum is the larger, thus making these angles satisfy equation (6) without disturbing the adjustment for equation (5). Proceed in the same way with each of the four angles involved in equation (7).

2. *Trigonometric Conditions.*—If the length of one line, as AB, is known, and the length of the opposite side CD is to be calculated, the computer may select one or another series of triangles for use in accomplishing this result. For example, a solution of triangle ABC gives the length of AC, then from triangle ACD the required length of CD is found; or in the triangle ABC the length BC is found, then in BCD the length CD is calculated. There are four possible choices of route through the figure and it now remains to be seen whether the angles, as so far adjusted, are so related as to make the value of the

length of a computed side independent of the route used. Assume that the length of AB is known and the length of CD is to be found.

$$AD = AB\frac{\sin c}{\sin h}$$

$$CD = AD\frac{\sin a}{\sin f} = AB\frac{\sin a \sin c}{\sin f \sin h}$$

Similarly

$$CD = AB\frac{\sin b \sin d}{\sin c \sin g}$$

Equating these two values of CD,

$$\frac{\sin a \sin c}{\sin f \sin h} = \frac{\sin b \sin d}{\sin e \sin g}$$

or

$$\frac{\sin a \sin c \sin e \sin g}{\sin b \sin d \sin f \sin h} = 1$$

Expressed in logarithmic form, this is

$$(\log \sin a + \log \sin c + \log \sin e + \log \sin g) \\ - (\log \sin b + \log \sin d + \log \sin f + \log \sin h) = 0 \quad (8)$$

The angles are tested for satisfaction of this equation by adding the logarithmic sines in the two groups as indicated and by finding the difference between the two sums.

Various adjustments by which this difference may be reduced to zero are possible. The *least squares* adjustment gives the most probable values to the adjusted angles, but it is somewhat more elaborate than is necessary for most surveys. A simple approximate method which gives an equal correction to each angle, and which does not disturb the geometric condition, is as follows: (*a*) record the log sines as shown in the following example; (*b*) for each angle, record the tabular logarithmic difference for 1″ opposite each logarithm; (*c*) find the average required change in log sine (A) by dividing the difference of the sums by 8; (*d*) find the average difference for 1″ (B); (*e*) the ratio $\frac{A}{B}$ gives the number of seconds of arc to be applied as a correction to each angle; (*f*) add this correction to each of the four angles the sum of whose log sines is the smaller, and subtract it from each of the angles the sum of whose log sines is the larger, and thus the corrected values are obtained.

Example: Given the angles as measured in the quadrilateral of Fig. 509, for which the station adjustment indicated as step A under the suggested procedure and indicated below under Adjustment *A*, has been made. Find the adjusted angles for both the geometric and the trigonometric conditions.

Geometric Conditions				Trigonometric Conditions
	Adjustment *A*	Adjustment *B*	Adjustment *C*	Adjustment *D*
a	38°44′06″	38°44′05″	38°44′06″	38°44′04″
b	23 44 38	37	35	37
c	42 19 09	08	06	04
d	44 52 01	00	51 59	52 01
e	69 04 21	20	19	17
f	39 37 48	47	49	51
g	26 25 51	50	52	50
h	75 12 14	13	14	16
	360 00 08	360 00 00	360 00 00	360 00 00

Adjustment B.—The sum of the angles *a*, *b*, *c*, etc., resulting from the station adjustment *A*, is found to differ from 360° by the amount of 08″. This amount divided by the number of angles gives the amount of the correction 01″, to be subtracted from each angle as shown in the second column in the above table.

Adjustment C.—The sum of the angles $b + c$ = 66°03′45″
The sum of the angles $f + g$ = 66°03′37″

Dividing this difference by 4, the correction to each angle is found to be 02″, to be added to the angles *b* and *c*, and subtracted from the angles *f* and *g*. In like manner, the corrections to each of the angles *d*, *e*, *h*, and *a*, is found to be 01″ to be added to *h* and *a*, and subtracted from *d* and *e*. The resultant angles are shown in the third column of the above table.

TRIGONOMETRIC CONDITIONS

Adjustment D.—The sums of the logarithmic sines of the angles (as given by Adjustment C) as indicated in equation (8) are recorded as shown below, and the tabular difference for 1″ is recorded for each angle.

		Difference for 1″	
log sin *a*	9.796479	2.6	
log sin *c*	9.828175	2.3	
log sin *e*	9.970353	0.8	
log sin *g*	9.648478	4.2	
	9.243486		8)45
	41		5.6 = *A*
	45		
log sin *b*	9.604912	4.8	
log sin *d*	9.848470	2.1	
log sin *f*	9.804705	2.6	
log sin *h*	9.985354	0.6	
	9.243441	20.1	8)20.1
			2.5 = *B*

The difference between the two sums is 45 units of the six places of logarithms used. This value, divided by 8, gives the average change required $5.6 = A$. The average tabular difference for 1″ is seen to be $2.5 = B$. Hence $\frac{A}{B} = \frac{5.6}{2.5} = 2''$ (very nearly), which is the correction to be applied to the angles. Obviously, it will be subtracted from angles a, c, e, g, and added to angles b, d, f, and h.

Since this adjustment is applied with opposite sign to alternate angles, it does not disturb the geometric conditions. The final adjustment is given in the fourth column of the above table.

510. Reduction to Center.—At certain triangulation stations it is difficult, if not impossible, to place the instrument vertically beneath the object which has been observed from adjacent stations. At such a place, the instrument is set over any convenient point near the principal station and angles to the adjacent stations are measured with the same precision as other angles in the system. These angles will not be the same as those which would be observed if the instrument were occupying the main station; to obtain the corresponding main-station values, corrections are computed and applied to the measured angles. This procedure of correcting the observed angles is termed *reduction to center.*

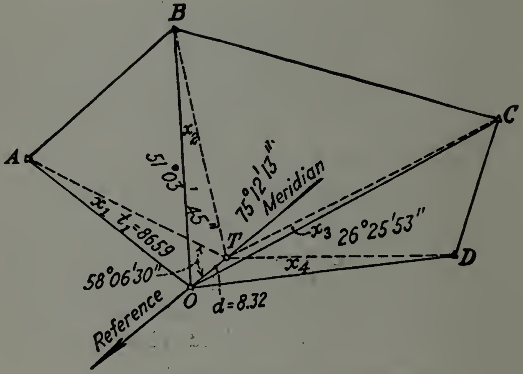


Fig. 510.—Reduction to center.

In addition to the measurement of the angles to adjacent stations, two other quantities are measured: (1) the distance from the main station to the occupied station, and (2) the clockwise angle at the occupied station between the main station and an adjacent station in the system. The situation is illustrated by Fig. 510, where O represents a main station in the system $OABCD$; T represents the point occupied by the instrument; the distance d, and the angle ATO are the quantities measured, in addition to the angles between the adjacent stations of the main system. The lengths of all sides in the main system, as, for example, $t_1 = 8{,}659$ ft., are known approximately from the angles which have been measured at the stations A, B, C, D, and from the known sides AB, BC, etc.

In the triangle AOT, the angle T, and the two sides t_1 and d are known, hence the angle x_1 may be computed. This angle is seen to be the difference in direction at station A between the lines AT and AO. Therefore, if the direction (azimuth) of AT is known with respect to any reference meridian, the direction of AO with respect to the same meridian can now be calculated. Likewise, the directions of the lines BO, CO, and DO can be found, and since these directions are referred to the same meridian, the correct angles at O between these stations can be determined.

The value of x_1 in triangle AOT is given by the equation

$$\sin x_1 = \frac{d \sin T}{t_1} \tag{1}$$

and since the angle x_1 is a small angle for which the sine is nearly equal to the arc, the value of x_1 will be given in seconds of arc if both members of the equation are divided by the sine of 1″, or

$$x_1'' = \frac{d \sin T}{\sin 1'' t_1} \text{ (approximate)} \tag{2}$$

It will be noted that $\dfrac{d}{\sin 1''}$ is a constant for a given station, so that once its value has been determined, the successive correction angles, x_1, x_2, etc., can be computed by a single multiplication. Since the correction angles are usually small, the slide rule will ordinarily render values correct to seconds.

511. Computations of Triangles and Coordinates.—In computing the lengths of the sides and the coordinates of the stations in a triangulation system, it is desirable to follow an orderly procedure to expedite the work and to avoid mistakes. Convenient arrangements for these computations are given below.

Triangles.—A sketch of the figure is drawn and the vertices are lettered A, B, and C in a clockwise direction, beginning with the side whose length is known. The sides opposite the vertices are indicated by the corresponding lower case letters, as a, b, and c. The sine relation states that,

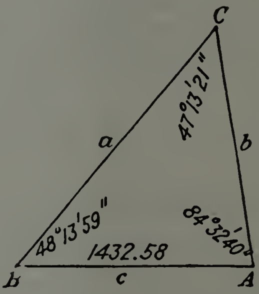
$$b = \frac{\sin B}{\sin C}c, \text{ or } \log b = \log c - \log \sin C + \log \sin B.$$

Accordingly, if the logarithms are recorded in the column of logarithms in the order, log c, colog sin C, log sin A, and log sin B, then log a is found by covering log sin A with a narrow strip of paper and by adding the other three values. Also, to find log b, cover log sin B, and add the remaining three values. Finally, the distances a and b are found as the numbers corresponding to their respective logarithms.

Following is an example of calculations:

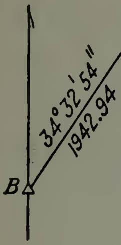
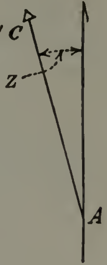
Example 1.—It is desired to calculate lengths of the sides a and b, the known data being given in the figure of the triangle below.

The logarithmic calculations are shown in the following tabulation:

Stations, lines	Angles and distances	Logarithms	Figure
c	1,432.58 ft.	3.156119	
C	47°13′21″	0.134306 (colog)	
A	84°32′40″	9.998028	
B	48°13′59″	9.872657	
a	1,942.94 ft.	3.288453	
b	1,455.74 ft.	3.163082	

Coordinates.—Example 2 shows a form for calculating the coordinates of station C from each of the Stations B and A. For each calculation a sketch is drawn and the known data are recorded, the azimuth of the side being changed to the bearing angle Z. The computation is then carried out as indicated, due regard being paid to the signs of the latitude and longitude values.

Example 2: It is desired to compute the coordinates of C, the coordinates of A and B and the lengths a and b being given. The logarithmic calculations are given below:

From Station B	From Station A
 Z = 34°32′54″ BC = 1,942.94 Latitude = +661.36 Longitude = +1,590.94	Z = 12°40′27″ AC = 1,455.74 Latitude = +841.37 Longitude = −169.71 

Mean Latitude C = +2261.64

(Check)

Latitude C	+2,261.66	Latitude C	+2,261.63
Latitude B	661.36	Latitude A	+ 841.37
$L \cos Z$	1,600.30	$L \cos Z$	+1,420.26
Log $L \cos Z$	3.204200	Log $L \cos Z$	3.152372
Log $\cos Z$	9.915741	Log $\cos Z$	9.989288
Log L	3.288459	Log L	3.163084
Log $\sin Z$	9.753660	Log $\sin Z$	9.341248
Log $L \sin Z$	3.042119	Log $L \sin Z$	2.504332
$L \sin Z$	+1,101.84	$L \sin Z$	−319.40
Longitude B	−1,590.94	Longitude A	−169.71
Longitude C	− 489.10	Longitude C	−489.11

(Check)

Mean Longitude C = −489.11

512. The Three-point Problem.—The purpose of the three-point problem has been stated and a graphical solution has been given in Chap. XXII. In triangulation work the position of an instrument station as O (Fig. 512) is determined by measuring each of the two angles subtended by three visible stations, as A, B, and C, and by solving the triangles involved. Thus, in Fig. 512, all parts of the triangle formed by the stations A, B, and C are known. The angles α and β are measured at the station O. The problem is solved when the values of the angles x and y have been determined; for the remaining parts in each of the triangles ABO and ACO can then be computed. A check is afforded if the same value for the side AO results from each of these triangles.

The problem is indeterminate if the station O lies on or near the great circle passing through the stations A, B, and C. This condition will be evidenced by the condition that $\alpha + \beta + A = 180°$.

There are many solutions for this problem. The one presented here follows that given by the U. S. Coast and Geodetic Survey.

Solution: Given the sides a, b, c, and the angle A, also the observed angles α and β (Fig. 512).
Let

$$S = 180° - \tfrac{1}{2}(A + \alpha + \beta) = \tfrac{1}{2}(x + y)$$

If the stations A and O lie on the same side of the side a, and if the station O is outside the triangle ABC, then,

$$S = \tfrac{1}{2}(A - \alpha - \beta) = \tfrac{1}{2}(x + y)$$

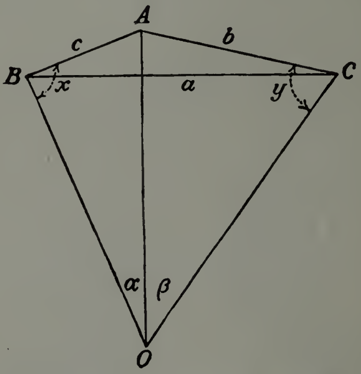


Fig. 512.—Three-point problem.

Let

$$\tan \phi = \frac{c \sin \beta}{b \sin \alpha}, \text{ and let } \Delta = \frac{1}{2}(x - y)$$

then

$$\tan \Delta = \cot (\phi + 45°) \tan S$$

If $\tan \Delta$ is positive, then $x = S + \Delta$, and $y = S - \Delta$.
If $\tan \Delta$ is negative, then $x = S - \Delta$, and $y = S + \Delta$.
This solution is impossible when $\alpha + \beta = A$.

513. Problems.

1. In the triangulation system illustrated in Fig. 499, estimate the values of the angles in each of the two base nets; and by use of Fig. 13, or by an examination of the tabular differences for the sines of the angles, determine which base net is the stronger.

2. For the measurement of a base line the following data are given: The Bureau of Standards certificate states that the tape has a length of 99.946 ft. at 68°F., when supported at the 0- and 100-ft. points and under a tension of 10 lb.; the coefficient of expansion of the tape is

0.00000645 per degree Fahrenheit; the tape weighs 1½ lb. The field records give the measured length as 1,418.314 ft.; the average temperature was 63°.6F.; the stakes 0 to 8 were set on a 1 per cent grade, the stakes 8 to end were set on a 3 per cent grade; the sum of the "set forwards" was 0.234 ft., the sum of the "set backs" was 0.114 ft. The interval and tension used were the same as those used for the standard comparison. Calculate the length of the line.

3. For the conditions given in problem 2, calculate the normal tension.

4. For the conditions given in problem 2, the interval between supports is assumed to be 50 ft. instead of 100 ft. Calculate the corresponding change in the length of the tape.

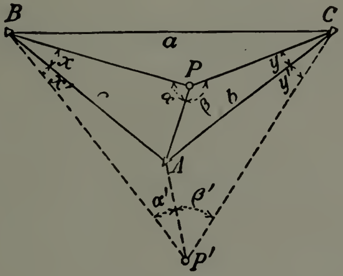


Fig. 513.

5. The angles in a quadrilateral *ABCD* resulting from the station adjustments, are as follows: *CAD* = 45°30′55″, *CAB* = 42°11′39″, *ABD* = 41°54′40″, *DBC* = 62°40′53″, *ACB* = 33°12′51″, *ACD* = 28°-05′30″, *CDB* = 56°00′50″, *BDA* = 50°22′51″. The length of the side *AD* is 2,910.63 ft. (*a*) Calculate the adjustment for geometric conditions. (*b*) Calculate the adjustment for trigonometric conditions.

6. In measuring the angles at a triangulation station *O*, it was necessary to set the transit over another point *T*, at a distance of 13.25 ft. from *O*. The angle measured at *T* from *O* to the first distant station *W* was 95°10′30″. The angles between the distant stations *W*, *X*, *Y*, and *Z* were as follows: *WTX* = 39°37′48″; *XTY* = 69°04′20″; *YTZ* = 83°16′08″. The distances to the stations are found to be: *OA* = 8,949 ft.; *OB* = 14,334 ft.; *OC* = 5,647 ft.; and *OD* = 7,326 ft.

Correct the angles measured at station *T*, to those which would have been measured if the transit had been set at station *O*.

7. In the quadrilateral *ABCD*, of Art. 509, it is assumed that the side *AB* has a length of 13,100.3 ft. Use the finally adjusted angles and calculate the length of the side *CD* by two independent series of computations.

8. For the quadrilateral of problem 7, the azimuth of *AB* is assumed to be 102°35′18″, and the coordinates of station *A* are latitude = +50,000 ft.; longitude = +40,000 ft. Calculate the coordinates of stations *B*, *C*, and *D*.

9. Given the data shown below (see Fig. 513): Assume the instrument at station P, within the triangle ABC, and solve the three-point problem for the angles x and y.

$$
\begin{aligned}
A &= 102°45'20'' \\
a &= 10{,}521.4 \\
b &= 6{,}383.4 \\
c &= 6{,}605.3 \\
\alpha &= 89°15'30'' \\
\beta &= 128°20'10'' \\
\alpha' &= 26°34'50'' \\
\beta' &= 44°15'15''
\end{aligned}
$$

10. Given the data of problem 9. Assume the instrument at station P' (Fig. 513), outside the triangle ABC, and solve the three-point problem for the angles x' and y'.

TABLES

TABLE I.—CORRECTION TO OBSERVED ALTITUDE OF THE SUN FOR REFRACTION AND PARALLAX

App't alt.	Temperature										App't alt.
	−10° C. +14° F.	−5° C. +23° F.	0° C. +32° F.	+5° C. +41° F.	+10° C. +50° F.	+15° C. +59° F.	+20° C. +68° F.	+25° C. +77° F.	+30° C. +86° F.	+35° C. +95° F.	
°	′	′	′	′	′	′	′	′	′	′	°
10	5.52	5.42	5.30	5.20	5.10	5.00	4.92	4.83	4.75	4.67	10
11	5.02	4.92	4.82	4.73	4.63	4.55	4.47	4.38	4.32	4.23	11
12	4.60	4.50	4.42	4.33	4.25	4.17	4.10	4.03	3.97	3.88	12
13	4.23	4.15	4.07	4.00	3.92	3.85	3.78	3.72	3.65	3.58	13
14	3.92	3.83	3.77	3.70	3.62	3.55	3.50	3.45	3.37	3.32	14
15	3.65	3.58	3.50	3.43	3.37	3.32	3.25	3.20	3.13	3.08	15
16	3.43	3.35	3.30	3.23	3.17	3.12	3.07	3.00	2.95	2.90	16
17	3.22	3.15	3.10	3.03	2.98	2.92	2.88	2.82	2.77	2.72	17
18	3.02	2.95	2.90	2.85	2.80	2.75	2.70	2.65	2.60	2.55	18
19	2.83	2.78	2.73	2.68	2.63	2.58	2.53	2.48	2.43	2.40	19
20	2.68	2.63	2.58	2.53	2.48	2.43	2.38	2.33	2.30	2.27	20
21	2.53	2.48	2.43	2.38	2.35	2.30	2.27	2.22	2.17	2.13	21
22	2.38	2.35	2.30	2.25	2.22	2.18	2.13	2.08	2.05	2.02	22
23	2.28	2.25	2.20	2.15	2.12	2.08	2.03	1.98	1.95	1.93	23
24	2.17	2.13	2.08	2.05	2.02	1.98	1.93	1.88	1.87	1.83	24
25	2.07	2.03	1.98	1.95	1.92	1.88	1.83	1.80	1.77	1.75	25
26	1.99	1.95	1.90	1.87	1.83	1.80	1.75	1.72	1.70	1.67	26
27	1.88	1.85	1.82	1.78	1.75	1.72	1.68	1.63	1.62	1.60	27
28	1.80	1.77	1.72	1.70	1.67	1.63	1.60	1.57	1.53	1.52	28
29	1.72	1.68	1.65	1.63	1.60	1.57	1.53	1.50	1.47	1.46	29
30	1.65	1.62	1.58	1.57	1.53	1.50	1.47	1.45	1.42	1.40	30
32	1.53	1.50	1.47	1.45	1.42	1.38	1.35	1.33	1.30	1.28	32
34	1.41	1.37	1.35	1.32	1.30	1.27	1.25	1.23	1.20	1.18	34
36	1.30	1.27	1.25	1.22	1.20	1.18	1.15	1.13	1.10	1.08	36
38	1.20	1.18	1.15	1.13	1.12	1.10	1.07	1.05	1.02	1.02	38
40	1.11	1.10	1.07	1.05	1.03	1.02	0.98	0.97	0.95	0.93	40
42	1.03	1.00	0.98	0.97	0.95	0.93	0.90	0.88	0.87	0.87	42
44	0.96	0.93	0.92	0.90	0.88	0.87	0.85	0.83	0.82	0.80	44
46	0.89	0.88	0.87	0.85	0.83	0.82	0.80	0.78	0.77	0.75	46
48	0.83	0.82	0.80	0.78	0.77	0.75	0.73	0.72	0.70	0.68	48
50	0.77	0.75	0.73	0.72	0.70	0.68	0.67	0.67	0.65	0.63	50
55	0.63	0.62	0.60	0.60	0.58	0.57	0.57	0.55	0.53	0.52	55
60	0.52	0.52	0.50	0.50	0.48	0.47	0.47	0.45	0.45	0.43	60
65	0.42	0.40	0.40	0.40	0.38	0.38	0.37	0.37	0.35	0.33	65
70	0.32	0.32	0.32	0.30	0.30	0.30	0.28	0.28	0.28	0.27	70
75	0.23	0.23	0.23	0.22	0.22	0.22	0.20	0.20	0.20	0.18	75
80	0.15	0.15	0.13	0.13	0.13	0.13	0.13	0.12	0.12	0.12	80
85	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.05	0.05	0.05	85
90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	90

TABLE II.—CORRECTION TO OBSERVED ALTITUDE OF A STAR FOR REFRACTION

App't alt.	Temperature										App't alt.
	−10° C. +14° F.	−5° C. +23° F.	0° C. +32° F.	+5° C. +41° F.	+10° C. +50° F.	+15° C. +59° F.	+20° C. +68° F.	+25° C. +77° F.	+30° C. +86° F.	+35° C. +95° F.	
°	′	′	′	′	′	′	′	′	′	′	°
10	5.67	5.57	5.45	5.35	5.25	5.15	5.07	4.98	4.90	4.82	10
11	5.17	5.07	4.97	4.88	4.78	4.70	4.62	4.53	4.47	4.38	11
12	4.75	4.65	4.57	4.48	4.40	4.32	4.25	4.18	4.12	4.03	12
13	4.38	4.30	4.22	4.15	4.07	4.00	3.93	3.87	3.80	3.73	13
14	4.06	3.97	3.91	3.84	3.76	3.69	3.64	3.59	3.51	3.46	14
15	3.79	3.72	3.64	3.57	3.51	3.46	3.39	3.34	3.27	3.22	15
16	3.57	3.49	3.44	3.37	3.31	3.26	3.21	3.14	3.09	3.04	16
17	3.36	3.29	3.24	3.17	3.12	3.06	3.02	2.96	2.91	2.86	17
18	3.16	3.09	3.04	2.99	2.94	2.89	2.84	2.79	2.74	2.69	18
19	2.97	2.92	2.87	2.82	2.77	2.72	2.67	2.62	2.57	2.54	19
20	2.82	2.77	2.72	2.67	2.62	2.57	2.52	2.47	2.44	2.41	20
21	2.67	2.62	2.57	2.52	2.49	2.44	2.41	2.36	2.31	2.27	21
22	2.52	2.49	2.44	2.39	2.36	2.32	2.27	2.22	2.19	2.16	22
23	2.42	2.39	2.34	2.29	2.26	2.22	2.17	2.12	2.09	2.07	23
24	2.31	2.27	2.22	2.19	2.16	2.12	2.07	2.02	2.01	1.97	24
25	2.21	2.17	2.12	2.09	2.06	2.02	1.97	1.94	1.91	1.89	25
26	2.12	2.08	2.03	2.00	1.96	1.93	1.88	1.85	1.83	1.80	26
27	2.01	1.98	1.95	1.91	1.88	1.85	1.81	1.76	1.75	1.73	27
28	1.93	1.90	1.85	1.83	1.80	1.76	1.73	1.70	1.66	1.65	28
29	1.85	1.81	1.78	1.76	1.73	1.70	1.66	1.63	1.60	1.59	29
30	1.78	1.75	1.71	1.70	1.66	1.63	1.60	1.58	1.55	1.53	30
32	1.65	1.62	1.59	1.57	1.54	1.50	1.47	1.45	1.42	1.40	32
34	1.53	1.49	1.47	1.44	1.42	1.39	1.35	1.35	1.32	1.30	34
36	1.42	1.39	1.37	1.34	1.32	1.30	1.27	1.25	1.22	1.20	36
38	1.32	1.30	1.27	1.25	1.24	1.22	1.19	1.17	1.14	1.14	38
40	1.22	1.21	1.18	1.16	1.14	1.13	1.09	1.08	1.06	1.04	40
42	1.14	1.11	1.09	1.08	1.06	1.04	1.01	0.99	0.98	0.98	42
44	1.07	1.04	1.03	1.01	0.99	0.98	0.96	0.94	0.93	0.91	44
46	0.99	0.98	0.97	0.95	0.93	0.92	0.90	0.88	0.87	0.85	46
48	0.93	0.92	0.90	0.88	0.87	0.85	0.83	0.82	0.80	0.78	48
50	0.86	0.84	0.82	0.81	0.79	0.77	0.76	0.76	0.74	0.72	50
55	0.72	0.71	0.69	0.69	0.67	0.66	0.66	0.64	0.62	0.61	55
60	0.59	0.59	0.57	0.57	0.55	0.54	0.54	0.52	0.52	0.50	60
65	0.48	0.46	0.46	0.46	0.44	0.44	0.43	0.43	0.41	0.39	65
70	0.37	0.37	0.37	0.35	0.35	0.35	0.33	0.33	0.33	0.32	70
75	0.27	0.27	0.27	0.26	0.26	0.26	0.24	0.24	0.24	0.22	75
80	0.18	0.18	0.16	0.16	0.16	0.16	0.16	0.15	0.15	0.15	80
85	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.06	0.06	0.06	85
90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	90

TABLE III.—REFRACTION CORRECTIONS TO BE APPLIED TO APPARENT DECLINATIONS

(*To be used with solar attachment*)

January	Hour angle	Refraction correction lat. 40°	February	Hour angle	Refraction correction lat. 40°	March	Hour angle	Refraction correction lat. 40°	April	Hour angle	Refraction correction lat. 40°	May	Hour angle	Refraction correction lat. 40°	June	Hour angle	Refraction correction lat. 40°
		′ ″		See Jan. 30				′ ″			′ ″			′ ″			′ ″
1	1	1 58	1		′ ″	1	1	1 03	1	3	0 57	1	3	0 39	1	4	0 44
2	2	2 16	2	3	2 13	2	2	1 10	2	4	1 19	2	4	0 55	2	5	1 11
3	3	3 04	3	4	3 41	3	3	1 27	3	5	2 18	3	5	1 30	3		
4	4	6 23	4			4	4	2 06	4			4			4	1	0 19
5			5	1	1 26	5			5	1	0 39	5	1	0 26	5	2	0 23
	1	1 54		2	1 37		1	0 59		2	0 44		2	0 30		3	0 30
6	2	2 11	6	3	2 04	6	2	1 06	6	3	0 54	6	3	0 37	6	4	0 43
7	3	2 59	7	4	3 21	7	3	1 21	7	4	1 14	7	4	0 53	7	5	1 10
8	4	6 01	8			8	4	1 56	8	5	2 08	8	5	1 26	8		
9			9	1	1 21	9	5	4 04	9			9			9	1	0 18
10			10	2	1 31	10			10	1	0 36	10	1	0 25	10	2	0 22
	1	1 50		3	1 56		1	0 55		2	0 41		2	0 29		3	0 29
11	2	2 06	11	4	3 04	11	2	1 02	11	3	0 51	11	3	0 36	11	4	0 43
12	3	2 49	12			12	3	1 15	12	4	1 10	12	4	0 51	12	5	1 09
13	4	5 33	13			13	4	1 47	13	5	1 58	13	5	1 22	13		
14			14	1	1 16	14	5	3 34	14			14			14	1	0 18
15			15	2	1 25	15			15	1	0 34	15	1	0 23	15	2	0 22
	1	1 46		3	1 48		1	0 52		2	0 38		2	0 27		3	0 29
16	2	2 01	16	4	2 47	16	2	0 58	16	3	0 48	16	3	0 34	16	4	0 42
17	3	2 40	17	5	8 39	17	3	1 10	17	4	1 06	17	4	0 49	17	5	1 08
18	4	5 00	18			18	4	1 39	18	5	1 49	18	5	1 18	18		
19			19	1	1 12	19	5	3 08	19			19			19	1	0 18
20			20	2	1 20	20			20	1	0 32	20	1	0 22	20	2	0 22
	1	1 42		3	1 40		1	0 48		2	0 36		2	0 26		3	0 29
21	2	1 56	21	4	2 31	21	2	0 54	21	3	0 45	21	3	0 33	21	4	0 42
22	3	2 31	22	5	6 49	22	3	1 05	22	4	1 02	22	4	0 47	22	5	1 08
23	4	4 35	23			23	4	1 32	23	5	1 42	23	5	1 15	23		
24			24	1	1 07	24	5	2 51	24			24			24	1	0 18
25	1	1 37	25	2	1 15	25			25	1	0 30	25	1	0 21	25	2	0 22
	2	1 50		3	1 33		1	0 45		2	0 34		2	0 25		3	0 29
26	3	2 22	26	4	2 18	26	2	0 50	26	3	0 42	26	3	0 32	26	4	0 42
27	4	4 07	27	5	5 29	27	3	1 01	27	4	0 58	27	4	0 46	27	5	1 08
28			28			28	4	1 25	28	5	1 36	28	5	1 13	28		
29			29	See Mar.		29	5	2 34	29			29			29	1	0 18
30	1	1 32			1	30			30	1	0 28	30	1	0 20	30	2	0 22
31	2	1 44				31	1	0 42		2	0 32	31	2	0 24		3	0 29
							2	0 47					3	0 31			

For latitudes other than 40° multiply by latitude coefficient, Table III(a).

TABLE III (*Cont'd*).—REFRACTION CORRECTIONS TO BE APPLIED TO APPARENT DECLINATIONS

(*To be used with solar attachment*)

July	Hour angle	Refraction correction lat. 40°	August	Hour angle	Refraction correction lat. 40°	September	Hour angle	Refraction correction lat. 40°	October	Hour angle	Refraction correction lat. 40°	November	Hour angle	Refraction correction lat. 40°	December	Hour angle	Refraction correction lat. 40°
		′ ″			′ ″			′ ″			′ ″			′ ″			′ ″
1	4	0 43	1	5	1 22	1	1	0 39	1	1	0 56	1			1	1	1 54
2	5	1 09	2			2	2	0 44	2	2	1 06	2	1	1 26	2	2	2 11
3			3	1	0 26	3	3	0 54	3	3	1 21	3	2	1 37	3	3	2 59
4	1	0 19	4	2	0 30	4	4	1 14	4	4	1 56	4	3	2 04	4	4	6 01
5	2	0 23	5	3	0 37	5	5	2 08	5	5	4 04	5	4	3 21	5		
	3	0 30		4	0 53												
6	4	0 43	6	5	1 26	6	1	0 42	6	1	1 03	6	1	1 32	6	1	1 58
7	5	1 10	7			7	2	0 47	7	2	1 10	7	2	1 44	7	2	2 16
8			8	1	0 28	8	3	0 57	8	3	1 27	8	3	2 13	8	3	3 04
9	1	0 20	9	2	0 32	9	4	1 19	9	4	2 06	9	4	3 41	9	4	6 23
10	2	0 24	10	3	0 39	10	5	2 18	10	5	4 39	10			10		
	3	0 31		4	0 55												
11	4	0 44	11	5	1 30	11	1	0 45	11	1	1 07	11	1	1 37	11	1	2 00
12	5	1 11	12			12	2	0 50	12	2	1 15	12	2	1 50	12	2	2 19
13			13	1	0 30	13	3	1 01	13	3	1 33	13	3	2 22	13	3	3 09
14	1	0 21	14	2	0 34	14	4	1 25	14	4	2 18	14	4	4 07	14	4	6 43
15	2	0 25	15	3	0 42	15	5	2 34	15	5	5 29	15			15		
	3	0 32		4	0 58												
16	4	0 46	16	5	1 36	16	1	0 48	16	1	1 12	16	1	1 42	16	1	2 01
17	5	1 13	17			17	2	0 54	17	2	1 20	17	2	1 56	17	2	2 20
18			18	1	0 32	18	3	1 05	18	3	1 40	18	3	2 31	18	3	3 11
19	1	0 22	19	2	0 36	19	4	1 32	19	4	2 31	19	4	4 35	19	4	6 47
20	2	0 26	20	3	0 45	20	5	2 51	20	5	6 49	20			20		
	3	0 33		4	1 02												
21	4	0 47	21	5	1 42	21	1	0 52	21	1	1 16	21	1	1 46	21	1	2 01
22	5	1 15	22			22	2	0 58	22	2	1 25	22	2	2 01	22	2	2 20
23			23	1	0 34	23	3	1 10	23	3	1 48	23	3	2 40	23	3	3 11
24	1	0 23	24	2	0 38	24	4	1 39	24	4	2 47	24	4	4 59	24	4	6 49
25	2	0 27	25	3	0 48	25	5	3 08	25	5	8 39	25			25		
	3	0 34		4	1 06												
26	4	0 49	26	5	1 49	26	1	0 55	26	1	1 21	26	1	1 50	26	1	2 00
27	5	1 18	27			27	2	1 02	27	2	1 31	27	2	2 06	27	2	2 19
28			28	1	0 36	28	3	1 15	28	3	1 56	28	3	2 49	28	3	3 09
29	1	0 25	29	2	0 41	29	4	1 47	29	4	3 04	29	4	5 33	29	4	6 43
30	2	0 29	30	3	0 51	30	5	3 34	30	5	11 01	30			30		
31	3	0 36	31	4	1 10				31	See Nov. 1					31	See Jan. 1	
	4	0 51		5	1 58												

For latitudes other than 40° multiply by latitude coefficient, Table III(a).

TABLE III(a).—LATITUDE COEFFICIENTS

Latitude	Coefficient	Latitude	Coefficient	Latitude	Coefficient
15°	0.30	30°	0.65	45°	1.20
16	0.32	31	0.68	46	1.24
17	0.34	32	0.71	47	1.29
18	0.36	33	0.75	48	1.33
19	0.38	34	0.78	49	1.38
20	0.40	35	0.82	50	1.42
21	0.42	36	0.85	51	1.47
22	0.44	37	0.89	52	1.53
23	0.46	38	0.92	53	1.58
24	0.48	39	0.96	54	1.64
25	0.50	40	1.00	55	1.70
26	0.53	41	1.04	56	1.76
27	0.56	42	1.08	57	1.82
28	0.59	43	1.12	58	1.88
29	0.62	44	1.16	59	1.94

To obtain the refraction correction (to be applied to declination) for any other latitude than 40° multiply the refraction correction for latitude 40° (Table III) by the coefficient corresponding to the latitude of observation.

Table IV.—Local Civil Time of Upper Culmination of Polaris in the Year 1926

(*Computed for* 90°, *or* 6 *hours west of Greenwich*)

Date, 1926	Civil time of upper culmination			Variation per day		Date, 1926	Civil time of upper culmination			Variation per day	
	h	m	s	m	s		h	m	s	m	s
Jan. 1	18	51	29	−3	57	July 10	6	24	11	−3	55
Jan. 11	18	11	59	3	57	July 20	5	45	03	3	55
Jan. 21	17	32	28	3	57	July 30	5	05	55	3	55
Jan. 31	16	52	58	3	57	Aug. 9	4	26	46	3	55
Feb. 10	16	13	28	3	57	Aug. 19	3	47	37	3	55
Feb. 20	15	33	58	3	57	Aug. 29	3	08	27	3	55
Mar. 2	14	54	30	3	57	Sept. 8	2	29	16	3	55
Mar. 12	14	15	04	3	57	Sept. 18	1	50	04	3	55
Mar. 22	13	35	40	3	56	Sept. 28	1	10	50	3	55
Apr. 1	12	56	17	3	56	Oct. 8	0	31	35	3	56
Apr. 11	12	16	57	3	56	Oct. 17	23	52	18	3	56
Apr. 21	11	37	39	3	56	Oct. 27	23	12	59	3	56
May 1	10	58	23	3	55	Nov. 6	22	33	39	3	56
May 11	10	19	09	3	55	Nov. 16	21	54	17	3	56
May 21	9	39	56	3	55	Nov. 26	21	14	52	3	57
May 31	9	00	45	3	55	Dec. 6	20	35	26	3	57
June 10	8	21	35	3	55	Dec. 16	19	55	58	3	57
June 20	7	42	26	3	55	Dec. 26	19	16	30	3	57
June 30	7	03	19	3	55	Jan. 5, 1927	18	37	00	3	57

Table IV(a).—Mean Time Interval between Upper Culmination and Elongation

Latitude	Time interval		Latitude	Time interval		Latitude	Time interval		Latitude	Time interval	
°	h	m	°	h	m	°	h	m	°	h	m
10	5	58.2	35	5	56.0	48	5	54.2	58	5	52.1
15	5	57.8	40	5	55.4	50	5	53.8	60	5	51.5
20	5	57.4	42	5	55.1	52	5	53.4	62	5	50.8
25	5	57.0	44	5	54.8	54	5	53.0	64	5	50.1
30	5	56.5	46	5	54.5	56	5	52.6			

Eastern elongation precedes and western elongation follows upper culmination by the time interval given in Table IV(a). Lower culmination precedes or follows upper culmination by $11^h\ 58.0^m$. It should be noted that there are two upper culminations on one day in October (Oct. 16 in 1926) and two lower culminations in April (April 15 in 1926). There are also two western elongations on one day in January and two eastern elongations on one day in July.

A. *To refer the times in Table IV to other years:*

FOR YEAR		m	
1927	add	1.3	
1928	add	2.6	up to Mar. 1
1928	subtract	1.3	on and after Mar. 1
1929	add	0.1	
1930	add	1.6	
1931	add	3.0	
1932	add	4.4	up to Mar. 1
1932	add	0.5	on and after Mar. 1
1933	add	2.1	
1934	add	3.7	
1935	add	5.3	

B. *To refer to other than the tabular days:* SUBTRACT from the time for the preceding tabular day the product of the variation per day and the days elapsed, as given below:

Days elapsed	Variation per day						Days elapsed	Variation per day					
	3^m	57^s	3	56^s	3^m	55^s		3^m	57^s	3^m	56^s	3^m	55^s
	m	s	m	s	m	s		m	s	m	s	m	s
1	3	57	3	56	3	55	6	23	42	23	36	23	30
2	7	54	7	52	7	50	7	27	39	27	32	27	25
3	11	51	11	48	11	45	8	31	36	31	28	31	20
4	15	48	15	44	15	40	9	35	33	35	24	35	15
5	19	45	19	40	19	35							

C. *To refer to any other than the tabular longitude* (90°)*:* ADD 0.1^m for each 10° east of the ninetieth meridian or SUBTRACT 0.1^m for each 10° west of the ninetieth meridian.

D. *To refer to standard time:* ADD to the quantities in Table IV four minutes for every degree of longitude the place of observation is west of the standard meridian (60°, 75°, 90°, etc.). SUBTRACT when the place is east of the standard meridian.

TABLE V.—AZIMUTH OF POLARIS AT ELONGATION, 1928 TO 1935

Latitude	1928	1929	1930	1931	1932	1933	1934	1935
	° '	° '	° '	° '	° '	° '	° '	° '
10°	1 05.9	1 05.6	1 05.3	1 05.0	1 04.7	1 04.4	1 04.1	1 03.8
11	1 06.1	1 05.8	1 05.5	1 05.2	1 04.9	1 04.6	1 04.3	1 04.0
12	1 06.4	1 06.0	1 05.7	1 05.4	1 05.1	1 04.8	1 04.5	1 04.2
13	1 06.6	1 06.3	1 06.0	1 05.7	1 05.3	1 05.0	1 04.7	1 04.4
14	1 06.9	1 06.6	1 06.3	1 05.9	1 05.6	1 05.3	1 05.0	1 04.7
15	1 07.2	1 06.9	1 06.6	1 06.2	1 05.9	1 05.6	1 05.3	1 05.0
16	1 07.5	1 07.2	1 06.9	1 06.6	1 06.2	1 05.9	1 05.6	1 05.3
17	1 07.9	1 07.5	1 07.2	1 06.9	1 06.6	1 06.3	1 06.0	1 05.7
18	1 08.2	1 07.9	1 07.6	1 07.3	1 07.0	1 06.6	1 06.3	1 06.0
19	1 08.6	1 08.3	1 08.0	1 07.7	1 07.3	1 07.0	1 06.7	1 06.4
20	1 09.1	1 08.7	1 08.4	1 08.1	1 07.8	1 07.4	1 07.1	1 06.8
21	1 09.5	1 09.2	1 08.9	1 08.5	1 08.2	1 07.9	1 07.6	1 07.2
22	1 10.0	1 09.7	1 09.3	1 09.0	1 08.7	1 08.4	1 08.0	1 07.7
23	1 10.5	1 10.2	1 09.8	1 09.5	1 09.2	1 08.9	1 08.5	1 08.2
24	1 11.0	1 10.7	1 10.4	1 10.0	1 09.7	1 09.4	1 09.0	1 08.7
25	1 11.6	1 11.3	1 10.9	1 10.6	1 10.3	1 09.9	1 09.6	1 09.3
26	1 12.2	1 11.9	1 11.5	1 11.2	1 10.9	1 10.5	1 10.2	1 09.9
27	1 12.8	1 12.5	1 12.2	1 11.8	1 11.5	1 11.1	1 10.8	1 10.5
28	1 13.5	1 13.2	1 12.8	1 12.5	1 12.1	1 11.8	1 11.4	1 11.1
29	1 14.2	1 13.9	1 13.5	1 13.2	1 12.8	1 12.5	1 12.1	1 11.8
30	1 14.9	1 14.6	1 14.2	1 13.9	1 13.5	1 13.2	1 12.8	1 12.5
31	1 15.7	1 15.4	1 15.0	1 14.6	1 14.3	1 13.9	1 13.6	1 13.2
32	1 16.5	1 16.2	1 15.8	1 15.4	1 15.1	1 14.7	1 14.4	1 14.0
33	1 17.4	1 17.0	1 16.7	1 16.3	1 15.9	1 15.6	1 15.2	1 14.9
34	1 18.3	1 17.9	1 17.5	1 17.2	1 16.8	1 16.4	1 16.1	1 15.7
35	1 19.2	1 18.9	1 18.5	1 18.1	1 17.7	1 17.4	1 17.0	1 16.6
36	1 20.2	1 19.8	1 19.5	1 19.1	1 18.7	1 18.3	1 18.0	1 17.6
37	1 21.3	1 20.9	1 20.5	1 20.1	1 19.7	1 19.4	1 19.0	1 18.6
38	1 22.4	1 22.0	1 21.6	1 21.2	1 20.8	1 20.4	1 20.0	1 19.7
39	1 23.5	1 23.1	1 22.7	1 22.3	1 21.9	1 21.6	1 21.2	1 20.8
40	1 24.7	1 24.3	1 23.9	1 23.5	1 23.1	1 22.7	1 22.3	1 22.0
41	1 26.0	1 25.6	1 25.2	1 24.8	1 24.4	1 24.0	1 23.6	1 23.2
42	1 27.3	1 26.9	1 26.5	1 26.1	1 25.7	1 25.3	1 24.9	1 24.5
43	1 28.7	1 28.3	1 27.9	1 27.5	1 27.1	1 26.7	1 26.3	1 25.8
44	1 30.2	1 29.8	1 29.4	1 29.0	1 28.5	1 28.1	1 27.7	1 27.3
45	1 31.8	1 31.4	1 30.9	1 30.5	1 30.1	1 29.6	1 29.2	1 28.8
46	1 33.4	1 33.0	1 32.5	1 32.1	1 31.7	1 31.2	1 30.8	1 30.4
47	1 35.2	1 34.7	1 34.3	1 33.8	1 33.4	1 32.9	1 32.5	1 32.0
48	1 37.0	1 36.5	1 36.1	1 35.6	1 35.2	1 34.7	1 34.3	1 33.8
49	1 38.9	1 38.5	1 38.0	1 37.5	1 37.1	1 36.6	1 36.1	1 35.7
50	1 41.0	1 40.5	1 40.0	1 39.5	1 39.1	1 38.6	1 38.1	1 37.7

Table V was computed using the mean declination of Polaris for each year. A more accurate result will be obtained by applying to the tabular values the following corrections, which depend on the difference between the mean and apparent place of the star:

Month	Correction	Month	Correction	Month	Correction
	′		′		′
January.........	−0.3	May..........	+0.2	September.....	0.0
February........	−0.3	June..........	+0.3	October........	−0.2
March..........	−0.2	July...........	+0.3	November.....	−0.5
April...........	0.0	August........	+0.2	December......	−0.7

TABLE V(a). FOR REDUCING TO ELONGATION OBSERVATIONS MADE NEAR ELONGATION

Azimuth at elongation * Time	1° 0′	1° 10′	1° 20′	1° 30′	1° 40′	1° 50′	2° 0′	2° 10′	Azimuth at elongation * Time
m	″	″	″	″	″	″	″	″	m
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0
1	0.0	0.0	0.0	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	1
2	+ 0.1	+ 0.2	+ 0.2	0.2	0.2	0.3	0.3	0.3	2
3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	0.7	3
4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	4
5	+ 0.9	+ 1.0	+ 1.1	+ 1.3	+ 1.4	+ 1.6	+ 1.7	+ 1.9	5
6	1.2	1.4	1.6	1.8	2.1	2.3	2.5	2.7	6
7	1.7	2.0	2.2	2.5	2.8	3.1	3.4	3.7	7
8	2.2	2.6	2.9	3.3	3.7	4.0	4.4	4.8	8
9	2.8	3.2	3.7	4.2	4.6	5.1	5.6	6.0	9
10	+ 3.4	+ 4.0	+ 4.6	+ 5.1	+ 5.7	+ 6.3	+ 6.9	+ 7.4	10
11	4.1	4.8	5.5	6.2	6.9	7.6	8.3	9.0	11
12	4.9	5.8	6.6	7.4	8.2	9.0	9.9	10.7	12
13	5.8	6.8	7.7	8.7	9.7	10.6	11.6	12.6	13
14	6.7	7.8	9.0	10.1	11.2	12.3	13.4	14.6	14
15	+ 7.7	+ 9.0	+10.3	+11.6	+12.8	+14.1	+15.4	+16.7	15
16	8.8	10.2	11.7	13.2	14.6	16.1	17.5	19.0	16
17	9.9	11.5	13.2	14.9	16.5	18.2	19.8	21.5	17
18	11.1	12.9	14.8	16.7	18.5	20.4	22.2	24.1	18
19	12.4	14.4	16.5	18.6	20.6	22.7	24.7	26.8	19
20	+13.7	+16.0	+18.3	+20.6	+22.8	+25.1	+27.4	+29.7	20
21	15.1	17.6	20.1	22.7	25.2	27.7	30.2	32.7	21
22	16.6	19.3	22.1	24.9	27.6	30.4	33.2	35.9	22
23	18.1	21.1	24.2	27.2	30.2	33.2	36.2	39.3	23
24	19.7	23.0	26.3	29.6	32.9	36.2	39.5	42.8	24
25	+21.4	+25.0	+28.5	+32.1	+35.7	+39.2	+42.8	+46.4	25

Azimuth at elongation * Time	2° 10′	2° 20′	2° 30′	2° 40′	2° 50′	3° 0′	3° 10′	3° 20′	Azimuth at elongation * Time
m	″	″	″	″	″	″	″	″	m
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0
1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	1
2	0.3	0.3	0.4	0.4	0.4	0.4	0.4	0.5	2
3	0.7	0.7	0.8	0.8	0.9	0.9	1.0	1.0	3
4	1.2	1.3	1.4	1.5	1.6	1.6	1.7	1.8	4
5	+ 1.9	+ 2.0	+ 2.1	+ 2.3	+ 2.4	+ 2.6	+ 2.7	+ 2.9	5
6	2.7	2.9	3.1	3.3	3.5	3.7	3.9	4.1	6
7	3.7	3.9	4.2	4.5	4.8	5.0	5.3	5.6	7
8	4.8	5.1	5.5	5.9	6.2	6.6	7.0	7.3	8
9	6.0	6.5	7.0	7.4	7.9	8.3	8.8	9.3	9
10	+ 7.4	+ 8.0	+ 8.6	+ 9.2	+ 9.7	+10.3	+10.9	+11.4	10
11	9.0	9.7	10.4	11.1	11.8	12.4	13.1	13.8	11
12	10.7	11.5	12.3	13.2	14.0	14.8	15.6	16.5	12
13	12.6	13.5	14.5	15.4	16.4	17.4	18.4	19.3	13
14	14.6	15.7	16.8	17.9	19.0	20.2	21.3	22.4	14
15	+16.7	+18.0	+19.3	+20.6	+21.9	+23.1	+24.4	+25.7	15
16	19.0	20.5	21.9	23.4	24.9	26.3	27.8	29.3	16
17	21.5	23.1	24.8	26.4	28.1	29.7	31.4	33.0	17
18	24.1	25.9	27.8	29.6	31.5	33.3	35.2	37.0	18
19	26.8	28.9	30.9	33.0	35.1	37.1	39.2	41.3	19
20	+29.7	+32.0	+34.3	+36.6	+38.8	+41.1	+43.4	+45.7	20
21	32.7	35.3	37.8	40.3	42.8	45.3	47.9	50.4	21
22	35.9	38.7	41.5	44.2	47.0	49.8	52.5	55.3	22
23	39.3	42.3	45.3	48.3	51.4	54.4	57.4	60.4	23
24	42.8	46.0	49.3	52.6	55.9	59.2	62.5	65.8	24
25	+46.4	+49.9	+53.5	+57.1	+60.7	+64.2	+67.8	+71.4	25

* Sidereal time from elongation.

TABLE VI.—TOTAL SOLAR-DIURNAL VARIATION OF THE MAGNETIC DECLINATION, ON THE YEARLY AVERAGE, AT PROMINENT PLACES IN NORTH AMERICA

[*A + sign indicates a deflection of the north-seeking end of the magnet toward the east, a — sign the contrary direction*]

Local mean time	1 Key West, Fla.	2 Los Angeles, Cal.	3 Washington, D. C.	4 Philadelphia, Pa.	5 Madison, Wis.	6 Toronto, Canada	7 Sitka, Alaska	8 Uglaamie, Point Barrow	9 Plover Point, Point Barrow	10 Fort Rae, Great Slave Lake	11 Kingua Fjord, Cumberland Sound	12 Fort Conger, Grinnell Land	Average values, stations 1 to 6, inclusive
	′	′	′	′	′	′	′	′	′	′	′	′	′
1 a.m. ..	+0.0	+0.0	+0.7	+0.6	+0.1	+0.6	+0.2	−12.8	− 8.0	−11.0	+11.7	+43.2	+0.35
2 a.m. ..	−0.0	+0.1	+0.7	+0.5	0.0	+0.5	+1.0	− 4.9	− 1.9	− 6.6	+15.8	+43.1	+0.05
3 a.m. ..	+0.1	+0.2	+0.9	+0.6	+0.2	+0.8	+1.4	+ 3.3	+ 3.6	+ 0.8	+18.0	+41.2	+0.07
4 a.m. ..	+0.2	+0.3	+1.2	+1.0	+0.5	+1.2	+2.0	+ 6.2	+10.9	+ 7.4	+19.1	+25.7	+0.75
5 a.m. ..	+0.4	+0.6	+1.7	+1.5	+1.0	+1.8	+2.9	+14.3	+16.6	+13.6	+19.3	+31.6	+1.19
6 a.m. ..	+1.0	+1.3	+2.1	+2.1	+1.4	+2.7	+4.2	+21.6	+19.3	+21.0	+20.1	+19.7	+1.79
7 a.m. ..	+2.1	+2.4	+2.8	+3.3	+2.6	+3.5	+5.3	+26.1	+27.1	+26.2	+19.9	+26.6	+2.80
8 a.m. ..	+2.5	+3.1	+3.2	+3.5	+3.2	+3.8	+6.0	+26.7	+27.0	+29.4	+17.4	+18.7	+3.24
9 a.m. ..	+2.2	+2.6	+2.3	+2.8	+3.0	+3.0	+5.3	+26.1	+19.9	+25.5	+10.8	+ 1.2	+2.67
10 a.m. ..	+1.1	+1.1	+0.9	+0.8	+1.7	−0.8	+3.0	+ 9.9	+ 9.3	+16.8	+ 3.7	−12.7	+1.09
11 a.m. ..	−0.2	−0.8	−1.3	−1.6	−0.7	−2.0	+0.6	+ 1.4	− 0.4	+ 8.0	+ 1.3	−21.4	−1.08
Noon.....	−1.4	−2.2	−3.2	−3.4	−2.5	−4.2	−2.1	− 5.9	− 8.2	− 0.9	− 9.0	−40.7	−2.80
1 p.m. ..	−2.1	−2.7	−4.3	−4.3	−3.5	−5.0	−3.2	− 7.3	−10.7	− 4.0	−15.1	−45.6	−3.63
2 p.m. ..	−2.2	−2.6	−4.3	−4.1	−3.5	−4.8	−4.2	− 7.7	− 9.8	− 8.1	−21.2	−49.2	−3.56
3 p.m. ..	−1.9	−2.0	−3.5	−3.1	−2.6	−3.8	−4.6	− 7.3	− 9.9	−10.6	−20.4	−45.8	−2.80
4 p.m. ..	−1.3	−1.1	−2.5	−2.2	−1.6	−2.5	−4.6	− 9.1	− 9.8	−11.3	−20.6	−53.7	−1.85
5 p.m. ..	−0.8	−0.5	−1.5	−1.0	−0.7	−1.3	−3.8	− 9.9	−10.2	−12.1	−23.6	−23.7	−0.95
6 p.m. ..	−0.4	−0.2	−0.8	−0.4	−0.2	−0.3	−3.2	− 9.9	− 9.7	−12.9	−19.4	−17.3	−0.36
7 p.m. ..	−0.2	−0.0	0.0	+0.0	+0.2	+0.2	−2.4	− 8.4	− 8.4	−12.5	−16.1	−27.2	+0.05
8 p.m. ..	+0.1	+0.1	+0.6	+0.8	+0.2	+0.7	−1.4	− 6.0	− 9.0	−11.0	−15.5	− 3.5	+0.44
9 p.m. ..	+0.2	+0.1	+1.0	+0.6	+0.6	+1.2	−0.8	− 8.1	− 7.5	−12.0	− 8.8	+ 3.5	+0.64
10 p.m. ..	+0.2	+0.1	+1.1	+1.2	+0.7	+1.3	−0.4	−10.9	− 7.9	−11.9	− 0.6	+22.4	+0.79
11 p.m. ..	+0.2	+0.1	+1.1	+0.7	+0.2	+1.2	−0.6	− 9.1	−11.5	−11.9	+ 3.9	−30.0	+0.60
Midnight.	+0.1	+0.0	+1.0	+0.6	+0.1	+0.8	−0.6	−13.4	−10.8	−12.0	+ 9.2	+32.6	+0.45
Range..	4.7	5.8	7.5	7.8	6.7	8.8	10.6	40.1	38.6	41.4	43.7	98.8	6.9

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 10°	Latitude 11°	Latitude 12°	Latitude 13°	Latitude 14°	Latitude 15°	Latitude 16°	Latitude 17°	Latitude 18°	Latitude 19°	Latitude 20°	Latitude 10°	Latitude 20°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0 15	0 04 36	0 04 37	0 04 38	0 04 39	0 04 41	0 04 42	0 04 43	0 04 45	0 04 47	0 04 48	0 04 50	− 4	− 4
0 30	0 09 11	0 09 13	0 09 15	0 09 17	0 09 20	0 09 23	0 09 25	0 09 29	0 09 32	0 09 36	0 09 39	− 8	− 8
0 45	0 13 43	0 13 46	0 13 49	0 13 53	0 13 57	0 14 01	0 14 05	0 14 10	0 14 15	0 14 20	0 14 26	−12	−13
1 00	0 18 12	0 18 16	0 18 20	0 18 25	0 18 30	0 18 35	0 18 41	0 18 47	0 18 54	0 19 01	0 19 09	−16	−17
1 15	0 22 36	0 22 41	0 22 46	0 22 52	0 22 58	0 23 05	0 23 12	0 23 20	0 23 28	0 23 37	0 23 46	−20	−21
1 30	0 26 54	0 27 00	0 27 06	0 27 13	0 27 21	0 27 29	0 27 37	0 27 46	0 27 56	0 28 07	0 28 18	−24	−25
1 45	0 31 05	0 31 12	0 31 19	0 31 27	0 31 36	0 31 45	0 31 55	0 32 06	0 32 17	0 32 29	0 32 42	−27	−29
2 00	0 35 09	0 35 16	0 35 24	0 35 33	0 35 43	0 35 53	0 36 04	0 36 16	0 36 29	0 36 43	0 36 57	−31	−32
2 15	0 39 03	0 39 11	0 39 20	0 39 30	0 39 41	0 39 52	0 40 04	0 40 18	0 40 32	0 40 47	0 41 03	−34	−36
2 30	0 42 47	0 42 56	0 43 06	0 43 17	0 43 28	0 43 41	0 43 54	0 44 09	0 44 24	0 44 40	0 44 58	−37	−39
2 45	0 46 19	0 46 29	0 46 40	0 46 52	0 47 04	0 47 18	0 47 32	0 47 48	0 48 04	0 48 22	0 48 41	−40	−42
3 00	0 49 40	0 49 51	0 50 02	0 50 15	0 50 28	0 50 42	0 50 58	0 51 15	0 51 33	0 51 52	0 52 12	−43	−45
3 15	0 52 48	0 52 59	0 53 11	0 53 24	0 53 39	0 53 54	0 54 11	0 54 28	0 54 47	0 55 07	0 55 29	−46	−48
3 30	0 55 43	0 55 54	0 56 07	0 56 21	0 56 36	0 56 52	0 57 09	0 57 28	0 57 47	0 58 09	0 58 31	−49	−51
3 45	0 58 22	0 58 34	0 58 48	0 59 02	0 59 18	0 59 35	0 59 53	1 00 12	1 00 33	1 00 55	1 01 18	−51	−53
4 00	1 00 47	1 01 00	1 01 13	1 01 28	1 01 44	1 02 02	1 02 21	1 02 41	1 03 02	1 03 25	1 03 49	−53	−55
4 15	1 02 56	1 03 09	1 03 23	1 03 38	1 03 55	1 04 13	1 04 33	1 04 53	1 05 15	1 05 39	1 06 04	−55	−57
4 30	1 04 49	1 05 02	1 05 17	1 05 33	1 05 50	1 06 08	1 06 28	1 06 49	1 07 12	1 07 36	1 08 02	−56	−59
4 45	1 06 25	1 06 39	1 06 53	1 07 10	1 07 27	1 07 46	1 08 06	1 08 28	1 08 51	1 09 15	1 09 42	−58	−61
5 00	1 07 44	1 07 58	1 08 13	1 08 29	1 08 47	1 09 06	1 09 26	1 09 48	1 10 12	1 10 37	1 11 03	−59	−62
5 15	1 08 46	1 08 59	1 09 15	1 09 31	1 09 49	1 10 08	1 10 29	1 10 51	1 11 15	1 11 40	1 12 07	−60	−62
5 30	1 09 30	1 09 43	1 09 59	1 10 15	1 10 33	1 10 52	1 11 13	1 11 35	1 11 59	1 12 25	1 12 52	−60	−63
5 45	1 09 56	1 10 09	1 10 25	1 10 41	1 10 59	1 11 18	1 11 39	1 12 01	1 12 25	1 12 51	1 13 18	−61	−63
6 00	1 10 04	1 10 17	1 10 32	1 10 49	1 11 07	1 11 26	1 11 47	1 12 09	1 12 33	1 12 58	1 13 26	−61	−64
6 15	1 09 54	1 10 07	1 10 22	1 10 39	1 10 55	1 11 15	1 11 36	1 11 58	1 12 22	1 12 47	1 13 14	−61	−63
6 30	1 09 26	1 09 39	1 09 54	1 10 10	1 10 28	1 10 46	1 11 07	1 11 29	1 11 52	1 12 17	1 12 44	−60	−63
6 45	1 08 40	1 08 53	1 09 08	1 09 23	1 09 41	1 09 59	1 10 19	1 10 41	1 11 04	1 11 29	1 11 55	−60	−62

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 10°	Latitude 11°	Latitude 12°	Latitude 13°	Latitude 14°	Latitude 15°	Latitude 16°	Latitude 17°	Latitude 18°	Latitude 19°	Latitude 20°	Latitude 10°	Latitude 20°
h *m*	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
7 00	1 07 37	1 07 50	1 08 05	1 08 19	1 08 36	1 08 54	1 09 14	1 09 35	1 09 58	1 10 22	1 10 47	−59	−61
7 15	1 06 16	1 06 29	1 06 42	1 06 57	1 07 14	1 07 32	1 07 51	1 08 11	1 08 33	1 08 57	1 09 22	−58	−60
7 30	1 04 39	1 04 51	1 05 04	1 05 19	1 05 35	1 05 52	1 06 10	1 06 30	1 06 52	1 07 15	1 07 39	−56	−58
7 45	1 02 44	1 02 56	1 03 09	1 03 23	1 03 38	1 03 55	1 04 13	1 04 32	1 04 53	1 05 15	1 05 39	−54	−56
8 00	1 00 34	1 00 45	1 00 58	1 01 11	1 01 26	1 01 42	1 01 59	1 02 18	1 02 38	1 02 59	1 03 22	−52	−55
8 15	0 58 09	0 58 19	0 58 31	0 58 44	0 58 58	0 59 13	0 59 30	0 59 47	1 00 06	1 00 27	1 00 49	−50	−53
8 30	0 55 28	0 55 38	0 55 49	0 56 02	0 56 15	0 56 29	0 56 45	0 57 02	0 57 20	0 57 39	0 58 00	−48	−50
8 45	0 52 33	0 52 43	0 52 53	0 53 05	0 53 18	0 53 31	0 53 46	0 54 02	0 54 19	0 54 37	0 54 57	−45	−48
9 00	0 49 25	0 49 33	0 49 44	0 49 55	0 50 07	0 50 19	0 50 33	0 50 48	0 51 04	0 51 21	0 51 40	−42	−45
9 15	0 46 05	0 46 13	0 46 22	0 46 32	0 46 43	0 46 55	0 47 08	0 47 21	0 47 36	0 47 52	0 48 09	−40	−42
9 30	0 42 32	0 42 40	0 42 48	0 42 57	0 43 07	0 43 18	0 43 30	0 43 43	0 43 57	0 44 11	0 44 27	−37	−38
9 45	0 38 49	0 38 56	0 39 03	0 39 12	0 39 21	0 39 31	0 39 42	0 39 53	0 40 06	0 40 19	0 40 33	−34	−35
10 00	0 34 56	0 35 02	0 35 09	0 35 16	0 35 24	0 35 33	0 35 43	0 35 53	0 36 04	0 36 16	0 36 29	−31	−31
10 15	0 30 54	0 30 59	0 31 05	0 31 12	0 31 19	0 31 27	0 31 35	0 31 44	0 31 54	0 32 05	0 32 16	−28	−28
10 30	0 26 44	0 26 48	0 26 54	0 26 59	0 27 06	0 27 12	0 27 20	0 27 28	0 27 36	0 27 45	0 27 55	−24	−24
10 45	0 22 27	0 22 31	0 22 35	0 22 40	0 22 45	0 22 51	0 22 57	0 23 04	0 23 11	0 23 18	0 23 27	−20	−20
11 00	0 18 04	0 18 08	0 18 11	0 18 15	0 18 19	0 18 24	0 18 29	0 18 34	0 18 40	0 18 46	0 18 52	−16	−16
11 15	0 13 37	0 13 40	0 13 42	0 13 45	0 13 48	0 13 52	0 13 56	0 14 00	0 14 04	0 14 09	0 14 13	−12	−12
11 30	0 09 07	0 09 08	0 09 10	0 09 12	0 09 14	0 09 16	0 09 19	0 09 22	0 09 25	0 09 28	0 09 31	− 8	− 8
11 45	0 04 34	0 04 35	0 04 36	0 04 37	0 04 38	0 04 39	0 04 40	0 04 41	0 04 43	0 04 44	0 04 46	− 4	− 4
Elongation:													
Azimuth	1 10 04	1 10 18	1 10 33	1 10 49	1 11 07	1 11 26	1 11 47	1 12 09	1 12 33	1 12 58	1 13 26	−61	−64
	h *m* *s*	*h* *m* *s*	*h* *m* *s*	*h* *m* *s*	*h* *m* *s*	*h* *m* *s*	*h* *m* *s*	*h* *m* *s*	*h* *m* *s*	*h* *m* *s*	*h* *m* *s*	*s*	*s*
Hour angle	5 59 11	5 59 06	5 59 02	5 58 56	5 58 51	5 58 46	5 58 41	5 58 36	5 58 30	5 58 25	5 58 19	+ 2	+ 2

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 20°	Latitude 21°	Latitude 22°	Latitude 23°	Latitude 24°	Latitude 25°	Latitude 26°	Latitude 27°	Latitude 28°	Latitude 29°	Latitude 30°	Latitude 20°	Latitude 30°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
0 15	0 04 50	0 04 52	0 04 55	0 04 57	0 04 59	0 05 02	0 05 04	0 05 07	0 05 10	0 05 13	0 05 17	− 4	− 5
0 30	0 09 39	0 09 43	0 09 48	0 09 52	0 09 57	0 10 02	0 10 07	0 10 13	0 10 19	0 10 25	0 10 31	− 8	− 9
0 45	0 14 26	0 14 32	0 14 38	0 14 45	0 14 52	0 15 00	0 15 07	0 15 16	0 15 25	0 15 34	0 15 44	−13	−14
1 00	0 19 09	0 19 16	0 19 25	0 19 34	0 19 43	0 19 53	0 20 04	0 20 15	0 20 26	0 20 39	0 20 51	−17	−18
1 15	0 23 46	0 23 56	0 24 07	0 24 18	0 24 29	0 24 42	0 24 55	0 25 08	0 25 23	0 25 38	0 25 54	−21	−23
1 30	0 28 18	0 28 29	0 28 42	0 28 55	0 29 09	0 29 24	0 29 39	0 29 55	0 30 12	0 30 30	0 30 49	−25	−27
1 45	0 32 42	0 32 55	0 33 10	0 33 25	0 33 41	0 33 58	0 34 16	0 34 34	0 34 54	0 35 15	0 35 37	−29	−31
2 00	0 36 57	0 37 12	0 37 29	0 37 46	0 38 04	0 38 23	0 38 43	0 39 04	0 39 26	0 39 50	0 40 15	−32	−35
2 15	0 41 03	0 41 20	0 41 38	0 41 57	0 42 17	0 42 38	0 43 00	0 43 24	0 43 49	0 44 14	0 44 42	−36	−39
2 30	0 44 58	0 45 16	0 45 36	0 45 57	0 46 19	0 46 42	0 47 06	0 47 32	0 47 59	0 48 27	0 48 57	−39	−43
2 45	0 48 41	0 49 01	0 49 22	0 49 45	0 50 08	0 50 33	0 51 00	0 51 27	0 51 57	0 52 27	0 53 00	−42	−46
3 00	0 52 12	0 52 33	0 52 56	0 53 20	0 53 45	0 54 12	0 54 40	0 55 10	0 55 41	0 56 14	0 56 48	−45	−50
3 15	0 55 29	0 55 51	0 56 15	0 56 41	0 57 08	0 57 36	0 58 06	0 58 37	0 59 10	0 59 45	1 00 22	−48	−53
3 30	0 58 31	0 58 55	0 59 20	0 59 47	1 00 15	1 00 45	1 01 16	1 01 49	1 02 24	1 03 01	1 03 40	−51	−56
3 45	1 01 18	1 01 43	1 02 09	1 02 37	1 03 07	1 03 38	1 04 11	1 04 46	1 05 22	1 06 00	1 06 41	−53	−58
4 00	1 03 49	1 04 15	1 04 43	1 05 12	1 05 42	1 06 15	1 06 49	1 07 25	1 08 02	1 08 42	1 09 24	−55	−61
4 15	1 06 04	1 06 31	1 06 59	1 07 29	1 08 01	1 08 34	1 09 09	1 09 46	1 10 25	1 11 06	1 11 49	−57	−63
4 30	1 08 02	1 08 29	1 08 58	1 09 29	1 10 01	1 10 35	1 11 12	1 11 50	1 12 30	1 13 12	1 13 56	−59	−64
4 45	1 09 42	1 10 10	1 10 39	1 11 10	1 11 44	1 12 19	1 12 55	1 13 34	1 14 15	1 14 58	1 15 44	−61	−66
5 00	1 11 03	1 11 32	1 12 02	1 12 34	1 13 07	1 13 43	1 14 20	1 15 00	1 15 41	1 16 25	1 17 11	−62	−68
5 15	1 12 07	1 12 36	1 13 06	1 13 38	1 14 12	1 14 48	1 15 26	1 16 06	1 16 48	1 17 32	1 18 19	−62	−69
5 30	1 12 52	1 13 21	1 13 51	1 14 24	1 14 58	1 15 34	1 16 13	1 16 53	1 17 35	1 18 20	1 19 07	−63	−69
5 45	1 13 18	1 13 47	1 14 18	1 14 50	1 15 25	1 16 01	1 16 39	1 17 19	1 18 02	1 18 47	1 19 34	−63	−69
6 00	1 13 26	1 13 54	1 14 25	1 14 57	1 15 32	1 16 08	1 16 46	1 17 26	1 18 09	1 18 53	1 19 41	−64	−70
6 15	1 13 14	1 13 43	1 14 13	1 14 45	1 15 19	1 15 55	1 16 33	1 17 13	1 17 55	1 18 40	1 19 26	−63	−69
6 30	1 12 44	1 13 12	1 13 42	1 14 14	1 14 48	1 15 23	1 16 01	1 16 40	1 17 22	1 18 06	1 18 52	−63	−68
6 45	1 11 55	1 12 23	1 12 52	1 13 24	1 13 57	1 14 32	1 15 09	1 15 48	1 16 29	1 17 12	1 17 58	−62	−67

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 20°	Latitude 21°	Latitude 22°	Latitude 23°	Latitude 24°	Latitude 25°	Latitude 26°	Latitude 27°	Latitude 28°	Latitude 29°	Latitude 30°	Latitude 20°	Latitude 30°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
7 00	1 10 47	1 11 15	1 11 44	1 12 15	1 12 47	1 13 22	1 13 58	1 14 36	1 15 16	1 15 59	1 16 44	−61	−66
7 15	1 09 22	1 09 49	1 10 17	1 10 47	1 11 19	1 11 53	1 12 28	1 13 05	1 13 45	1 14 26	1 15 10	−60	−65
7 30	1 07 39	1 08 05	1 08 32	1 09 02	1 09 33	1 10 05	1 10 40	1 11 16	1 11 54	1 12 35	1 13 17	−58	−64
7 45	1 05 39	1 06 04	1 06 30	1 06 59	1 07 28	1 08 00	1 08 33	1 09 08	1 09 46	1 10 24	1 11 06	−56	−62
8 00	1 03 22	1 03 46	1 04 11	1 04 38	1 05 07	1 05 38	1 06 10	1 06 43	1 07 19	1 07 57	1 08 36	−55	−60
8 15	1 00 49	1 01 11	1 01 36	1 02 02	1 02 30	1 02 59	1 03 29	1 04 02	1 04 36	1 05 12	1 05 49	−53	−57
8 30	0 58 00	0 58 22	0 58 45	0 59 10	0 59 35	1 00 04	1 00 33	1 01 04	1 01 36	1 02 10	1 02 46	−50	−54
8 45	0 54 57	0 55 17	0 55 40	0 56 03	0 56 28	0 56 54	0 57 22	0 57 51	0 58 21	0 58 53	0 59 27	−48	−51
9 00	0 51 40	0 51 59	0 52 20	0 52 41	0 53 04	0 53 29	0 53 55	0 54 22	0 54 51	0 55 21	0 55 53	−45	−48
9 15	0 48 09	0 48 27	0 48 46	0 49 07	0 49 28	0 49 51	0 50 15	0 50 40	0 51 07	0 51 35	0 52 05	−42	−45
9 30	0 44 27	0 44 43	0 45 01	0 45 20	0 45 40	0 46 01	0 46 22	0 46 46	0 47 11	0 47 37	0 48 04	−38	−42
9 45	0 40 33	0 40 48	0 41 04	0 41 21	0 41 39	0 41 58	0 42 19	0 42 40	0 43 02	0 43 26	0 43 51	−35	−38
10 00	0 36 29	0 36 43	0 36 57	0 37 12	0 37 29	0 37 46	0 38 04	0 38 23	0 38 43	0 39 04	0 39 27	−31	−34
10 15	0 32 16	0 32 28	0 32 41	0 32 54	0 33 08	0 33 24	0 33 40	0 33 57	0 34 14	0 34 34	0 34 53	−28	−30
10 30	0 27 55	0 28 05	0 28 16	0 28 28	0 28 40	0 28 53	0 29 07	0 29 22	0 29 37	0 29 53	0 30 10	−24	−26
10 45	0 23 27	0 23 35	0 23 44	0 23 54	0 24 05	0 24 16	0 24 27	0 24 39	0 24 52	0 25 06	0 25 20	−20	−22
11 00	0 18 52	0 18 59	0 19 07	0 19 15	0 19 23	0 19 32	0 19 41	0 19 51	0 20 01	0 20 12	0 20 24	−16	−18
11 15	0 14 13	0 14 19	0 14 24	0 14 30	0 14 37	0 14 43	0 14 50	0 14 58	0 15 05	0 15 14	0 15 22	−12	−13
11 30	0 09 31	0 09 34	0 09 38	0 09 42	0 09 46	0 09 51	0 09 56	0 10 01	0 10 06	0 10 11	0 10 17	− 8	− 9
11 45	0 04 46	0 04 48	0 04 50	0 04 52	0 04 54	0 04 56	0 04 58	0 05 00	0 05 03	0 05 06	0 05 09	− 4	− 4
Elongation: Azimuth	1 13 26	1 13 55	1 14 25	1 14 58	1 15 32	1 16 08	1 16 46	1 17 27	1 18 09	1 18 54	1 19 41	−64	−69
	h m s	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*s*	*s*
Hour angle	5 58 19	5 58 14	5 58 08	5 58 03	5 57 57	5 57 51	5 57 45	5 57 39	5 57 33	5 57 27	5 57 21	+ 2	+ 2

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 30°	Latitude 31°	Latitude 32°	Latitude 33°	Latitude 34°	Latitude 35°	Latitude 36°	Latitude 37°	Latitude 38°	Latitude 39°	Latitude 40°	Latitude 30°	Latitude 40°
h *m*	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
0 15	0 05 17	0 05 20	0 05 23	0 05 27	0 05 31	0 05 35	0 05 40	0 05 44	0 05 49	0 05 54	0 06 00	− 5	− 5
0 30	0 10 31	0 10 38	0 10 45	0 10 53	0 11 01	0 11 09	0 11 18	0 11 27	0 11 37	0 11 47	0 11 57	− 9	−10
0 45	0 15 44	0 15 54	0 16 04	0 16 16	0 16 27	0 16 40	0 16 53	0 17 07	0 17 21	0 17 36	0 17 52	−14	−16
1 00	0 20 51	0 21 05	0 21 19	0 21 34	0 21 50	0 22 06	0 22 24	0 22 42	0 23 01	0 23 21	0 23 42	−18	−21
1 15	0 25 54	0 26 11	0 26 28	0 26 47	0 27 06	0 27 27	0 27 48	0 28 11	0 28 34	0 28 59	0 29 26	−23	−26
1 30	0 30 49	0 31 09	0 31 30	0 31 52	0 32 15	0 32 40	0 33 05	0 33 32	0 34 00	0 34 30	0 35 01	−27	−31
1 45	0 35 37	0 36 00	0 36 24	0 36 49	0 37 16	0 37 44	0 38 14	0 38 44	0 39 17	0 39 51	0 40 27	−31	−36
2 00	0 40 15	0 40 41	0 41 08	0 41 37	0 42 07	0 42 38	0 43 12	0 43 47	0 44 23	0 45 02	0 45 42	−35	−40
2 15	0 44 42	0 45 11	0 45 41	0 46 13	0 46 46	0 47 21	0 47 58	0 48 37	0 49 18	0 50 00	0 50 45	−39	−45
2 30	0 48 57	0 49 29	0 50 02	0 50 37	0 51 13	0 51 52	0 52 32	0 53 14	0 53 59	0 54 46	0 55 35	−43	−49
2 45	0 53 00	0 53 34	0 54 10	0 54 47	0 55 27	0 56 08	0 56 52	0 57 37	0 58 25	0 59 16	1 00 09	−46	−53
3 00	0 56 48	0 57 25	0 58 03	0 58 43	0 59 25	1 00 10	1 00 56	1 01 45	1 02 37	1 03 31	1 04 28	−50	−57
3 15	1 00 22	1 01 01	1 01 41	1 02 24	1 03 08	1 03 55	1 04 45	1 05 37	1 06 31	1 07 29	1 08 29	−53	−60
3 30	1 03 40	1 04 20	1 05 03	1 05 48	1 06 35	1 07 24	1 08 16	1 09 11	1 10 08	1 11 09	1 12 12	−56	−63
3 45	1 06 41	1 07 23	1 08 08	1 08 54	1 09 44	1 10 35	1 11 30	1 12 27	1 13 27	1 14 30	1 15 36	−58	−66
4 00	1 09 24	1 10 08	1 10 54	1 11 43	1 12 34	1 13 28	1 14 24	1 15 23	1 16 26	1 17 31	1 18 40	−61	−69
4 15	1 11 49	1 12 35	1 13 23	1 14 13	1 15 06	1 16 01	1 16 59	1 18 00	1 19 05	1 20 12	1 21 23	−63	−72
4 30	1 13 56	1 14 43	1 15 32	1 16 23	1 17 18	1 18 14	1 19 14	1 20 17	1 21 23	1 22 32	1 23 45	−64	−74
4 45	1 15 44	1 16 31	1 17 21	1 18 14	1 19 09	1 20 07	1 21 08	1 22 12	1 23 20	1 24 31	1 25 45	−66	−75
5 00	1 17 11	1 18 00	1 18 51	1 19 44	1 20 40	1 21 39	1 22 41	1 23 46	1 24 55	1 26 07	1 27 23	−68	−76
5 15	1 18 19	1 19 08	1 19 59	1 20 54	1 21 50	1 22 50	1 23 53	1 24 59	1 26 08	1 27 21	1 28 38	−69	−77
5 30	1 19 07	1 19 56	1 20 48	1 21 42	1 22 40	1 23 40	1 24 43	1 25 49	1 26 59	1 28 12	1 29 30	−69	−78
5 45	1 19 34	1 20 23	1 21 15	1 22 10	1 23 07	1 24 08	1 25 11	1 26 17	1 27 27	1 28 41	1 29 58	−69	−78
6 00	1 19 41	1 20 30	1 21 22	1 22 16	1 23 13	1 24 14	1 25 17	1 26 23	1 27 33	1 28 47	1 30 04	−70	−78
6 15	1 19 26	1 20 15	1 21 07	1 22 01	1 22 58	1 23 58	1 25 01	1 26 07	1 27 17	1 28 30	1 29 46	−69	−78
6 30	1 18 52	1 19 41	1 20 32	1 21 26	1 22 22	1 23 21	1 24 24	1 25 29	1 26 38	1 27 50	1 29 06	−68	−77
6 45	1 17 58	1 18 46	1 19 36	1 20 29	1 21 25	1 22 23	1 23 24	1 24 29	1 25 37	1 26 48	1 28 03	−67	−76

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 30°	Latitude 31°	Latitude 32°	Latitude 33°	Latitude 34°	Latitude 35°	Latitude 36°	Latitude 37°	Latitude 38°	Latitude 39°	Latitude 40°	Latitude 30°	Latitude 40°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
7 00	1 16 44	1 17 31	1 18 20	1 19 12	1 20 06	1 21 04	1 22 04	1 23 07	1 24 14	1 25 24	1 26 37	−66	−75
7 15	1 15 10	1 15 56	1 16 44	1 17 35	1 18 28	1 19 24	1 20 23	1 21 25	1 22 30	1 23 38	1 24 50	−65	−73
7 30	1 13 17	1 14 02	1 14 49	1 15 38	1 16 30	1 17 24	1 18 21	1 19 21	1 20 25	1 21 31	1 22 41	−64	−72
7 45	1 11 06	1 11 49	1 12 34	1 13 22	1 14 12	1 15 05	1 16 00	1 16 58	1 17 59	1 19 04	1 20 11	−62	−69
8 00	1 08 36	1 09 18	1 10 01	1 10 47	1 11 36	1 12 26	1 13 20	1 14 16	1 15 14	1 16 16	1 17 21	−60	−66
8 15	1 05 49	1 06 29	1 07 11	1 07 55	1 08 41	1 09 30	1 10 21	1 11 14	1 12 11	1 13 10	1 14 12	−57	−64
8 30	1 02 46	1 03 24	1 04 04	1 04 46	1 05 29	1 06 16	1 07 04	1 07 55	1 08 49	1 09 45	1 10 44	−54	−61
8 45	0 59 27	1 00 03	1 00 40	1 01 20	1 02 01	1 02 45	1 03 31	1 04 19	1 05 10	1 06 03	1 06 59	−51	−58
9 00	0 55 53	0 56 26	0 57 02	0 57 39	0 58 18	0 58 59	0 59 43	1 00 27	1 01 14	1 02 04	1 02 57	−48	−54
9 15	0 52 05	0 52 36	0 53 09	0 53 43	0 54 20	0 54 58	0 55 38	0 56 20	0 57 04	0 57 50	0 58 39	−45	−50
9 30	0 48 04	0 48 33	0 49 03	0 49 35	0 50 08	0 50 43	0 51 20	0 51 58	0 52 39	0 53 22	0 54 07	−42	−46
9 45	0 43 51	0 44 17	0 44 44	0 45 13	0 45 44	0 46 16	0 46 49	0 47 24	0 48 01	0 48 40	0 49 21	−38	−42
10 00	0 39 27	0 39 50	0 40 15	0 40 41	0 41 08	0 41 37	0 42 07	0 42 39	0 43 12	0 43 47	0 44 24	−34	−38
10 15	0 34 53	0 35 14	0 35 35	0 35 58	0 36 22	0 36 48	0 37 14	0 37 42	0 38 12	0 38 43	0 39 15	−30	−34
10 30	0 30 10	0 30 28	0 30 47	0 31 07	0 31 28	0 31 50	0 32 12	0 32 37	0 33 02	0 33 29	0 33 57	−26	−29
10 45	0 25 20	0 25 35	0 25 51	0 26 08	0 26 25	0 26 43	0 27 03	0 27 23	0 27 44	0 28 07	0 28 30	−22	−24
11 00	0 20 24	0 20 36	0 20 49	0 21 02	0 21 16	0 21 31	0 21 46	0 22 03	0 22 20	0 22 38	0 22 57	−18	−20
11 15	0 15 22	0 15 31	0 15 41	0 15 51	0 16 02	0 16 13	0 16 24	0 16 37	0 16 50	0 17 03	0 17 17	−13	−15
11 30	0 10 17	0 10 23	0 10 29	0 10 36	0 10 43	0 10 51	0 10 58	0 11 07	0 11 15	0 11 24	0 11 34	− 9	−10
11 45	0 05 09	0 05 12	0 05 15	0 05 19	0 05 22	0 05 26	0 05 30	0 05 34	0 05 38	0 05 43	0 05 47	− 4	− 5
Elongation: Azimuth.	1 19 41	1 20 30	1 21 22	1 22 17	1 23 14	1 24 14	1 25 18	1 26 24	1 27 34	1 28 48	1 30 05	−69	−78
	h m s	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*s*	*s*
Hour angle.	5 57 21	5 57 14	5 57 08	5 57 01	5 56 54	5 56 47	5 56 39	5 56 32	5 56 24	5 56 16	5 56 08	+ 2	+ 3

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 40°	Latitude 41°	Latitude 42°	Latitude 43°	Latitude 44°	Latitude 45°	Latitude 46°	Latitude 47°	Latitude 48°	Latitude 49°	Latitude 50°	Latitude 40°	Latitude 50°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
0 15	0 06 00	0 06 05	0 06 11	0 06 17	0 06 24	0 06 31	0 06 38	0 06 46	0 06 54	0 07 03	0 07 12	− 5	− 6
0 30	0 11 57	0 12 09	0 12 21	0 12 33	0 12 46	0 13 00	0 13 14	0 13 30	0 13 46	0 14 03	0 14 21	−10	−13
0 45	0 17 52	0 18 09	0 18 27	0 18 45	0 19 05	0 19 25	0 19 47	0 20 10	0 20 34	0 21 00	0 21 27	−16	−19
1 00	0 23 42	0 24 04	0 24 28	0 24 52	0 25 18	0 25 45	0 26 14	0 26 45	0 27 17	0 27 51	0 28 27	−21	−25
1 15	0 29 26	0 29 53	0 30 22	0 30 53	0 31 25	0 31 59	0 32 34	0 33 12	0 33 52	0 34 34	0 35 18	−26	−32
1 30	0 35 01	0 35 34	0 36 08	0 36 45	0 37 23	0 38 03	0 38 46	0 39 30	0 40 18	0 41 08	0 42 01	−31	−38
1 45	0 40 27	0 41 05	0 41 45	0 42 27	0 43 11	0 43 57	0 44 46	0 45 38	0 46 33	0 47 30	0 48 31	−36	−43
2 00	0 45 42	0 46 25	0 47 10	0 47 57	0 48 47	0 49 39	0 50 35	0 51 33	0 52 35	0 53 40	0 54 49	−40	−49
2 15	0 50 45	0 51 33	0 52 22	0 53 15	0 54 10	0 55 08	0 56 09	0 57 14	0 58 22	0 59 35	1 00 51	−45	−54
2 30	0 55 35	0 56 26	0 57 21	0 58 18	0 59 18	1 00 22	1 01 29	1 02 40	1 03 54	1 05 13	1 06 37	−49	−59
2 45	1 00 09	1 01 05	1 02 04	1 03 06	1 04 11	1 05 20	1 06 32	1 07 48	1 09 09	1 10 34	1 12 04	−53	−64
3 00	1 04 28	1 05 28	1 06 30	1 07 36	1 08 46	1 10 00	1 11 17	1 12 39	1 14 05	1 15 36	1 17 12	−57	−68
3 15	1 08 29	1 09 32	1 10 39	1 11 49	1 13 03	1 14 21	1 15 43	1 17 10	1 18 41	1 20 18	1 22 00	−60	−72
3 30	1 12 12	1 13 19	1 14 29	1 15 43	1 17 00	1 18 22	1 19 49	1 21 20	1 22 56	1 24 37	1 26 25	−63	−76
3 45	1 15 36	1 16 46	1 17 59	1 19 16	1 20 37	1 22 03	1 23 33	1 25 08	1 26 49	1 28 35	1 30 27	−66	−80
4 00	1 18 40	1 19 52	1 21 08	1 22 28	1 23 53	1 25 21	1 26 54	1 28 34	1 30 18	1 32 08	1 34 05	−69	−83
4 15	1 21 23	1 22 38	1 23 56	1 25 19	1 26 46	1 28 17	1 29 54	1 31 33	1 33 24	1 35 17	1 37 17	−72	−86
4 30	1 23 45	1 25 02	1 26 22	1 27 47	1 29 16	1 30 50	1 32 30	1 34 14	1 36 06	1 38 01	1 40 04	−74	−88
4 45	1 25 45	1 27 03	1 28 26	1 29 52	1 31 23	1 32 59	1 34 41	1 36 28	1 38 20	1 40 19	1 42 25	−75	−90
5 00	1 27 23	1 28 42	1 30 06	1 31 34	1 33 06	1 34 44	1 36 27	1 38 16	1 40 10	1 42 11	1 44 19	−76	−91
5 15	1 28 38	1 29 58	1 31 23	1 32 52	1 34 25	1 36 04	1 37 48	1 39 38	1 41 34	1 43 36	1 45 45	−77	−92
5 30	1 29 30	1 30 50	1 32 16	1 33 45	1 35 20	1 36 59	1 38 44	1 40 34	1 42 31	1 44 34	1 46 44	−78	−93
5 45	1 29 58	1 31 20	1 32 45	1 34 15	1 35 50	1 37 29	1 39 14	1 41 05	1 43 02	1 45 05	1 47 16	−78	−94
6 00	1 30 04	1 31 25	1 32 50	1 34 20	1 35 55	1 37 34	1 39 19	1 41 09	1 43 06	1 45 09	1 47 19	−78	−93
6 15	1 29 46	1 31 07	1 32 32	1 34 01	1 35 35	1 37 14	1 38 58	1 40 48	1 42 44	1 44 46	1 46 56	−78	−93
6 30	1 29 06	1 30 26	1 31 50	1 33 18	1 34 51	1 36 29	1 38 12	1 40 01	1 41 56	1 43 57	1 46 04	−77	−92
6 45	1 28 03	1 29 20	1 30 44	1 32 11	1 33 43	1 35 19	1 37 01	1 38 48	1 40 41	1 42 40	1 44 46	−76	−91

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Concluded*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of polaris	
	Latitude 40°	Latitude 41°	Latitude 42°	Latitude 43°	Latitude 44°	Latitude 45°	Latitude 46°	Latitude 47°	Latitude 48°	Latitude 49°	Latitude 50°	Latitude 40°	Latitude 50°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
7 00	1 26 37	1 27 54	1 29 15	1 30 41	1 32 11	1 33 45	1 35 25	1 37 10	1 39 01	1 40 58	1 43 02	−75	−89
7 15	1 24 50	1 26 05	1 27 24	1 28 48	1 30 16	1 31 48	1 33 25	1 35 08	1 36 56	1 38 51	1 40 51	−73	−87
7 30	1 22 41	1 23 54	1 25 11	1 26 32	1 27 58	1 29 27	1 31 02	1 32 42	1 34 27	1 36 18	1 38 16	−72	−85
7 45	1 20 11	1 21 22	1 22 36	1 23 55	1 25 17	1 26 44	1 28 16	1 29 52	1 31 34	1 33 22	1 35 15	−69	−82
8 00	1 17 21	1 18 29	1 19 41	1 20 57	1 22 16	1 23 40	1 25 08	1 26 41	1 28 19	1 30 02	1 31 51	−65	−79
8 15	1 14 12	1 15 17	1 16 26	1 17 38	1 18 54	1 20 14	1 21 39	1 23 07	1 24 41	1 26 20	1 28 05	−64	−76
8 30	1 10 44	1 11 46	1 12 52	1 14 00	1 15 12	1 16 29	1 17 49	1 19 14	1 20 43	1 22 17	1 23 56	−61	−72
8 45	1 06 59	1 07 57	1 08 59	1 10 04	1 11 12	1 12 23	1 13 40	1 15 00	1 16 24	1 17 53	1 19 27	−58	−68
9 00	1 02 57	1 03 52	1 04 50	1 05 50	1 06 55	1 08 02	1 09 13	1 10 28	1 11 47	1 13 10	1 14 38	−54	−64
9 15	0 58 39	0 59 30	1 00 24	1 01 21	1 02 20	1 03 23	1 04 29	1 05 39	1 06 52	1 08 10	1 09 32	−50	−59
9 30	0 54 07	0 54 54	0 55 44	0 56 36	0 57 31	0 58 28	0 59 29	1 00 34	1 01 41	1 02 53	1 04 08	−46	−55
9 45	0 49 21	0 50 04	0 50 49	0 51 37	0 52 27	0 53 20	0 54 15	0 55 13	0 56 15	0 57 20	0 58 29	−42	−50
10 00	0 44 24	0 45 02	0 45 43	0 46 25	0 47 10	0 47 58	0 48 47	0 49 40	0 50 35	0 51 34	0 52 35	−38	−45
10 15	0 39 15	0 39 49	0 40 25	0 41 03	0 41 42	0 42 24	0 43 08	0 43 54	0 44 43	0 45 35	0 46 29	−34	−40
10 30	0 33 57	0 34 26	0 34 57	0 35 30	0 36 04	0 36 40	0 37 18	0 37 58	0 38 40	0 39 24	0 40 12	−29	−34
10 45	0 28 30	0 28 55	0 29 21	0 29 48	0 30 17	0 30 47	0 31 19	0 31 53	0 32 28	0 33 05	0 33 45	−24	−29
11 00	0 22 57	0 23 16	0 23 37	0 23 59	0 24 22	0 24 47	0 25 12	0 25 39	0 26 08	0 26 38	0 27 09	−20	−23
11 15	0 17 17	0 17 32	0 17 48	0 18 05	0 18 22	0 18 40	0 19 00	0 19 20	0 19 42	0 20 04	0 20 28	−15	−18
11 30	0 11 34	0 11 44	0 11 54	0 12 06	0 12 17	0 12 29	0 12 42	0 12 56	0 13 10	0 13 25	0 13 43	−10	−12
11 45	0 05 47	0 05 52	0 05 58	0 06 04	0 06 09	0 06 15	0 06 22	0 06 29	0 06 36	0 06 44	0 06 51	− 5	− 6
Elongation: Azimuth.	1 30 05	1 31 26	1 32 51	1 34 21	1 35 56	1 37 35	1 39 20	1 41 11	1 43 08	1 45 11	1 47 21	−78	−93
	h m s	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*h m s*	*s*	*h m s*
Hour angle.	5 56 08	5 56 00	5 55 51	5 55 43	5 55 33	5 55 24	5 55 14	5 55 04	5 54 53	5 54 42	5 54 31	+ 3	+ 3

Table VIII.—Polar Distance of Polaris for the Beginning of Each Year, 1926–1935

Year	Mean polar distance	Year	Mean polar distance
1926	1° 05′ 30.″5	1931	1° 03′ 59.″0
1927	1 05 12.2	1932	1 03 40.7
1928	1 04 53.8	1933	1 03 22.5
1929	1 04 35.5	1934	1 03 04.3
1930	1 04 17.2	1935	1 02 46.1

Mean Polar Distances of Polaris are found by subtracting from 90° the corresponding *Mean* declinations. For precise observations the *Apparent* declination (to be found in the American Ephemeris) should be used since the variation between Mean and Apparent Declinations is constantly changing, and has as a maximum value about ½′.

TABLE IX.[1] HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS

Minutes	0°		1°		2°		3°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	100.00	0.00	99.97	1.74	99.88	3.49	99.73	5.23
2	100.00	0.06	99.97	1.80	99.87	3.55	99.72	5.28
4	100.00	0.12	99.97	1.86	99.87	3.60	99.71	5.34
6	100.00	0.17	99.96	1.92	99.87	3.66	99.71	5.40
8	100.00	0.23	99.96	1.98	99.86	3.72	99.70	5.46
10	100.00	0.29	99.96	2.04	99.86	3.78	99.69	5.52
12	100.00	0.35	99.96	2.09	99.85	3.84	99.69	5.57
14	100.00	0.41	99.95	2.15	99.85	3.90	99.68	5.63
16	100.00	0.47	99.95	2.21	99.84	3.95	99.68	5.69
18	100.00	0.52	99.95	2.27	99.84	4.01	99.67	5.75
20	100.00	0.58	99.95	2.33	99.83	4.07	99.66	5.80
22	100.00	0.64	99.94	2.38	99.83	4.13	99.66	5.86
24	100.00	0.70	99.94	2.44	99.82	4.18	99.65	5.92
26	99.99	0.76	99.94	2.50	99.82	4.24	99.64	5.98
28	99.99	0.81	99.93	2.56	99.81	4.30	99.63	6.04
30	99.99	0.87	99.93	2.62	99.81	4.36	99.63	6.09
32	99.99	0.93	99.93	2.67	99.80	4.42	99.62	6.15
34	99.99	0.99	99.93	2.73	99.80	4.48	99.62	6.21
36	99.99	1.05	99.92	2.79	99.79	4.53	99.61	6.27
38	99.99	1.11	99.92	2.85	99.79	4.59	99.60	6.33
40	99.99	1.16	99.92	2.91	99.78	4.65	99.59	6.38
42	99.99	1.22	99.91	2.97	99.78	4.71	99.59	6.44
44	99.98	1.28	99.91	3.02	99.77	4.76	99.58	6.50
46	99.98	1.34	99.90	3.08	99.77	4.82	99.57	6.56
48	99.98	1.40	99.90	3.14	99.76	4.88	99.56	6.61
50	99.98	1.45	99.90	3.20	99.76	4.94	99.56	6.67
52	99.98	1.51	99.89	3.26	99.75	4.99	99.55	6.73
54	99.98	1.57	99.89	3.31	99.74	5.05	99.54	6.78
56	99.97	1.63	99.89	3.37	99.74	5.11	99.53	6.84
58	99.97	1.69	99.88	3.43	99.73	5.17	99.52	6.90
60	99.97	1.74	99.88	3.49	99.73	5.23	99.51	6.96
c = 0.75.	0.75	0.01	0.75	0.02	0.75	0.03	0.75	0.05
c = 1.00.	1.00	0.01	1.00	0.03	1.00	0.04	1.00	0.06
c = 1.25.	1.25	0.02	1.25	0.03	1.25	0.05	1.25	0.08

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[1] From "Theory and Practice of Surveying," by Prof. J. B. Johnson, New York; John Wiley & Sons.

Table IX. Horizontal Distances and Elevations from Stadia Readings.—*Continued*

	4°		5°		6°		7°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	99.51	6.96	99.24	8.68	98.91	10.40	98.51	12.10
2	99.51	7.02	99.23	8.74	98.90	10.45	98.50	12.15
4	99.50	7.07	99.22	8.80	98.88	10.51	98.48	12.21
6	99.49	7.13	99.21	8.85	98.87	10.57	98.47	12.26
8	99.48	7.19	99.20	8.91	98.86	10.62	98.46	12.32
10	99.47	7.25	99.19	8.97	98.85	10.68	98.44	12.38
12	99.46	7.30	99.18	9.03	98.83	10.74	98.43	12.43
14	99.46	7.36	99.17	9.08	98.82	10.79	98.41	12.49
16	99.45	7.42	99.16	9.14	98.81	10.85	98.40	12.55
18	99.44	7.48	99.15	9.20	98.80	10.91	98.39	12.60
20	99.43	7.53	99.14	9.25	98.78	10.96	98.37	12.66
22	99.42	7.59	99.13	9.31	98.77	11.02	98.36	12.72
24	99.41	7.65	99.11	9.37	98.76	11.08	98.34	12.77
26	99.40	7.71	99.10	9.43	98.74	11.13	98.33	12.83
28	99.39	7.76	99.09	9.48	98.73	11.19	98.31	12.88
30	99.38	7.82	99.08	9.54	98.72	11.25	98.29	12.94
32	99.38	7.88	99.07	9.60	98.71	11.30	98.28	13.00
34	99.37	7.94	99.06	9.65	98.69	11.36	98.27	13.05
36	99.36	7.99	99.05	9.71	98.68	11.42	98.25	13.11
38	99.35	8.05	99.04	9.77	98.67	11.47	98.24	13.17
40	99.34	8.11	99.03	9.83	98.65	11.53	98.22	13.22
42	99.33	8.17	99.01	9.88	98.64	11.59	98.20	13.28
44	99.32	8.22	99.00	9.94	98.63	11.64	98.19	13.33
46	99.31	8.23	98.99	10.00	98.61	11.70	98.17	13.39
48	99.30	8.34	98.98	10.05	98.60	11.76	98.16	13.45
50	99.29	8.40	98.97	10.11	98.58	11.81	98.14	13.50
52	99.28	8.45	98.96	10.17	98.57	11.87	98.13	13.56
54	99.27	8.51	98.94	10.22	98.56	11.93	98.11	13.61
56	99.26	8.57	98.93	10.28	98.54	11.98	98.10	13.67
58	99.25	8.63	98.92	10.34	98.53	12.04	98.08	13.73
60	99.24	8.68	98.91	10.40	98.51	12.10	98.06	13.78
c = 0.75.	0.75	0.06	0.75	0.07	0.75	0.08	0.74	0.10
c = 1.00.	1.00	0.08	0.99	0.09	0.99	0.11	0.99	0.13
c = 1.25.	1.25	0.10	1.24	0.11	1.24	0.14	1.24	0.16

TABLE IX. HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

	8°		9°		10°		11°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	98.06	13.78	97.55	15.45	96.98	17.10	96.36	18.73
2	98.05	13.84	97.53	15.51	96.96	17.16	96.34	18.78
4	98.03	13.89	97.52	15.56	96.94	17.21	96.32	18.84
6	98.01	13.95	97.50	15.62	96.92	17.26	96.29	18.89
8	98.00	14.01	97.48	15.67	96.90	17.32	96.27	18.95
10	97.98	14.06	97.46	15.73	96.88	17.37	96.25	19.00
12	97.97	14.12	97.44	15.78	96.86	17.43	96.23	19.05
14	97.95	14.17	97.43	15.84	96.84	17.48	96.21	19.11
16	97.93	14.23	97.41	15.89	96.82	17.54	96.18	19.16
18	97.92	14.28	97.39	15.95	96.80	17.59	96.16	19.21
20	97.90	14.34	97.37	16.00	96.78	17.65	96.14	19.27
22	97.88	14.40	97.35	16.06	96.76	17.70	96.12	19.32
24	97.87	14.45	97.33	16.11	96.74	17.76	96.09	19.38
26	97.85	14.51	97.31	16.17	96.72	17.81	96.07	19.43
28	97.83	14.56	97.29	16.22	96.70	17.86	96.05	19.48
30	97.82	14.62	97.28	16.28	96.68	17.92	96.03	19.54
32	97.80	14.67	97.26	16.33	96.66	17.97	96.00	19.59
34	97.78	14.73	97.24	16.39	96.64	18.03	95.98	19.64
36	97.76	14.79	97.22	16.44	96.62	18.08	95.96	19.70
38	97.75	14.84	97.20	16.50	96.60	18.14	95.93	19.75
40	97.73	14.90	97.18	16.55	96.57	18.19	95.91	19.80
42	97.71	14.95	97.16	16.61	96.55	18.24	95.89	19.86
44	97.69	15.01	97.14	16.66	96.53	18.30	95.86	19.91
46	97.68	15.06	97.12	16.72	96.51	18.35	95.84	19.96
48	97.66	15.12	97.10	16.77	96.49	18.41	95.82	20.02
50	97.64	15.17	97.08	16.83	96.47	18.46	95.79	20.07
52	97.62	15.23	97.06	16.88	96.45	18.51	95.77	20.12
54	97.61	15.28	97.04	16.94	96.42	18.57	95.75	20.18
56	97.59	15.34	97.02	16.99	96.40	18.62	95.72	20.23
58	97.57	15.40	97.00	17.05	96.38	18.68	95.70	20.28
60	97.55	15.45	96.98	17.10	96.36	18.73	95.68	20.34
c = 0.75.	0.74	0.11	0.74	0.12	0.74	0.14	0.73	0.15
c = 1.00.	0.99	0.15	0.99	0.16	0.98	0.18	0.98	0.20
c = 1.25.	1.23	0.18	1.23	0.21	1.23	0.23	1.22	0.25

TABLE IX. HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

	12°		13°		14°		15°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	95.68	20.34	94.94	21.92	94.15	23.47	93.30	25.00
2	95.65	20.39	94.91	21.97	94.12	23.52	93.27	25.05
4	95.63	20.44	94.89	22.02	94.09	23.58	93.24	25.10
6	95.61	20.50	94.86	22.08	94.07	23.63	93.21	25.15
8	95.58	20.55	94.84	22.13	94.04	23.68	93.18	25.20
10	95.56	20.60	94.81	22.18	94.01	23.73	93.16	25.25
12	95.53	20.66	94.79	22.23	93.98	23.78	93.13	25.30
14	95.51	20.71	94.76	22.28	93.95	23.83	93.10	25.35
16	95.49	20.76	94.73	22.34	93.93	23.88	93.07	25.40
18	95.46	20.81	94.71	22.39	93.90	23.93	93.04	25.45
20	95.44	20.87	94.68	22.44	93.87	23.99	93.01	25.50
22	95.41	20.92	94.66	22.49	93.84	24.04	92.98	25.55
24	95.39	20.97	94.63	22.54	93.81	24.09	92.95	25.60
26	95.36	21.03	94.60	22.60	93.79	24.14	92.92	25.65
28	95.34	21.08	94.58	22.65	93.76	24.19	92.89	25.70
30	95.32	21.13	94.55	22.70	93.73	24.24	92.86	25.75
32	95.29	21.18	94.52	22.75	93.70	24.29	92.83	25.80
34	95.27	21.24	94.50	22.80	93.67	24.34	92.80	25.85
36	95.24	21.29	94.47	22.85	93.65	24.39	92.77	25.90
38	95.22	21.34	94.44	22.91	93.62	24.44	92.74	25.95
40	95.19	21.39	94.42	22.96	93.59	24.49	92.71	26.00
42	95.17	21.45	94.39	23.01	93.56	24.55	92.68	26.05
44	95.14	21.50	94.36	23.06	93.53	24.60	92.65	26.10
46	95.12	21.55	94.34	23.11	93.50	24.65	92.62	26.15
48	95.09	21.60	94.31	23.16	93.47	24.70	92.59	26.20
50	95.07	21.66	94.28	23.22	93.45	24.75	92.56	26.25
52	95.04	21.71	94.26	23.27	93.42	24.80	92.53	26.30
54	95.02	21.76	94.23	23.32	93.39	24.85	92.49	26.35
56	94.99	21.81	94.20	23.37	93.36	24.90	92.46	26.40
58	94.97	21.87	94.17	23.42	93.33	24.95	92.43	26.45
60	94.94	21.92	94.15	23.47	93.30	25.00	92.40	26.50
c = 0.75.	0.73	0.16	0.73	0.17	0.73	0.19	0.72	0.20
c = 1.00.	0.98	0.22	0.97	0.23	0.97	0.25	0.96	0.27
c = 1.25.	1.22	0.27	1.21	0.29	1.21	0.31	1.20	0.34

TABLE IX. HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

	16°		17°		18°		19°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	92.40	26.50	91.45	27.96	90.45	29.39	89.40	30.78
2	92.37	26.55	91.42	28.01	90.42	29.44	89.36	30.83
4	92.34	26.59	91.39	28.06	90.38	29.48	89.33	30.87
6	92.31	26.64	91.35	28.10	90.35	29.53	89.29	30.92
8	92.28	26.69	91.32	28.15	90.31	29.58	89.26	30.97
10	92.25	26.74	91.29	28.20	90.28	29.62	89.22	31.01
12	92.22	26.79	91.26	28.25	90.24	29.67	89.18	31.06
14	92.19	26.84	91.22	28.30	90.21	29.72	89.15	31.10
16	92.15	26.89	91.19	28.34	90.18	29.76	89.11	31.15
18	92.12	26.94	91.16	28.39	90.14	29.81	89.08	31.19
20	92.09	26.99	91.12	28.44	90.11	29.86	89.04	31.24
22	92.06	27.04	91.09	28.49	90.07	29.90	89.00	31.28
24	92.03	27.09	91.06	28.54	90.04	29.95	88.96	31.33
26	92.00	27.13	91.02	28.58	90.00	30.00	88.93	31.38
28	91.97	27.18	90.99	28.63	89.97	30.04	88.89	31.42
30	91.93	27.23	90.96	28.68	89.93	30.09	88.86	31.47
32	91.90	27.28	90.92	28.73	89.90	30.14	88.82	31.51
34	91.87	27.33	90.89	28.77	89.86	30.19	88.78	31.56
36	91.84	27.38	90.86	28.82	89.83	30.23	88.75	31.60
38	91.81	27.43	90.82	28.87	89.79	30.28	88.71	31.65
40	91.77	27.48	90.79	28.92	89.76	30.32	88.67	31.69
42	91.74	27.52	90.76	28.96	89.72	30.37	88.64	31.74
44	91.71	27.57	90.72	29.01	89.69	30.41	88.60	31.78
46	91.68	27.62	90.69	29.06	89.65	30.46	88.56	31.83
48	91.65	27.67	90.66	29.11	89.61	30.51	88.53	31.87
50	91.61	27.72	90.62	29.15	89.58	30.55	88.49	31.92
52	91.58	27.77	90.59	29.20	89.54	30.60	88.45	31.96
54	91.55	27.81	90.55	29.25	89.51	30.65	88.41	32.01
56	91.52	27.86	90.52	29.30	89.47	30.69	88.38	32.05
58	91.48	27.91	90.48	29.34	89.44	30.74	88.34	32.09
60	91.45	27.96	90.45	29.39	89.40	30.78	88.30	32.14
c = 0.75.	0.72	0.21	0.72	0.23	0.71	0.24	0.71	0.25
c = 1.00.	0.96	0.28	0.95	0.30	0.95	0.32	0.94	0.33
c = 1.25.	1.20	0.35	1.19	0.38	1.19	0.40	1.18	0.42

TABLE IX. HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

	20°		21°		22°		23°	
Minutes.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	88.30	32.14	87.16	33.46	85.97	34.73	84.73	35.97
2	88.26	32.18	87.12	33.50	85.93	34.77	84.69	36.01
4	88.23	32.23	87.08	33.54	85.89	34.82	84.65	36.05
6	88.19	32.27	87.04	33.59	85.85	34.86	84.61	36.09
8	88.15	32.32	87.00	33.63	85.80	34.90	84.57	36.13
10	88.11	32.36	86.96	33.67	85.76	34.94	84.52	36.17
12	88.08	32.41	86.92	33.72	85.72	34.98	84.48	36.21
14	88.04	32.45	86.88	33.76	85.68	35.02	84.44	36.25
16	88.00	32.49	86.84	33.80	85.64	35.07	84.40	36.29
18	87.96	32.54	86.80	33.84	85.60	35.11	84.35	36.33
20	87.93	32.58	86.77	33.89	85.56	35.15	84.31	36.37
22	87.89	32.63	86.73	33.93	85.52	35.19	84.27	36.41
24	87.85	32.67	86.69	33.97	85.48	35.23	84.23	36.45
26	87.81	32.72	86.65	34.01	85.44	35.27	84.18	36.49
28	87.77	32.76	86.61	34.06	85.40	35.31	84.14	36.53
30	87.74	32.80	86.57	34.10	85.36	35.36	84.10	36.57
32	87.70	32.85	86.53	34.14	85.31	35.40	84.06	36.61
34	87.66	32.89	86.49	34.18	85.27	35.44	84.01	36.65
36	87.62	32.93	86.45	34.23	85.23	35.48	83.97	36.69
38	87.58	32.98	86.41	34.27	85.19	35.52	83.93	36.73
40	87.54	33.02	86.37	34.31	85.15	35.56	83.89	36.77
42	87.51	33.07	86.33	34.35	85.11	35.60	83.84	36.80
44	87.47	33.11	86.29	34.40	85.07	35.64	83.80	36.84
46	87.43	33.15	86.25	34.44	85.02	35.68	83.76	36.88
48	87.39	33.20	86.21	34.48	84.98	35.72	83.72	36.92
50	87.35	33.24	86.17	34.52	84.94	35.76	83.67	36.96
52	87.31	33.28	86.13	34.57	84.90	35.80	83.63	37.00
54	87.27	33.33	86.09	34.61	84.86	35.85	83.59	37.04
56	87.24	33.37	86.05	34.65	84.82	35.89	83.54	37.08
58	87.20	33.41	86.01	34.69	84.77	35.93	83.50	37.12
60	87.16	33.46	85.97	34.73	84.73	35.97	83.46	37.16
c = 0.75.	0.70	0.26	0.70	0.27	0.69	0.29	0.69	0.30
c = 1.00.	0.94	0.35	0.93	0.37	0.92	0.38	0.92	0.40
c = 1.25.	1.17	0.44	1.16	0.46	1.15	0.48	1.15	0.50

TABLE IX. HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

Minutes	24°		25°		26°		27°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff Elev.	Hor. Dist.	Diff. Elev.
0	83.46	37.16	82.14	38.30	80.78	39.40	79.39	40.45
2	83.41	37.20	82.09	38.34	80.74	39.44	79.34	40.49
4	83.37	37.23	82.05	38.38	80.69	39.47	79.30	40.52
6	83.33	37.27	82.01	38.41	80.65	39.51	79.25	40.55
8	83.28	37.31	81.96	38.45	80.60	39.54	79.20	40.59
10	83.24	37.35	81.92	38.49	80.55	39.58	79.15	40.62
12	83.20	37.39	81.87	38.53	80.51	39.61	79.11	40.66
14	83.15	37.43	81.83	38.56	80.46	39.65	79.06	40.69
16	83.11	37.47	81.78	38.60	80.41	39.69	79.01	40.72
18	83.07	37.51	81.74	38.64	80.37	39.72	78.96	40.76
20	83.02	37.54	81.69	38.67	80.32	39.76	78.92	40.79
22	82.98	37.58	81.65	38.71	80.28	39.79	78.87	40.82
24	82.93	37.62	81.60	38.75	80.23	39.83	78.82	40.86
26	82.89	37.66	81.56	38.78	80.18	39.86	78.77	40.89
28	82.85	37.70	81.51	38.82	80.14	39.90	78.73	40.92
30	82.80	37.74	81.47	38.86	80.09	39.93	78.68	40.96
32	82.76	37.77	81.42	38.89	80.04	39.97	78.63	40.99
34	82.72	37.81	81.38	38.93	80.00	40.00	78.58	41.02
36	82.67	37.85	81.33	38.97	79.95	40.04	78.54	41.06
38	82.63	37.89	81.28	39.00	79.90	40.07	78.49	41.09
40	82.58	37.93	81.24	39.04	79.86	40.11	78.44	41.12
42	82.54	37.96	81.19	39.08	79.81	40.14	78.39	41.16
44	82.49	38.00	81.15	39.11	79.76	40.18	78.34	41.19
46	82.45	38.04	81.10	39.15	79.72	40.21	78.30	41.22
48	82.41	38.08	81.06	39.18	79.67	40.24	78.25	41.26
50	82.36	38.11	81.01	39.22	79.62	40.28	78.20	41.29
52	82.32	38.15	80.97	39.26	79.58	40.31	78.15	41.32
54	82.27	38.19	80.92	39.29	79.53	40.35	78.10	41.35
56	82.23	38.23	80.87	39.33	79.48	40.38	78.06	41.39
58	82.18	38.26	80.83	39.36	79.44	40.42	78.01	41.42
60	82.14	38.30	80.78	39.40	79.39	40.45	77.96	41.45
c = 0.75 .	0.68	0.31	0.68	0.32	0.67	0.33	0.66	0.35
c = 1.00 .	0.91	0.41	0.90	0.43	0.89	0.45	0.89	0.46
c = 1.25 .	1.14	0.52	1.13	0.54	1.12	0.56	1.11	0.58

TABLE IX. HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Concluded*

Minutes	28°		29°		30°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	77.96	41.45	76.50	42.40	75.00	43.30
2	77.91	41.48	76.45	42.43	74.95	43.33
4	77.86	41.52	76.40	42.46	74.90	43.36
6	77.81	41.55	76.35	42.49	74.85	43.39
8	77.77	41.58	76.30	42.53	74.80	43.42
10	77.72	41.61	76.25	42.56	74.75	43.45
12	77.67	41.65	76.20	42.59	74.70	43.47
14	77.62	41.68	76.15	42.62	74.65	43.50
16	77.57	41.71	76.10	42.65	74.60	43.53
18	77.52	41.74	76.05	42.68	74.55	43.56
20	77.48	41.77	76.00	42.71	74.49	43.59
22	77.42	41.81	75.95	42.74	74.44	43.62
24	77.38	41.84	75.90	42.77	74.39	43.65
26	77.33	41.87	75.85	42.80	74.34	43.67
28	77.28	41.90	75.80	42.83	74.29	43.70
30	77.23	41.93	75.75	42.86	74.24	43.73
32	77.18	41.97	75.70	42.89	74.19	43.76
34	77.13	42.00	75.65	42.92	74.14	43.79
36	77.09	42.03	75.60	42.95	74.09	43.82
38	77.04	42.06	75.55	42.98	74.04	43.84
40	76.99	42.09	75.50	43.01	73.99	43.87
42	76.94	42.12	75.45	43.04	73.93	43.90
44	76.89	42.15	75.40	43.07	73.88	43.93
46	76.84	42.19	75.35	43.10	73.83	43.95
48	76.79	42.22	75.30	43.13	73.78	43.98
50	76.74	42.25	75.25	43.16	73.73	44.01
52	76.69	42.28	75.20	43.18	73.68	44.04
54	76.64	42.31	75.15	43.21	73.63	44.07
56	76.59	42.34	75.10	43.24	73.58	44.09
58	76.55	42.37	75.05	43.27	73.52	44.12
60	76.50	42.40	75.00	43.30	73.47	44.15
c = 0.75..	0.66	0.36	0.65	0.37	0.65	0.38
c = 1.00..	0.88	0.48	0.87	0.49	0.86	0.51
c = 1.25..	1.10	0.60	1.09	0.62	1.08	0.64

TABLE X.*—MINUTES IN DECIMALS OF A DEGREE

′	0″	10″	15″	20″	30″	40″	45″	50″	′
0	.00000	.00278	.00417	.00556	.00833	.01111	.01250	.01389	0
1	.01667	.01944	.02083	.02222	.02500	.02778	.02917	.03055	1
2	.03333	.03611	.03750	.03889	.04167	.04444	.04583	.04722	2
3	.05000	.05278	.05417	.05556	.05833	.06111	.06250	.06389	3
4	.06667	.06944	.07083	.07222	.07500	.07778	.07917	.08056	4
5	.08333	.08611	.08750	.08889	.09167	.09444	.09583	.09722	5
6	.10000	.10278	.10417	.10556	.10833	.11111	.11250	.11389	6
7	.11667	.11944	.12083	.12222	.12500	.12778	.12917	.13056	7
8	.13333	.13611	.13750	.13889	.14167	.14444	.14583	.14722	8
9	.15000	.15278	.15417	.15556	.15833	.16111	.16250	.16389	9
10	.16667	.16944	.17083	.17222	.17500	.17778	.17917	.18056	10
11	.18333	.18611	.18750	.18889	.19167	.19444	.19583	.19722	11
12	.20000	.20278	.20417	.20556	.20833	.21111	.21250	.21389	12
13	.21667	.21944	.22083	.22222	.22500	.22778	.22917	.23056	13
14	.23333	.23611	.23750	.23889	.24167	.24444	.24583	.24722	14
15	.25000	.25278	.25417	.25556	.25833	.26111	.26250	.26389	15
16	.26667	.26944	.27083	.27222	.27500	.27778	.27917	.28056	16
17	.28333	.28611	.28750	.28889	.29167	.29444	.29583	.29722	17
18	.30000	.30278	.30417	.30556	.30833	.31111	.31250	.31389	18
19	.31667	.31944	.32083	.32222	.32500	.32778	.32917	.33056	19
20	.33333	.33611	.33750	.33889	.34167	.34444	.34583	.34722	20
21	.35000	.35278	.35417	.35556	.35833	.36111	.36250	.36389	21
22	.36667	.36944	.37083	.37222	.37500	.37778	.37917	.38056	22
23	.38333	.38611	.38750	.38889	.39167	.39444	.39583	.39722	23
24	.40000	.40278	.40417	.40556	.40833	.41111	.41250	.41389	24
25	.41667	.41944	.42083	.42222	.42500	.42778	.42917	.43056	25
26	.43333	.43611	.43750	.43889	.44167	.44444	.44583	.44722	26
27	.45000	.45278	.45417	.45556	.45833	.46111	.46250	.46389	27
28	.46667	.46944	.47083	.47222	.47500	.47778	.47917	.48056	28
29	.48333	.48611	.48750	.48889	.49167	.49444	.49583	.49722	29
30	.50000	.50278	.50417	.50556	.50833	.51111	.51250	.51389	30
31	.51667	.51944	.52083	.52222	.52500	.52778	.52917	.53056	31
32	.53333	.53611	.53750	.53889	.54167	.54444	.54583	.54722	32
33	.55000	.55278	.55417	.55556	.55833	.56111	.56250	.56389	33
34	.56667	.56944	.57083	.57222	.57500	.57778	.57917	.58056	34
35	.58333	.58611	.58750	.58889	.59167	.59444	.59583	.59722	35
36	.60000	.60278	.60417	.60556	.60833	.61111	.61250	.61389	36
37	.61667	.61944	.62083	.62222	.62500	.62778	.62917	.63056	37
38	.63333	.63611	.63750	.63889	.64167	.64444	.64583	.64722	38
39	.65000	.65278	.65417	.65556	.65833	.66111	.66250	.66389	39
40	.66667	.66944	.67083	.67222	.67500	.67778	.67917	.68056	40
41	.68333	.68611	.68750	.68889	.69167	.69444	.69583	.69722	41
42	.70000	.70278	.70417	.70556	.70833	.71111	.71250	.71389	42
43	.71667	.71944	.72083	.72222	.72500	.72778	.72917	.73056	43
44	.73333	.73611	.73750	.73889	.74167	.74444	.74583	.74722	44
45	.75000	.75278	.75417	.75556	.75833	.76111	.76250	.76389	45
46	.76667	.76944	.77083	.77222	.77500	.77778	.77917	.78056	46
47	.78333	.78611	.78750	.78889	.79167	.79444	.79583	.79722	47
48	.80000	.80278	.80417	.80556	.80833	.81111	.81250	.81389	48
49	.81667	.81944	.82083	.82222	.82500	.82778	.82917	.83056	49
50	.83333	.83611	.83750	.83889	.84167	.84444	.84583	.84722	50
51	.85000	.85278	.85417	.85556	.85833	.86111	.86250	.86389	51
52	.86667	.86944	.87083	.87222	.87500	.87778	.87917	.88056	52
53	.88333	.88611	.88750	.88889	.89167	.89444	.89583	.89722	53
54	.90000	.90278	.90417	.90556	.90833	.91111	.91250	.91389	54
55	.91667	.91944	.92083	.92222	.92500	.92778	.92917	.93056	55
56	.93333	.93611	.93750	.93889	.94167	.94444	.94583	.94722	56
57	.95000	.95278	.95417	.95556	.95833	.96111	.96250	.96389	57
58	.96667	.96944	.97083	.97222	.97500	.97778	.97917	.98056	58
59	.98333	.98611	.98750	.98889	.99167	.99444	.99583	.99722	59
′	0″	10″	15″	20″	30″	40″	45″	50″	′

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No. 100
Log. 000

Table XI.

No. 109
Log. 040

N.	0	1	2	3	4	5	6	7	8	9	Diff.
100	00 0000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	0300	0724	1147	1570	1993	2415	424
3	01 2837	3259	3680	4100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8700	9116	9532	9947	0361	0775	416
105	02 1189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	0195	0600	1004	1408	1812	2216	2619	3021	404
8	03 3424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426 04	7825	8223	8620	9017	9414	9811	0207	0602	0998	397

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
434	43.4	86.8	130.2	173.6	217.0	260.4	303.8	347.2	390.6
433	43.3	86.6	129.9	173.2	216.5	259.8	303.1	346.4	389.7
432	43.2	86.4	129.6	172.8	216.0	259.2	302.4	345.6	388.8
431	43.1	86.2	129.3	172.4	215.5	258.6	301.7	344.8	387.9
430	43.0	86.0	129.0	172.0	215.0	258.0	301.0	344.0	387.0
429	42.9	85.8	128.7	171.6	214.5	257.4	300.3	343.2	386.1
428	42.8	85.6	128.4	171.2	214.0	256.8	299.6	342.4	385.2
427	42.7	85.4	128.1	170.8	213.5	256.2	298.9	341.6	384.3
426	42.6	85.2	127.8	170.4	213.0	255.6	298.2	340.8	383.4
425	42.5	85.0	127.5	170.0	212.5	255.0	297.5	340.0	382.5
424	42.4	84.8	127.2	169.6	212.0	254.4	296.8	339.2	381.6
423	42.3	84.6	126.9	169.2	211.5	253.8	296.1	338.4	380.7
422	42.2	84.4	126.6	168.8	211.0	253.2	295.4	337.6	379.8
421	42.1	84.2	126.3	168.4	210.5	252.6	294.7	336.8	378.9
420	42.0	84.0	126.0	168.0	210.0	252.0	294.0	336.0	378.0
419	41.9	83.8	125.7	167.6	209.5	251.4	293.3	335.2	377.1
418	41.8	83.6	125.4	167.2	209.0	250.8	292.6	334.4	376.2
417	41.7	83.4	125.1	166.8	208.5	250.2	291.9	333.6	375.3
416	41.6	83.2	124.8	166.4	208.0	249.6	291.2	332.8	374.4
415	41.5	83.0	124.5	166.0	207.5	249.0	290.5	332.0	373.5
414	41.4	82.8	124.2	165.6	207.0	248.4	289.8	331.2	372.6
413	41.3	82.6	123.9	165.2	206.5	247.8	289.1	330.4	371.7
412	41.2	82.4	123.6	164.8	206.0	247.2	288.4	329.6	370.8
411	41.1	82.2	123.3	164.4	205.5	246.6	287.7	328.8	369.9
410	41.0	82.0	123.0	164.0	205.0	246.0	287.0	328.0	369.0
409	40.9	81.8	122.7	163.6	204.5	245.4	286.3	327.2	368.1
408	40.8	81.6	122.4	163.2	204.0	244.8	285.6	326.4	367.2
407	40.7	81.4	122.1	162.8	203.5	244.2	284.9	325.6	366.3
406	40.6	81.2	121.8	162.4	203.0	243.6	284.2	324.8	365.4
405	40.5	81.0	121.5	162.0	202.5	243.0	283.5	324.0	364.5
404	40.4	80.8	121.2	161.6	202.0	242.4	282.8	323.2	363.6
403	40.3	80.6	120.9	161.2	201.5	241.8	282.1	322.4	362.7
402	40.2	80.4	120.6	160.8	201.0	241.2	281.4	321.6	361.8
401	40.1	80.2	120.3	160.4	200.5	240.6	280.7	320.8	360.9
400	40.0	80.0	120.0	160.0	200.0	240.0	280.0	320.0	360.0
399	39.9	79.8	119.7	159.6	199.5	239.4	279.3	319.2	359.1
398	39.8	79.6	119.4	159.2	199.0	238.8	278.6	318.4	358.2
397	39.7	79.4	119.1	158.8	198.5	238.2	277.9	317.6	357.3
396	39.6	79.2	118.8	158.4	198.0	237.6	277.2	316.8	356.4
395	39.5	79.0	118.5	158.0	197.5	237.0	276.5	316.0	351.5

No. 110
Log. 041

TABLE XI.—*Continued.*

No. 119
Log. 078

N.	0	1	2	3	4	5	6	7	8	9	Diff.
110	04 1393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	0380	0766	1153	1538	1924	2309	2694	386
3	05 3078	3463	3846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	0320	379
115	06 0698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	0038	0407	0776	1145	1514	370
8	07 1882	2250	2617	2935	3352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
395	39.5	79.0	118.5	158.0	197.5	237.0	276.5	316.0	355.5
394	39.4	78.8	118.2	157.6	197.0	236.4	275.8	315.2	354.6
393	39.3	78.6	117.9	157.2	196.5	235.8	275.1	314.4	353.7
392	39.2	78.4	117.6	156.8	196.0	235.2	274.4	313.6	352.8
391	39.1	78.2	117.3	156.4	195.5	234.6	273.7	312.8	351.9
390	39.0	78.0	117.0	156.0	195.0	234.0	273.0	312.0	351.0
389	38.9	77.8	116.7	155.6	194.5	233.4	272.3	311.2	350.1
388	38.8	77.6	116.4	155.2	194.0	232.8	271.6	310.4	349.2
387	38.7	77.4	116.1	154.8	193.5	232.2	270.9	309.6	348.3
386	38.6	77.2	115.8	154.4	193.0	231.6	270.2	308.8	347.4
385	38.5	77.0	115.5	154.0	192.5	231.0	269.5	308.0	346.5
384	38.4	76.8	115.2	153.6	192.0	230.4	268.8	307.2	345.6
383	38.3	76.6	114.9	153.2	191.5	229.8	268.1	306.4	344.7
382	38.2	76.4	114.6	152.8	191.0	229.2	267.4	305.6	343.8
381	38.1	76.2	114.3	152.4	190.5	228.6	266.7	304.8	342.9
380	38.0	76.0	114.0	152.0	190.0	228.0	266.0	304.0	342.0
379	37.9	75.8	113.7	151.6	189.5	227.4	265.3	303.2	341.1
378	37.8	75.6	113.4	151.2	189.0	226.8	264.6	302.4	340.2
377	37.7	75.4	113.1	150.8	188.5	226.2	263.9	301.6	339.3
376	37.6	75.2	112.8	150.4	188.0	225.6	263.2	300.8	338.4
375	37.5	75.0	112.5	150.0	187.5	225.0	262.5	300.0	337.5
374	37.4	74.8	112.2	149.6	187.0	224.4	261.8	299.2	336.6
373	37.3	74.6	111.9	149.2	186.5	223.8	261.1	298.4	335.7
372	37.2	74.4	111.6	148.8	186.0	223.2	260.4	297.6	334.8
371	37.1	74.2	111.3	148.4	185.5	222.6	259.7	296.8	333.9
370	37.0	74.0	111.0	148.0	185.0	222.0	259.0	296.0	333.0
369	36.9	73.8	110.7	147.6	184.5	221.4	258.3	295.2	332.1
368	36.8	73.6	110.4	147.2	184.0	220.8	257.6	294.4	331.2
367	36.7	73.4	110.1	146.8	183.5	220.2	256.9	293.6	330.3
366	36.6	73.2	109.8	146.4	183.0	219.6	256.2	292.8	329.4
365	36.5	73.0	109.5	146.0	182.5	219.0	255.7	292.0	328.5
364	36.4	72.8	109.2	145.6	182.0	218.4	254.8	291.2	327.6
363	36.3	72.6	108.9	145.2	181.5	217.8	254.1	290.4	326.7
362	36.2	72.4	108.6	144.8	181.0	217.2	253.4	289.6	325.8
361	36.1	72.2	108.3	144.4	180.5	216.6	252.7	288.8	324.9
360	36.0	72.0	108.0	144.0	180.0	216.0	252.0	288.0	324.0
359	35.9	71.8	107.7	143.6	179.5	215.4	251.3	287.2	323.1
358	35.8	71.6	107.4	143.2	179.0	214.8	250.6	286.4	322.2
357	35.7	71.4	107.1	142.8	178.5	214.2	249.9	285.6	321.3
356	35.6	71.2	106.8	142.4	178.0	213.6	249.2	284.8	320.4

TABLE XI.—*Continued.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
120	07 9181	9543	9904	0266	0626	0987	1347	1707	2067	2426	360
1	08 2785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
3	9905	0258	0611	0963	1315	1667	2018	2370	2721	3071	352
4	09 3422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	0026	346
6	10 0371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	0253	338
9	11 0590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	0245	330
2	12 0574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	0012	323
	13										

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
355	35.5	71.0	106.5	142.0	177.5	213.0	248.5	284.0	319.5
354	35.4	70.8	106.2	141.6	177.0	212.4	247.8	283.2	318.6
353	35.3	70.6	105.9	141.2	176.5	211.8	247.1	282.4	317.7
352	35.2	70.4	105.6	140.8	176.0	211.2	246.4	281.6	316.8
351	35.1	70.2	105.3	140.4	175.5	210.6	245.7	280.8	315.9
350	35.0	70.0	105.0	140.0	175.0	210.0	245.0	280.0	315.0
349	34.9	69.8	104.7	139.6	174.5	209.4	244.3	279.2	314.1
348	34.8	69.6	104.4	139.2	174.0	208.8	243.6	278.4	313.2
347	34.7	69.4	104.1	138.8	173.5	208.2	242.9	277.6	312.3
346	34.6	69.2	103.8	138.4	173.0	207.6	242.2	276.8	311.4
345	34.5	69.0	103.5	138.0	172.5	207.0	241.5	276.0	310.5
344	34.4	68.8	103.2	137.6	172.0	206.4	240.8	275.2	309.6
343	34.3	68.6	102.9	137.2	171.5	205.8	240.1	274.4	308.7
342	34.2	68.4	102.6	136.8	171.0	205.2	239.4	273.6	307.8
341	34.1	68.2	102.3	136.4	170.5	204.6	238.7	272.8	306.9
340	34.0	68.0	102.0	136.0	170.0	204.0	238.0	272.0	306.0
339	33.9	67.8	101.7	135.6	169.5	203.4	237.3	271.2	305.1
338	33.8	67.6	101.4	135.2	169.0	202.8	236.6	270.4	304.2
337	33.7	67.4	101.1	134.8	168.5	202.2	235.9	269.6	303.3
336	33.6	67.2	100.8	134.4	168.0	201.6	235.2	268.8	302.4
335	33.5	67.0	100.5	134.0	167.5	201.0	234.5	268.0	301.5
334	33.4	66.8	100.2	133.6	167.0	200.4	233.8	267.2	300.6
333	33.3	66.6	99.9	133.2	166.5	199.8	233.1	266.4	299.7
332	33.2	66.4	99.6	132.8	166.0	199.2	232.4	265.6	298.8
331	33.1	66.2	99.3	132.4	165.5	198.6	231.7	264.8	297.9
330	33.0	66.0	99.0	132.0	165.0	198.0	231.0	264.0	297.0
329	32.9	65.8	98.7	131.6	164.5	197.4	230.3	263.2	296.1
328	32.8	65.6	98.4	131.2	164.0	196.8	229.6	262.4	295.2
327	32.7	65.4	98.1	130.8	163.5	196.2	228.9	261.6	294.3
326	32.6	65.2	97.8	130.4	163.0	195.6	228.2	260.8	293.4
325	32.5	65.0	97.5	130.0	162.5	195.0	227.5	260.0	292.5
324	32.4	64.8	97.2	129.6	162.0	194.4	226.8	259.2	291.6
323	32.3	64.6	96.9	129.2	161.5	193.8	226.1	258.4	290.7
322	32.2	64.4	96.6	128.8	161.0	193.2	225.4	257.6	289.8

TABLE XI.—*Continued.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
135	13 0334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
8	9879	0194	0508	0822	1136	1450	1763	2076	2389	2702	314
9	14 3015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
1	9219	9527	9835	0142	0449	0756	1063	1370	1676	1982	307
2	15 2288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
4	8362	8664	8965	9266	9567	9868	0168	0469	0769	1068	301
145	16 1368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
8	17 0262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
321	32.1	64.2	96.3	128.4	160.5	192.6	224.7	256.8	288.9
320	32.0	64.0	96.0	128.0	160.0	192.0	224.0	256.0	288.0
319	31.9	63.8	95.7	127.6	159.5	191.4	223.3	255.2	287.1
318	31.8	63.6	95.4	127.2	159.0	190.8	222.6	254.4	286.2
317	31.7	63.4	95.1	126.8	158.5	190.2	221.9	253.6	285.3
316	31.6	63.2	94.8	126.4	158.0	189.6	221.2	252.8	284.4
315	31.5	63.0	94.5	126.0	157.5	189.0	220.5	252.0	283.5
314	31.4	62.8	94.2	125.6	157.0	188.4	219.8	251.2	282.6
313	31.3	62.6	93.9	125.2	156.5	187.8	219.1	250.4	281.7
312	31.2	62.4	93.6	124.8	156.0	187.2	218.4	249.6	280.8
311	31.1	62.2	93.3	124.4	155.5	186.6	217.7	248.8	279.9
310	31.0	62.0	93.0	124.0	155.0	186.0	217.0	248.0	279.0
309	30.9	61.8	92.7	123.6	154.5	185.4	216.3	247.2	278.1
308	30.8	61.6	92.4	123.2	154.0	184.8	215.6	246.4	277.2
307	30.7	61.4	92.1	122.8	153.5	184.2	214.9	245.6	276.3
306	30.6	61.2	91.8	122.4	153.0	183.6	214.2	244.8	275.4
305	30.5	61.0	91.5	122.0	152.5	183.0	213.5	244.0	274.5
304	30.4	60.8	91.2	121.6	152.0	182.4	212.8	243.2	273.6
303	30.3	60.6	90.9	121.2	151.5	181.8	212.1	242.4	272.7
302	30.2	60.4	90.6	120.8	151.0	181.2	211.4	241.6	271.8
301	30.1	60.2	90.3	120.4	150.5	180.6	210.7	240.8	270.9
300	30.0	60.0	90.0	120.0	150.0	180.0	210.0	240.0	270.0
299	29.9	59.8	89.7	119.6	149.5	179.4	209.3	239.2	269.1
298	29.8	59.6	89.4	119.2	149.0	178.8	208.6	238.4	268.2
297	29.7	59.4	89.1	118.8	148.5	178.2	207.9	237.6	267.3
296	29.6	59.2	88.8	118.4	148.0	177.6	207.2	236.8	266.4
295	29.5	59.0	88.5	118.0	147.5	177.0	206.5	236.0	265.5
294	29.4	58.8	88.2	117.6	147.0	176.4	205.8	235.2	264.6
293	29.3	58.6	87.9	117.2	146.5	175.8	205.1	234.4	263.7
292	29.2	58.4	87.6	116.8	146.0	175.2	204.4	233.6	262.8
291	29.1	58.2	87.3	116.4	145.5	174.6	203.7	232.8	261.9
290	29.0	58.0	87.0	116.0	145.0	174.0	203.0	232.0	261.0
289	28.9	57.8	86.7	115.6	144.5	173.4	202.3	231.2	260.1
288	28.8	57.6	86.4	115.2	144.0	172.8	201.6	230.4	259.2
287	28.7	57.4	86.1	114.8	143.5	172.2	200.9	229.6	258.3
286	28.6	57.2	85.8	114.4	143.0	171.6	200.2	228.8	257.4

No. 150
Log. 176

TABLE XI.—*Continued.*

No. 169
Log. 230

N.	0	1	2	3	4	5	6	7	8	9	Diff.
150	17 6091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
1	8977	9264	9552	9839	0126	0413	0699	0986	1272	1558	287
2	18 1844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	0051	281
155	19 0332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
8	8657	8932	9206	9481	9755	0029	0303	0577	0850	1124	274
9	20 1397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783	0051	0319	0586	0353	1121	1388	1654	1921	267
3	21 2188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	22 0108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887 23	8144	8400	8657	8913	9170	9426	9682	9938	0193	256

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
285	28.5	57.0	85.5	114.0	142.5	171.0	199.5	228.0	256.5
284	28.4	56.8	85.2	113.6	142.0	170.4	198.8	227.2	255.6
283	28.3	56.6	84.9	113.2	141.5	169.8	198.1	226.4	254.7
282	28.2	56.4	84.6	112.8	141.0	169.2	197.4	225.6	253.8
281	28.1	56.2	84.3	112.4	140.5	168.6	196.7	224.8	252.9
280	28.0	56.0	84.0	112.0	140.0	168.0	196.0	224.0	252.0
279	27.9	55.8	83.7	111.6	139.5	167.4	195.3	223.2	251.1
278	27.8	55.6	83.4	111.2	139.0	166.8	194.6	222.4	250.2
277	27.7	55.4	83.1	110.8	138.5	166.2	193.9	221.6	249.3
276	27.6	55.2	82.8	110.4	138.0	165.6	193.2	220.8	248.4
275	27.5	55.0	82.5	110.0	137.5	165.0	192.5	220.0	247.5
274	27.4	54.8	82.2	109.6	137.0	164.4	191.8	219.2	246.6
273	27.3	54.6	81.9	109.2	136.5	163.8	191.1	218.4	245.7
272	27.2	54.4	81.6	108.8	136.0	163.2	190.4	217.6	244.8
271	27.1	54.2	81.3	108.4	135.5	162.6	189.7	216.8	243.9
270	27.0	54.0	81.0	108.0	135.0	162.0	189.0	216.0	243.0
269	26.9	53.8	80.7	107.6	134.5	161.4	188.3	215.2	242.1
268	26.8	53.6	80.4	107.2	134.0	160.8	187.6	214.4	241.2
267	26.7	53.4	80.1	106.8	133.5	160.2	186.9	213.6	240.3
266	26.6	53.2	79.8	106.4	133.0	159.6	186.2	212.8	239.4
265	26.5	53.0	79.5	106.0	132.5	159.0	185.5	212.0	238.5
264	26.4	52.8	79.2	105.6	132.0	158.4	184.8	211.2	237.6
263	26.3	52.6	78.9	105.2	131.5	157.8	184.1	210.4	236.7
262	26.2	52.4	78.6	104.8	131.0	157.2	183.4	209.6	235.8
261	26.1	52.2	78.3	104.4	130.5	156.6	182.7	208.8	234.9
260	26.0	52.0	78.0	104.0	130.0	156.0	182.0	208.0	234.0
259	25.9	51.8	77.7	103.6	129.5	155.4	181.3	207.2	233.1
258	25.8	51.6	77.4	103.2	129.0	154.8	180.6	206.4	232.2
257	25.7	51.4	77.1	102.8	128.5	154.2	179.9	205.6	231.3
256	25.6	51.2	76.8	102.4	128.0	153.6	179.2	204.8	230.4
255	25.5	51.0	76.5	102.0	127.5	153.0	178.5	204.0	229.5

No. 170
Log. 230

TABLE XI.—*Continued.*

No. 189
Log. 278

N.	0	1	2	3	4	5	6	7	8	9	Diff.
170	23 0449	0704	0960	1215	1470	1724	1979	2234	2488	2742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	0050	0300	250
4	24 0549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	0176	245
8	25 0420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
2	26 0071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
6	9513	9746	9980	0213	0446	0679	0912	1144	1377	1609	233
7	27 1842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
255	25.5	51.0	76.5	102.0	127.5	153.0	178.5	204.0	229.5
254	25.4	50.8	76.2	101.6	127.0	152.4	177.8	203.2	228.6
253	25.3	50.6	75.9	101.2	126.5	151.8	177.1	202.4	227.7
252	25.2	50.4	75.6	100.8	126.0	151.2	176.4	201.6	226.8
251	25.1	50.2	75.3	100.4	125.5	150.6	175.7	200.8	225.9
250	25.0	50.0	75.0	100.0	125.0	150.0	175.0	200.0	225.0
249	24.9	49.8	74.7	99.6	124.5	149.4	174.3	199.2	224.1
248	24.8	49.6	74.4	99.2	124.0	148.8	173.6	198.4	223.2
247	24.7	49.4	74.1	98.8	123.5	148.2	172.9	197.6	222.3
246	24.6	49.2	73.8	98.4	123.0	147.6	172.2	196.8	221.4
245	24.5	49.0	73.5	98.0	122.5	147.0	171.5	196.0	220.5
244	24.4	48.8	73.2	97.6	122.0	146.4	170.8	195.2	219.6
243	24.3	48.6	72.9	97.2	121.5	145.8	170.1	194.4	218.7
242	24.2	48.4	72.6	96.8	121.0	145.2	169.4	193.6	217.8
241	24.1	48.2	72.3	96.4	120.5	144.6	168.7	192.8	216.9
240	24.0	48.0	72.0	96.0	120.0	144.0	168.0	192.0	216.0
239	23.9	47.8	71.7	95.6	119.5	143.4	167.3	191.2	215.1
238	23.8	47.6	71.4	95.2	119.0	142.8	166.6	190.4	214.2
237	23.7	47.4	71.1	94.8	118.5	142.2	165.9	189.6	213.3
236	23.6	47.2	70.8	94.4	118.0	141.6	165.2	188.8	212.4
235	23.5	47.0	70.5	94.0	117.5	141.0	164.5	188.0	211.5
234	23.4	46.8	70.2	93.6	117.0	140.4	163.8	187.2	210.6
233	23.3	46.6	69.9	93.2	116.5	139.8	163.1	186.4	209.7
232	23.2	46.4	69.6	92.8	116.0	139.2	162.4	185.6	208.8
231	23.1	46.2	69.3	92.4	115.5	138.6	161.7	184.8	207.9
230	23.0	46.0	69.0	92.0	115.0	138.0	161.0	184.0	207.0
229	22.9	45.8	68.7	91.6	114.5	137.4	160.3	183.2	206.1
228	22.8	45.6	68.4	91.2	114.0	136.8	159.6	182.4	205.2
227	22.7	45.4	68.1	90.8	113.5	136.2	158.9	181.6	204.3
226	22.6	45.2	67.8	90.4	113.0	135.6	158.2	180.8	203.4

No. 190 · Log. 278

No. 214 · Log. 332

TABLE XI.—*Continued.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
190	27 8754	8982	9211	9439	9667	9895	0123	0351	0578	0806	228
1	28 1033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	29 0035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
9	8853	9071	9289	9507	9725	9943	0161	0378	0595	0813	218
200	30 1030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
4	9630	9843	0056	0268	0481	0693	0906	1118	1330	1542	212
205	31 1754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
6	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
8	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
9	32 0146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
3	8380	8583	8787	8991	9194	9398	9601	9805	0008	0211	203
4	33 0414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
225	22.5	45.0	67.5	90.0	112.5	135.0	157.5	180.0	202.5
224	22.4	44.8	67.2	89.6	112.0	134.4	156.8	179.2	201.6
223	22.3	44.6	66.9	89.2	111.5	133.8	156.1	178.4	200.7
222	22.2	44.4	66.6	88.8	111.0	133.2	155.4	177.6	199.8
221	22.1	44.2	66.3	88.4	110.5	132.6	154.7	176.8	198.9
220	22.0	44.0	66.0	88.0	110.0	132.0	154.0	176.0	198.0
219	21.9	43.8	65.7	87.6	109.5	131.4	153.3	175.2	197.1
218	21.8	43.6	65.4	87.2	109.0	130.8	152.6	174.4	196.2
217	21.7	43.4	65.1	86.8	108.5	130.2	151.9	173.6	195.3
216	21.6	43.2	64.8	86.4	108.0	129.6	151.2	172.8	194.4
215	21.5	43.0	64.5	86.0	107.5	129.0	150.5	172.0	193.5
214	21.4	42.8	64.2	85.6	107.0	128.4	149.8	171.2	192.6
213	21.3	42.6	63.9	85.2	106.5	127.8	149.1	170.4	191.7
212	21.2	42.4	63.6	84.8	106.0	127.2	148.4	169.6	190.8
211	21.1	42.2	63.3	84.4	105.5	126.6	147.7	168.8	189.9
210	21.0	42.0	63.0	84.0	105.0	126.0	147.0	168.0	189.0
209	20.9	41.8	62.7	83.6	104.5	125.4	146.3	167.2	188.1
208	20.8	41.6	62.4	83.2	104.0	124.8	145.6	166.4	187.2
207	20.7	41.4	62.1	82.8	103.5	124.2	144.9	165.6	186.3
206	20.6	41.2	61.8	82.4	103.0	123.6	144.2	164.8	185.4
205	20.5	41.0	61.5	82.0	102.5	123.0	143.5	164.0	184.5
204	20.4	40.8	61.2	81.6	102.0	122.4	142.8	163.2	183.6
203	20.3	40.6	60.9	81.2	101.5	121.8	142.1	162.4	182.7
202	20.2	40.4	60.6	80.8	101.0	121.2	141.4	161.6	181.8

Table XI.—*Continued.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
215	33 2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
8	8456	8656	8855	9054	9253	9451	9650	9849	0047	0246	199
9	34 0444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
3	8305	8500	8694	8889	9033	9278	9472	9666	9860	0054	194
4	35 0248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
9	9835	0025	0215	0404	0593	0783	0972	1161	1350	1539	189
230	36 1728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
4	9216	9401	9587	9772	9958	0143	0328	0513	0698	0883	185
235	37 1068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	0030	181
	38										

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
202	20.2	40.4	60.6	80.8	101.0	121.2	141.4	161.6	181.8
201	20.1	40.2	60.3	80.4	100.5	120.6	140.7	160.8	180.9
200	20.0	40.0	60.0	80.0	100.0	120.0	140.0	160.0	180.0
199	19.9	39.8	59.7	79.6	99.5	119.4	139.3	159.2	179.1
198	19.8	39.6	59.4	79.2	99.0	118.8	138.6	158.4	178.2
197	19.7	39.4	59.1	78.8	98.5	118.2	137.9	157.6	177.3
196	19.6	39.2	58.8	78.4	98.0	117.6	137.2	156.8	176.4
195	19.5	39.0	58.5	78.0	97.5	117.0	136.5	156.0	175.5
194	19.4	38.8	58.2	77.6	97.0	116.4	135.8	155.2	174.6
193	19.3	38.6	57.9	77.2	96.5	115.8	135.1	154.4	173.7
192	19.2	38.4	57.6	76.8	96.0	115.2	134.4	153.6	172.8
191	19.1	38.2	57.3	76.4	95.5	114.6	133.7	152.8	171.9
190	19.0	38.0	57.0	76.0	95.0	114.0	133.0	152.0	171.0
189	18.9	37.8	56.7	75.6	94.5	113.4	132.3	151.2	170.1
188	18.8	37.6	56.4	75.2	94.0	112.8	131.6	150.4	169.2
187	18.7	37.4	56.1	74.8	93.5	112.2	130.9	149.6	168.3
186	18.6	37.2	55.8	74.4	93.0	111.6	130.2	148.8	167.4
185	18.5	37.0	55.5	74.0	92.5	111.0	129.5	148.0	166.5
184	18.4	36.8	55.2	73.6	92.0	110.4	128.8	147.2	165.6
183	18.3	36.6	54.9	73.2	91.5	109.8	128.1	146.4	164.7
182	18.2	36.4	54.6	72.8	91.0	109.2	127.4	145.6	163.8
181	18.1	36.2	54.3	72.4	90.5	108.6	126.7	144.8	162.9
180	18.0	36.0	54.0	72.0	90.0	108.0	126.0	144.0	162.0
179	17.9	35.8	53.7	71.6	89.5	107.4	125.3	143.2	161.1

No. 240 TABLE XI.—*Continued.* No. 269
LOG. 380 LOG. 431

N.	0	1	2	3	4	5	6	7	8	9	Diff.
240	38 0211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
4	7390	7568	7746	7924	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	0051	0228	0405	0582	0759	177
6	39 0935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
1	9674	9847	0020	0192	0365	0538	0711	0883	1056	1228	173
2	40 1401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
7	9933	0102	0271	0440	0609	0777	0946	1114	1283	1451	169
8	41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
3	9956	0121	0286	0451	0616	0781	0945	1110	1275	1439	165
4	42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
6	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
9	9752 43	9914	0075	0236	0398	0559	0720	0881	1042	1203	161

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
178	17.8	35.6	53.4	71.2	89.0	106.8	124.6	142.4	160.2
177	17.7	35.4	53.1	70.8	88.5	106.2	123.9	141.6	159.3
176	17.6	35.2	52.8	70.4	83.0	105.6	123.2	140.8	158.4
175	17.5	35.0	52.5	70.0	87.5	105.0	122.5	140.0	157.5
174	17.4	34.8	52.2	69.6	87.0	104.4	121.8	139.2	156.6
173	17.3	34.6	51.9	69.2	86.5	103.8	121.1	138.4	155.7
172	17.2	34.4	51.6	68.8	86.0	103.2	120.4	137.6	154.8
171	17.1	34.2	51.3	68.4	85.5	102.6	119.7	136.8	153.9
170	17.0	34.0	51.0	68.0	85.0	102.0	119.0	136.0	153.0
169	16.9	33.8	50.7	67.6	84.5	101.4	118.3	135.2	152.1
168	16.8	33.6	50.4	67.2	84.0	100.8	117.6	134.4	151.2
167	16.7	33.4	50.1	66.8	83.5	100.2	116.9	133.6	150.3
166	16.6	33.2	49.8	66.4	83.0	99.6	116.2	132.8	149.4
165	16.5	33.0	49.5	66.0	82.5	99.0	115.5	132.0	148.5
164	16.4	32.8	49.2	65.6	82.0	98.4	114.8	131.2	147.6
163	16.3	32.6	48.9	65.2	81.5	97.8	114.1	130.4	146.7
162	16.2	32.4	48.5	64.8	81.0	97.2	113.4	129.6	145.8
161	16.1	32.2	48.3	64.4	80.5	96.6	112.7	128.8	144.9

No. 270
Log. 431

TABLE XI.—*Continued.*

No. 299
Log. 476

N.	0	1	2	3	4	5	6	7	8	9	Diff.
270	43 1364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
1	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
2	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
3	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
4	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	0122	0279	0437	0594	0752	158
6	44 0909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
9	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
1	8706	8861	9015	9170	9324	9478	9633	9787	9941	0095	154
2	45 0249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
4	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
6	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
7	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
8	9392	9543	9694	9845	9995	0146	0296	0447	0597	0748	151
9	46 0898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
1	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
2	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
3	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
4	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	0116	0263	0410	0557	0704	0851	0998	1145	147
6	47 1292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
8	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
9	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
161	16.1	32.2	48.3	64.4	80.5	96.6	112.7	128.8	144.9
160	16.0	32.0	48.0	64.0	80.0	96.0	112.0	128.0	144.0
159	15.9	31.8	47.7	63.6	79.5	95.4	111.3	127.2	143.1
158	15.8	31.6	47.4	63.2	79.0	94.8	110.6	126.4	142.2
157	15.7	31.4	47.1	62.8	78.5	94.2	109.9	125.6	141.3
156	15.6	31.2	46.8	62.4	78.0	93.6	109.2	124.8	140.4
155	15.5	31.0	46.5	62.0	77.5	93.0	108.5	124.0	139.5
154	15.4	30.8	46.2	61.6	77.0	92.4	107.8	123.2	138.6
153	15.3	30.6	45.9	61.2	76.5	91.8	107.1	122.4	137.7
152	15.2	30.4	45.6	60.8	76.0	91.2	106.4	121.6	136.8
151	15.1	30.2	45.3	60.4	75.5	90.6	105.7	120.8	135.9
150	15.0	30.0	45.0	60.0	75.0	90.0	105.0	120.0	135.0
149	14.9	29.8	44.7	59.6	74.5	89.4	104.3	119.2	134.1
148	14.8	29.6	44.4	59.2	74.0	88.8	103.6	118.4	133.2
147	14.7	29.4	44.1	58.8	73.5	88.2	102.9	117.6	132.3
146	14.6	29.2	43.8	58.4	73.0	87.6	102.2	116.8	131.4
145	14.5	29.0	43.5	58.0	72.5	87.0	101.5	116.0	130.5
144	14.4	28.8	43.2	57.6	72.0	86.4	100.8	115.2	129.6
143	14.3	28.6	42.9	57.2	71.5	85.8	100.1	114.4	128.7
142	14.2	28.4	42.6	56.8	71.0	85.2	99.4	113.6	127.8
141	14.1	28.2	42.3	56.4	70.5	84.6	98.7	112.8	126.9
140	14.0	28.0	42.0	56.0	70.0	84.0	98.0	112.0	126.0

No. 300
Log. 477

TABLE XI.—*Continued.*

No. 339
Log. 531

N.	0	1	2	3	4	5	6	7	8	9	Diff.
300	47 7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
1	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
2	48 0007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
3	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
4	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
8	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
9	9958	0099	0239	0380	0520	0661	0801	0941	1081	1222	140
310	49 1362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
1	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
2	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
3	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
4	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
6	9687	9824	9962	0099	0236	0374	0511	0648	0785	0922	137
7	50 1059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
8	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
9	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
1	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
2	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
3	9203	9337	9471	9606	9740	9874	0009	0143	0277	0411	134
4	51 0545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
6	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
7	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
8	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
9	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
1	9828	9959	0090	0221	0353	0484	0615	0745	0876	1007	131
2	52 1138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
3	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
4	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
6	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
7	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
8	8917	9045	9174	9302	9430	9559	9687	9815	9943	0072	128
9	53 0200	0328	0456	0584	0712	0840	0968	1096	1223	1351	128

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
139	13.9	27.8	41.7	55.6	69.5	83.4	97.3	111.2	125.1
138	13.8	27.6	41.4	55.2	69.0	82.8	96.6	110.4	124.2
137	13.7	27.4	41.1	54.8	68.5	82.2	95.9	109.6	123.3
136	13.6	27.2	40.8	54.4	68.0	81.6	95.2	108.8	122.4
135	13.5	27.0	40.5	54.0	67.5	81.0	94.5	108.0	121.5
134	13.4	26.8	40.2	53.6	67.0	80.4	93.8	107.2	120.6
133	13.3	26.6	39.9	53.2	66.5	79.8	93.1	106.4	119.7
132	13.2	26.4	39.6	52.8	66.0	79.2	92.4	105.6	118.8
131	13.1	26.2	39.3	52.4	65.5	78.6	91.7	104.8	117.9
130	13.0	26.0	39.0	52.0	65.0	78.0	91.0	104.0	117.0
129	12.9	25.8	38.7	51.6	64.5	77.4	90.3	103.2	116.1
128	12.8	25.6	38.4	51.2	64.0	76.8	89.6	102.4	115.2
127	12.7	25.4	38.1	50.8	63.5	76.2	88.9	101.6	114.3

TABLE XI.—*Continued.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
340	53 1479	1607	1734	1862	1990	2117	2245	2372	2500	2627	128
1	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
2	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
3	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
4	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
6	9076	9202	9327	9452	9578	9703	9829	9954	0079	0204	125
7	54 0329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
8	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
9	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
1	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
2	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
3	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
4	9003	9126	9249	9371	9494	9616	9739	9861	9984	0106	123
355	55 0228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
8	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
9	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
1	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
2	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
3	9907	0026	0146	0265	0385	0504	0624	0743	0863	0982	119
4	56 1101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
1	9374	9491	9608	9725	9842	9959	0076	0193	0309	0426	117
2	57 0543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
4	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
6	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
7	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
8	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
9	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
128	12.8	25.6	38.4	51.2	64.0	76.8	89.6	102.4	115.2
127	12.7	25.4	38.1	50.8	63.5	76.2	88.9	101.6	114.3
126	12.6	25.2	37.8	50.4	63.0	75.6	88.2	100.8	113.4
125	12.5	25.0	37.5	50.0	62.5	75.0	87.5	100.0	112.5
124	12.4	24.8	37.2	49.6	62.0	74.4	86.8	99.2	111.6
123	12.3	24.6	36.9	49.2	61.5	73.8	86.1	98.4	110.7
122	12.2	24.4	36.6	48.8	61.0	73.2	85.4	97.6	109.8
121	12.1	24.2	36.3	48.4	60.5	72.6	84.7	96.8	108.9
120	12.0	24.0	36.0	48.0	60.0	72.0	84.0	96.0	108.0
119	11.9	23.8	35.7	47.6	59.5	71.4	83.3	95.2	107.1

No. 380 No. 414
Log. 579 Table XI.—*Continued.* Log. 617

N.	0	1	2	3	4	5	6	7	8	9	Diff.
380	57 9784	9898	0012	0126	0241	0355	0469	0583	0697	0811	114
1	58 0925	1039	1153	1267	1381	1495	1608	1722	1836	1950	
2	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	
3	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	
4	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	
6	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	
7	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
8	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	
9	9950	0061	0173	0284	0396	0507	0619	0730	0842	0953	
390	59 1065	1176	1287	1399	1510	1621	1732	1843	1955	2066	
1	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
2	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	
3	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	
4	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
6	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	
7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	
8	9883	9992	0101	0210	0319	0428	0537	0646	0755	0864	109
9	60 0973	1082	1191	1299	1408	1517	1625	1734	1843	1951	
400	2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	
1	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
2	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	
3	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	
4	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
6	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	
7	9594	9701	9808	9914	0021	0128	0234	0341	0447	0554	
8	61 0660	0767	0873	0979	1086	1192	1298	1405	1511	1617	
9	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	
1	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	
2	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	
3	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
4	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
118	11.8	23.6	35.4	47.2	59.0	70.8	82.6	94.4	106.2
117	11.7	23.4	35.1	46.8	58.5	70.2	81.9	93.6	105.3
116	11.6	23.2	34.8	46.4	58.0	69.6	81.2	92.8	104.4
115	11.5	23.0	34.5	46.0	57.5	69.0	80.5	92.0	103.5
114	11.4	22.8	34.2	45.6	57.0	68.4	79.8	91.2	102.6
113	11.3	22.6	33.9	45.2	56.5	67.8	79.1	90.4	101.7
112	11.2	22.4	33.6	44.8	56.0	67.2	78.4	89.6	100.8
111	11.1	22.2	33.3	44.4	55.5	66.6	77.7	88.8	99.9
110	11.0	22.0	33.0	44.0	55.0	66.0	77.0	88.0	99.0
109	10.9	21.8	32.7	43.6	54.5	65.4	76.3	87.2	98.1
108	10.8	21.6	32.4	43.2	54.0	64.8	75.6	86.4	97.2
107	10.7	21.4	32.1	42.8	53.5	64.2	74.9	85.6	96.3
106	10.6	21.2	31.8	42.4	53.0	63.6	74.2	84.8	95.4
105	10.5	21.0	31.5	42.0	52.5	63.0	73.5	84.0	94.5
104	10.4	20.8	31.2	41.6	52.0	62.4	72.8	83.2	93.6

No. 415
Log. 618

TABLE XI.—*Continued.*

No. 459
Log. 662

N.	0	1	2	3	4	5	6	7	8	9	Diff.
415	61 8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
6	9093	9198	9302	9406	9511	9615	9719	9824	9928	0032	
7	62 0136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179	
1	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
2	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	
3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	
4	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
6	9410	9512	9613	9715	9817	9919	0021	0123	0224	0326	
7	63 0428	0530	0631	0733	0835	0936	1038	1139	1241	1342	
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	101
1	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	
2	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	
3	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	
4	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	100
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	
6	9486	9586	9686	9785	9885	9984	0084	0183	0283	0382	
7	64 0481	0581	0680	0779	0879	0978	1077	1177	1276	1375	
8	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	
9	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	
1	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	
2	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	
3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	
6	9335	9432	9530	9627	9724	9821	9919	0016	0113	0210	
7	65 0308	0405	0502	0599	0696	0793	0890	0987	1084	1181	
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	
450	3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	
1	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	
2	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
3	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	
4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	
6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	
7	9916	0011	0106	0201	0296	0391	0486	0581	0676	0771	95
8	66 0865	0960	1055	1150	1245	1339	1434	1529	1623	1718	
9	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
105	10.5	21.0	31.5	42.0	52.5	63.0	73.5	84.0	94.5
104	10.4	20.8	31.2	41.6	52.0	62.4	72.8	83.2	93.6
103	10.3	20.6	30.9	41.2	51.5	61.8	72.1	82.4	92.7
102	10.2	20.4	30.6	40.8	51.0	61.2	71.4	81.6	91.8
101	10.1	20.2	30.3	40.4	50.5	60.6	70.7	80.8	90.9
100	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0
99	9.9	19.8	29.7	39.6	49.5	59.4	69.3	79.2	89.1

No. 460
Log. 662

Table XI.—*Continued.*

No. 499
Log. 698

N.	0	1	2	3	4	5	6	7	8	9	Diff.
460	66 2758	2852	2947	3041	3135	3230	3324	3418	3512	3607	
1	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	
2	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	
6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	
7	9317	9410	9503	9596	9689	9782	9875	9967	0060	0153	93
8	67 0246	0339	0431	0524	0617	0710	0802	0895	0988	1080	
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	
1	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	
2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
8	9428	9519	9610	9700	9791	9882	9973	0063	0154	0245	
9	68 0336	0426	0517	0607	0698	0789	0879	0970	1060	1151	
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	
1	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	
2	3047	3137	3227	3317	3407	3497	3587	3677	3767	3357	90
3	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	
4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	
9	9309	9398	9486	9575	9664	9753	9841	9930	0019	0107	
490	69 0196	0285	0373	0462	0550	0639	0728	0816	0905	0993	
1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	
2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	
3	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	
6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	
9	8100	8188	8275	8362	8449	8535	8622	8709	8796	8883	87

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
98	9.8	19.6	29.4	39.2	49.0	58.8	68.6	78.4	88.2
97	9.7	19.4	29.1	38.8	48.5	58.2	67.9	77.6	87.3
96	9.6	19.2	28.8	38.4	48.0	57.6	67.2	76.8	86.4
95	9.5	19.0	28.5	38.0	47.5	57.0	66.5	76.0	85.5
94	9.4	18.8	28.2	37.6	47.0	56.4	65.8	75.2	84.6
93	9.3	18.6	27.9	37.2	46.5	55.8	65.1	74.4	83.7
92	9.2	18.4	27.6	36.8	46.0	55.2	64.4	73.6	82.8
91	9.1	18.2	27.3	36.4	45.5	54.6	63.7	72.8	81.9
90	9.0	18.0	27.0	36.0	45.0	54.0	63.0	72.0	81.0
89	8.9	17.8	26.7	35.6	44.5	53.4	62.3	71.2	80.1
88	8.8	17.6	26.4	35.2	44.0	52.8	61.6	70.4	79.2
87	8.7	17.4	26.1	34.8	43.5	52.2	60.9	69.6	78.3
86	8.6	17.2	25.8	34.4	43.0	51.6	60.2	68.8	77.4

No. 500
Log. 698

TABLE XI.—*Continued.*

No. 544
Log. 736

N.	0	1	2	3	4	5	6	7	8	9	Diff.
500	69 8970	9057	9144	9231	9317	9404	9491	9578	9664	9751	
1	9838	9924	0011	0098	0184	0271	0358	0444	0531	0617	
2	70 0704	0790	0877	0963	1050	1136	1222	1309	1395	1482	
3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	
4	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
1	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	
2	9270	9355	9440	9524	9609	9694	9779	9863	9948	0033	
3	71 0117	0202	0287	0371	0456	0540	0625	0710	0794	0879	
4	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	
9	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	
1	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	
2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	0077	
525	72 0159	0242	0325	0407	0490	0573	0655	0738	0821	0903	
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	
1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	
7	9974	0055	0136	0217	0298	0378	0459	0540	0621	0702	
8	73 0782	0863	0944	1024	1105	1186	1266	1347	1428	1508	
9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117	
1	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	
2	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
3	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	
4	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
87	8.7	17.4	26.1	34.8	43.5	52.2	60.9	69.6	78.3
86	8.6	17.2	25.8	34.4	43.0	51.6	60.2	68.8	77.4
85	8.5	17.0	25.5	34.0	42.5	51.0	59.5	68.0	76.5
84	8.4	16.8	25.2	33.6	42.0	50.4	58.8	67.2	75.6

TABLE XI.—*Continued.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
545	73 6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	
9	9572	9651	9731	9810	9889	9968	0047	0126	0205	0284	79
550	74 0363	0442	0521	0600	0678	0757	0836	0915	0994	1073	
1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	
1	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	
2	9736	9814	9891	9968	0045	0123	0200	0277	0354	0431	
3	75 0508	0586	0663	0740	0817	0894	0971	1048	1125	1202	
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	
1	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	
3	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	
4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	
575	9668	9743	9819	9894	9970	0045	0121	0196	0272	0347	
6	76 0422	0498	0573	0649	0724	0799	0875	0950	1025	1101	
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	
9	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	75
580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101	
1	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	
2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
83	8.3	16.6	24.9	33.2	41.5	49.8	58.1	66.4	74.7
82	8.2	16.4	24.6	32.8	41.0	49.2	57.4	65.6	73.8
81	8.1	16.2	24.3	32.4	40.5	48.6	56.7	64.8	72.9
80	8.0	16.0	24.0	32.0	40.0	48.0	56.0	64.0	72.0
79	7.9	15.8	23.7	31.6	39.5	47.4	55.3	63.2	71.1
78	7.8	15.6	23.4	31.2	39.0	46.8	54.6	62.4	70.2
77	7.7	15.4	23.1	30.8	38.5	46.2	53.9	61.6	69.3
76	7.6	15.2	22.8	30.4	38.0	45.6	53.2	60.8	68.4
75	7.5	15.0	22.5	30.0	37.5	45.0	52.5	60.0	67.5
74	7.4	14.8	22.2	29.6	37.0	44.4	51.8	59.2	66.6

No. 585
Log. 767

TABLE XI.—*Continued.*

No. 629
Log. 799

N.	0	1	2	3	4	5	6	7	8	9	Diff.
585	76 7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	0042	
9	77 0115	0189	0263	0336	0410	0484	0557	0631	0705	0778	
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	
1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	
600	8151	8224	8296	8368	8441	8513	8585	8658	8730	8802	
1	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	
2	9596	9669	9741	9813	9885	9957	0029	0101	0173	0245	72
3	78 0317	0389	0461	0533	0605	0677	0749	0821	0893	0965	
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	
6	9581	9651	9722	9792	9863	9933	0004	0074	0144	0215	
7	79 0285	0356	0426	0496	0567	0637	0707	0778	0848	0918	
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
1	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	
2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
75	7.5	15.0	22.5	30.0	37.5	45.0	52.5	60.0	67.5
74	7.4	14.8	22.2	29.6	37.0	44.4	51.8	59.2	66.6
73	7.3	14.6	21.9	29.2	36.5	43.8	51.1	58.4	65.7
72	7.2	14.4	21.6	28.8	36.0	43.2	50.4	57.6	64.8
71	7.1	14.2	21.3	28.4	35.5	42.6	49.7	56.8	63.9
70	7.0	14.0	21.0	28.0	35.0	42.0	49.0	56.0	63.0
69	6.9	13.8	20.7	27.6	34.5	41.4	48.3	55.2	62.1

TABLE XI.—*Continued.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
630	79 9341	9409	9478	9547	9616	9685	9754	9823	9892	9961	
1	80 0029	0098	0167	0236	0305	0373	0442	0511	0580	0648	
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	**68**
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	
640	80 6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	
1	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	
2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	
645	9560	9627	9694	9762	9829	9896	9964	0031	0098	0165	
6	81 0233	0300	0367	0434	0501	0569	0636	0703	0770	0837	
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	**67**
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	
650	2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	**66**
660	9544	9610	9676	9741	9807	9873	9939	0004	0070	0136	
1	82 0201	0267	0333	0399	0464	0530	0595	0661	0727	0792	
2	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	
3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	**65**
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
68	6.8	13.6	20.4	27.2	34.0	40.8	47.6	54.4	61.2
67	6.7	13.4	20.1	26.8	33.5	40.2	46.9	53.6	60.3
66	6.6	13.2	19.8	26.4	33.0	39.6	46.2	52.8	59.4
65	6.5	13.0	19.5	26.0	32.5	39.0	45.5	52.0	58.5
64	6.4	12.8	19.2	25.6	32.0	38.4	44.8	51.2	57.6

No. 675
Log. 829

TABLE XI.—*Continued.*

No. 719
Log. 857

N.	0	1	2	3	4	5	6	7	8	9	Diff.
675	82 9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	
6	9947	0011	0075	0139	0204	0268	0332	0396	0460	0525	
7	83 0589	0653	0717	0781	0845	0909	0973	1037	1102	1166	
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	
1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	
2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	
3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	
4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	
1	9478	9541	9604	9667	9729	9792	9855	9918	9981	0043	
2	84 0106	0169	0232	0294	0357	0420	0482	0545	0608	0671	
3	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	
4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	
9	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	
700	5098	5160	5222	5284	5346	5408	5470	5532	5594	5656	62
1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	
8	85 0033	0095	0156	0217	0279	0340	0401	0462	0524	0585	
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	
1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	
2	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
65	6.5	13.0	19.5	26.0	32.5	39.0	45.5	52.0	58.5
64	6.4	12.8	19.2	25.6	32.0	38.4	44.8	51.2	57.6
63	6.3	12.6	18.9	25.2	31.5	37.8	44.1	50.4	56.7
62	6.2	12.4	18.6	24.8	31.0	37.2	43.4	49.6	55.8
61	6.1	12.2	18.3	24.4	30.5	36.6	42.7	48.8	54.9
60	6.0	12.0	18.0	24.0	30.0	36.0	42.0	48.0	54.0

No. 720
Log. 857

TABLE XI.—*Continued.*

No. 764
Log. 883

N.	0	1	2	3	4	5	6	7	8	9	Diff.
720	85 7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	
1	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	
2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	
3	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
4	9739	9799	9859	9918	9978	0038	0098	0158	0218	0278	
725	86 0338	0398	0458	0518	0578	0637	0697	0757	0817	0877	
6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	
1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	
9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	
1	9818	9877	9935	9994	0053	0111	0170	0228	0287	0345	
2	87 0404	0462	0521	0579	0638	0696	0755	0813	0872	0930	
3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	
750	5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	
1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	
2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	
8	9669	9726	9784	9841	9898	9956	0013	0070	0127	0185	
9	88 0242	0299	0356	0413	0471	0528	0585	0642	0699	0756	
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	
1	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	
2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	
4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
59	5.9	11.8	17.7	23.6	29.5	35.4	41.3	47.2	53.1
58	5.8	11.6	17.4	23.2	29.0	34.8	40.6	46.4	52.2
57	5.7	11.4	17.1	22.8	28.5	34.2	39.9	45.6	51.3
56	5.6	11.2	16.8	22.4	28.0	33.6	39.2	44.8	50.4

No. 765
LOG. 883

TABLE XI.—*Continued.*

No. 809
LOG. 908

N.	0	1	2	3	4	5	6	7	8	9	Diff.
765	88 3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	
1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	
2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
6	9862	9918	9974	0030	0086	0141	0197	0253	0309	0365	
7	89 0421	0477	0533	0589	0645	0700	0756	0812	0868	0924	
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	
780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	
1	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	
2	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	
3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	
2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	
4	9821	9875	9930	9985	0039	0094	0149	0203	0258	0312	
795	90 0367	0422	0476	0531	0586	0640	0695	0749	0804	0859	
6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	
800	3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	
1	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	
2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
57	5.7	11.4	17.1	22.8	28.5	34.2	39.9	45.6	51.3
56	5.6	11.2	16.8	22.4	28.0	33.6	39.2	44.8	50.4
55	5.5	11.0	16.5	22.0	27.5	33.0	38.5	44.0	49.5
54	5.4	10.8	16.2	21.6	27.0	32.4	37.8	43.2	48.6

TABLE XI.—*Continued.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
810	90 8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	
1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	
2	9556	9610	9663	9716	9770	9823	9877	9930	9984	0037	
3	91 0091	0144	0197	0251	0304	0358	0411	0464	0518	0571	
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	
9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290	
1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	
2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	
1	9601	9653	9706	9758	9810	9862	9914	9967	0019	0071	
2	92 0123	0176	0228	0280	0332	0384	0436	0489	0541	0593	
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	
1	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	
2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	
850	9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
1	9930	9981	0032	0083	0134	0185	0236	0287	0338	0389	
2	93 0440	0491	0542	0592	0643	0694	0745	0796	0847	0898	
3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
53	5.3	10.6	15.9	21.2	26.5	31.8	37.1	42.4	47.7
52	5.2	10.4	15.6	20.8	26.0	31.2	36.4	41.6	46.8
51	5.1	10.2	15.3	20.4	25.5	30.6	35.7	40.8	45.9
50	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0

No. 855
Log. 931

Table XI.—*Continued.*

No. 899
Log. 954

N.	0	1	2	3	4	5	6	7	8	9	Diff.
855	93 1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953	
1	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	
2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	
3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	
4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	
865	7016	7066	7116	7167	7217	7267	7317	7367	7418	7468	
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	
9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	
1	94 0018	0068	0118	0168	0218	0267	0317	0367	0417	0467	
2	0516	0566	0616	0666	0716	0765	0315	0865	0915	0964	
3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	
4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	
6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	
8	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	
1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	
2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	
885	6943	6992	7041	7090	7139	7189	7238	7287	7336	7385	
6	7434	7483	7532	7531	7630	7679	7728	7777	7826	7875	49
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	
8	8413	8462	8511	8560	8608	8657	8706	8755	8304	8853	
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	
1	9878	9926	9975	0024	0073	0121	0170	0219	0267	0316	
2	95 0365	0414	0462	0511	0560	0608	0657	0706	0754	0803	
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
51	5.1	10.2	15.3	20.4	25.5	30.6	35.7	40.8	45.9
50	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0
49	4.9	9.8	14.7	19.6	24.5	29.4	34.3	39.2	44.1
48	4.8	9.6	14.4	19.2	24.0	28.8	33.6	38.4	43.2

No. 900
Log. 954

Table XI.—*Continued.*

No. 944
Log. 975

N.	0	1	2	3	4	5	6	7	8	9	Diff.
900	95 4243	4291	4339	4387	4435	4484	4532	4580	4628	4677	
1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	
2	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	
3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	
4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	
1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	
2	9995	0042	0090	0138	0185	0233	0280	0328	0376	0423	
3	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899	
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	
2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	
3	9882	9928	9975	0021	0068	0114	0161	0207	0254	0300	
4	97 0347	0393	0440	0486	0533	0579	0626	0672	0719	0765	
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
47	4.7	9.4	14.1	18.8	23.5	28.2	32.9	37.6	42.3
46	4,6	9,2	13.8	18.4	23.0	27.6	32,2	36,8	41.4

No. 945
Log. 975

TABLE XI.—*Continued.*

No. 989
Log. 995

N.	0	1	2	3	4	5	6	7	8	9	Diff.
945	97 5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	
950	7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	
955	98 0003	0049	0094	0140	0185	0231	0276	0322	0367	0412	
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	
7	9895	9939	9983	0028	0072	0117	0161	0206	0250	0294	
8	99 0339	0383	0428	0472	0516	0561	0605	0650	0694	0738	
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
46	4.6	9.2	13.8	18.4	23.0	27.6	32.2	36.8	41.4
45	4.5	9.0	13.5	18.0	22.5	27.0	31.5	36.0	40.5
44	4.4	8.8	13.2	17.6	22.0	26.4	30.8	35.2	39.6
43	4.3	8.6	12.9	17.2	21.5	25.8	30.1	34.4	38.7

Table XI.—*Concluded.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
990	99 5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43

VALUES OF *S*, *T*, AND *C* IN TABLE XII, PAGES 492 AND 493

If we were to plot the values of the logarithmic functions given in Table XII as ordinates and corresponding minutes as abscissas it would be found that the points for each function were on a curve with variable radius. It would further be noted that the curves for sines, tangents, and cotangents were of comparatively small radii when the angles were small; that the curves for cosines, cotangents, and tangents of angles near 90° respectively had the same shape as the curves for sines, tangents, and cotangents of the complements of the angles; and that other than the portions of the curves just mentioned were nearly straight lines for short distances.

When seconds are involved it will be sufficiently accurate to interpolate in the ordinary manner between adjacent values in the tables—or in other words, to assume that the curve joining two adjacent points is a straight line—for all functions between 2° and 88°, and also for sines of angles between 88° and 90° and for cosines of angles between 0° and 2°. The values in the columns headed *S*, *T*, and *C* provide a means (1) of accurately determining for any given angle between 0° and 2° the logarithmic sine, tangent, or cotangent, and for any given angle between 88° and 90°, the logarithmic cosine, cotangent, or tangent; or (2) for a given value of the logarithmic sine, tangent, or cotangent of accurately determining the angle when it lies between 0° and 2°, and for any given value of the cosine, cotangent, or tangent, the angle when it lies between 88° and 90°.

Given: *angle.* ***Required***: *logarithmic function.*

$$\left.\begin{aligned}\log\sin\alpha &= \log\alpha\ (\text{in seconds}) + S\\ \log\tan\alpha &= \log\alpha\ (\text{in seconds}) + T\\ \log\cot\alpha &= C - \log\alpha\ (\text{in seconds})\end{aligned}\right\}\ \text{In which } \alpha \text{ is less than } 2^\circ.$$

$$\left.\begin{aligned}\log\cos\beta &= \log(90^\circ-\beta)\ (\text{in seconds}) + S\\ \log\tan\beta &= C - \log(90^\circ-\beta)(\text{in seconds})\\ \log\cot\beta &= \log(90^\circ-\beta)\ (\text{in seconds}) + T\end{aligned}\right\}\ \text{In which } \beta \text{ lies between } 88^\circ \text{ and } 90^\circ$$

Given: *logarithmic function.* ***Required***: *angle.*

$$\left.\begin{aligned}\log\alpha\ (\text{in seconds}) &= \log\sin\alpha - S\\ &= \log\tan\alpha - T\\ &= C - \log\cot\alpha\end{aligned}\right\}\ \text{In which } \alpha \text{ is less than } 2^\circ.$$

$$\left.\begin{aligned}\log(90^\circ-\beta)\ (\text{in seconds}) &= \log\cos\beta - S\\ &= C - \log\tan\beta\\ &= \log\cot\beta - T\end{aligned}\right\}\ \text{In which } \beta \text{ lies between } 88^\circ \text{ and } 90^\circ.$$

EXAMPLES

Given: *angle.*

Required: *logarithmic function.*

$$\text{When}\ \left\{\begin{matrix}\alpha\\ 90^\circ-\beta\end{matrix}\right\} = 19'\,22'' = 1162''$$

$$\log 1162'' = 3.065206$$

$$S\ (\text{for } 19') = 4.685573$$

$$\left.\begin{matrix}\log\sin 19'\,22''\\ \log\cos 89^\circ\,40'\,38''\end{matrix}\right\} = 7.750779$$

$$\text{When}\ \left\{\begin{matrix}\alpha\\ 90^\circ-\beta\end{matrix}\right\} = 23'\,21'' = 1401''$$

$$\log 1401'' = 3.146438$$

$$C\ (\text{for } 23') = 15.314419$$

$$\left.\begin{matrix}\log\cot 23'\,21''\\ \log\tan 89^\circ\,36'\,39''\end{matrix}\right\} = 12.167981$$

Given: *logarithmic function.*

Required: *angle.*

$$\text{When}\ \left\{\begin{matrix}\log\sin\alpha\\ \log\cos\beta\end{matrix}\right\} = 7.750779$$

$$S\ (\text{for } 19') = 4.685573$$

$$\log\left\{\begin{matrix}\alpha\\ 90^\circ-\beta\end{matrix}\right\}(\text{in seconds}) = 3.065206$$

$$\text{or}\quad \begin{aligned}\alpha &= 1162'' = 19'\,22''\\ \beta &= 89^\circ\,40'\,38''\end{aligned}$$

$$\text{When}\ \left\{\begin{matrix}\log\cot\alpha\\ \log\tan\beta\end{matrix}\right\} = 12.167981$$

$$C\ (\text{for } 23') = 15.314419$$

$$\log\left\{\begin{matrix}\alpha\\ 90^\circ-\beta\end{matrix}\right\}(\text{in seconds}) = 3.146438$$

$$\text{or}\quad \begin{aligned}\alpha &= 1401'' = 23'\,21''\\ \beta &= 89^\circ\,36'\,39''\end{aligned}$$

0° TABLE XII. 179°

″	′	Sine.	S.*	T.*	Tang.	Cotang.	C.*	D. 1″.	Cosine.	′
			4.685				15.314			
0	0	Inf. neg.	575	575	Inf. neg.	Inf. pos.	425		10.00 0000	60
60	1	6.46 3726	575	575	6.46 3726	13.53 6274	425		0000	59
120	2	.76 4756	575	575	.76 4756	.23 5244	425		0000	58
180	3	6.94 0847	575	575	6.94 0847	13.05 9153	425		0000	57
240	4	7.06 5786	575	575	7.06 5786	12.93 4214	425		0000	56
300	**5**	**7.16 2696**	575	575	**7.16 2696**	**12.83 7304**	425	.02	**10.00 0000**	**55**
360	6	.24 1877	575	575	.24 1878	.75 8122	425	.00	9.99 9999	54
420	7	.30 8824	575	575	.30 8825	.69 1175	425	.00	9999	53
480	8	.36 6816	574	576	.36 6817	.63 3183	424	.00	9999	52
540	9	.41 7968	574	576	.41 7970	.58 2030	424	.02	9999	51
600	**10**	**7.46 3726**	574	576	**7.46 3727**	**12.53 6273**	424	.00	**9.99 9998**	**50**
660	11	.50 5118	574	576	.50 5120	.49 4880	424	.02	9998	49
720	12	.54 2906	574	577	.54 2909	.45 7091	423	.00	9997	48
780	13	.57 7668	574	577	.57 7672	.42 2328	423	.02	9997	47
840	14	.60 9853	574	577	.60 9857	.39 0143	423	.00	9996	46
900	**15**	**7.63 9816**	573	578	**7.63 9820**	**12.36 0180**	422	.02	**9.99 9996**	**45**
960	16	.66 7845	573	578	.66 7849	.33 2151	422	.00	9995	44
1020	17	.69 4173	573	578	.69 4179	.30 5821	422	.02	9995	43
1080	18	.71 8997	573	579	.71 9003	.28 0997	421	.02	9994	42
1140	19	.74 2478	573	579	.74 2484	.25 7516	421	.00	9993	41
1200	**20**	**7.76 4754**	572	580	**7.76 4761**	**12.23 5239**	420	.02	**9.99 9993**	**40**
1260	21	.78 5943	572	580	.78 5951	.21 4049	420	.02	9992	39
1320	22	.80 6146	572	581	.80 6155	.19 3845	419	.02	9991	38
1380	23	.82 5451	572	581	.82 5460	.17 4540	419	.02	9990	37
1440	24	.84 3934	571	582	.84 3944	.15 6056	418	.00	9989	36
1500	**25**	**7.86 1662**	571	583	**7.86 1674**	**12.13 8326**	417	.02	**9.99 9989**	**35**
1560	26	.87 8695	571	583	.87 8708	.12 1292	417	.02	9988	34
1620	27	.89 5085	570	584	.89 5099	.10 4901	416	.02	9987	33
1680	28	.91 0879	570	584	.91 0894	.08 9106	416	.02	9986	32
1740	29	.92 6119	570	585	.92 6134	.07 3866	415	.03	9985	31
1800	**30**	**7.94 0842**	569	586	**7.94 0858**	**12.05 9142**	414	.02	**9.99 9983**	**30**
1860	31	.95 5082	569	587	.95 5100	.04 4900	413	.02	9982	29
1920	32	.96 8870	569	587	.96 8889	.03 1111	413	.02	9981	28
1980	33	.98 2233	568	588	.98 2253	.01 7747	412	.02	9980	27
2040	34	7.99 5198	568	589	7.99 5219	12.00 4781	411	.03	9979	26
2100	**35**	**8.00 7787**	567	590	**8.00 7809**	**11.99 2191**	410	.02	**9.99 9977**	**25**
2160	36	.02 0021	567	591	.02 0044	.97 9956	409	.02	9976	24
2220	37	.03 1919	566	592	.03 1945	.96 8055	408	.03	9975	23
2280	38	.04 3501	566	593	.04 3527	.95 6473	407	.02	9973	22
2340	39	.05 4781	566	593	.05 4809	.94 5191	407	.02	9972	21
2400	**40**	**8.06 5576**	565	594	**8.06 5806**	**11.93 4194**	406	.03	**9.99 9971**	**20**
2460	41	.07 6500	565	595	.07 6531	.92 3469	405	.02	9969	19
2520	42	.08 6965	564	596	.08 6997	.91 3003	404	.03	9968	18
2580	43	.09 7183	564	598	.09 7217	.90 2783	402	.03	9966	17
2640	44	.10 7167	563	599	.10 7203	.89 2797	401	.02	9964	16
2700	**45**	**8.11 6926**	562	600	**8.11 6963**	**11.88 3037**	400	.03	**9.99 9963**	**15**
2760	46	.12 6471	562	601	.12 6510	.87 3490	399	.03	9961	14
2820	47	.13 5810	561	602	.13 5851	.86 4149	398	.02	9959	13
2880	48	.14 4953	561	603	.14 4996	.85 5004	397	.03	9958	12
2940	49	.15 3907	560	604	.15 3952	.84 6048	396	.03	9956	11
3000	**50**	**8.16 2681**	560	605	**8.16 2727**	**11.83 7273**	395	.03	**9.99 9954**	**10**
3060	51	.17 1280	559	607	.17 1328	.82 8672	393	.03	9952	9
3120	52	.17 9713	558	608	.17 9763	.82 0237	392	.03	9950	8
3180	53	.18 7985	558	609	.18 8036	.81 1964	391	.03	9948	7
3240	54	.19 6102	557	611	.19 6156	.80 3844	389	.03	9946	6
3300	**55**	**8.20 4070**	556	612	**8.20 4126**	**11.79 5874**	388	.03	**9.99 9944**	**5**
3360	56	.21 1895	556	613	.21 1953	.78 8047	387	.03	9942	4
3420	57	.21 9581	555	615	.21 9641	.77 0359	385	.03	9940	3
3480	58	.22 7134	554	616	.22 7195	.77 2805	384	.03	9938	2
3540	59	.23 4557	554	618	.23 4621	.76 5379	382	.03	9936	1
3600	**60**	**8.24 1855**	553	619	**8.24 1921**	**11.75 8079**	381		**9.99 9934**	**0**
			4.685				15.314			
″	′	Cosine.	S.*	T.*	Cotang.	Tang.	C.*	D. 1″.	Sine.	′

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* For use of S, T, and C see page 490.

1° TABLE XII.—*Continued.* 178°

″	′	Sine.	S.*	T.*	Tang.	Cotang.	C.*	D. 1″.	Cosine.	′
			4.685				15.314			
3600	0	8.24 1855	553	619	8.24 1921	11.75 8079	381	.03	9.99 9934	60
3660	1	.24 9033	552	620	.24 9102	.75 0898	380	.05	9932	59
3720	2	.25 6094	551	622	.25 6165	.74 3835	378	.03	9929	58
3780	3	.26 3042	551	623	.26 3115	.73 6885	377	.03	9927	57
3840	4	.26 9881	550	625	.26 9956	.73 0044	375	.05	9925	56
3900	5	8.27 6614	549	627	8.27 6691	11.72 3309	373	.03	9.99 9922	55
3960	6	.28 3243	548	628	.28 3323	.71 6677	372	.03	9920	54
4020	7	.28 9773	547	630	.28 9856	.71 0144	370	.05	9918	53
4080	8	.29 6207	546	632	.29 6292	.70 3708	368	.03	9915	52
4140	9	.30 2546	546	633	.30 2634	.69 7366	367	.05	9913	51
4200	10	8.30 8794	545	635	8.30 8884	11.69 1116	365	.05	9.99 9910	50
4260	11	.31 4954	544	637	.31 5046	.68 4954	363	.03	9907	49
4320	12	.32 1027	543	638	.32 1122	.67 8878	362	.05	9905	48
4380	13	.32 7016	542	640	.32 7114	.67 2886	360	.05	9902	47
4440	14	.33 2924	541	642	.33 3025	.66 6975	358	.03	9899	46
4500	15	8.33 8753	540	644	8.33 8856	11.66 1144	356	.05	9.99 9897	45
4560	16	.34 4504	539	646	.34 4610	.65 5390	354	.05	9894	44
4620	17	.35 0181	539	648	.35 0289	.64 9711	352	.05	9891	43
4680	18	.35 5783	538	649	.35 5895	.64 4105	351	.05	9888	42
4740	19	.36 1315	537	651	.36 1430	.63 8570	349	.05	9885	41
4800	20	8.36 6777	536	653	8.36 6895	11.63 3105	347	.05	9.99 9882	40
4860	21	.37 2171	535	655	.37 2292	.62 7708	345	.05	9879	39
4920	22	.37 7499	534	657	.37 7622	.62 2378	343	.05	9876	38
4980	23	.38 2762	533	659	.38 2889	.61 7111	341	.05	9873	37
5040	24	.38 7962	532	661	.38 8092	.61 1908	339	.05	9870	36
5100	25	8.39 3101	531	663	8.39 3234	11.60 6766	337	.05	9.99 9867	35
5160	26	.39 8179	530	666	.39 8315	.60 1685	334	.05	9864	34
5220	27	.40 3199	529	668	.40 3338	.59 6662	332	.05	9861	33
5280	28	.40 8161	527	670	.40 8304	.59 1696	330	.07	9858	32
5340	29	.41 3068	526	672	.41 3213	.58 6787	328	.05	9854	31
5400	30	8.41 7919	525	674	8.41 8068	11.58 1932	326	.05	9.99 9851	30
5460	31	.42 2717	524	676	.42 2869	.57 7131	324	.07	9848	29
5520	32	.42 7462	523	679	.42 7618	.57 2382	321	.05	9844	28
5580	33	.43 2156	522	681	.43 2315	.56 7685	319	.05	9841	27
5640	34	.43 6800	521	683	.43 6962	.56 3038	317	.07	9838	26
5700	35	8.44 1394	520	685	8.44 1560	11.55 8440	315	.05	9.99 9834	25
5760	36	.44 5941	518	688	.44 6110	.55 3890	312	.07	9831	24
5820	37	.45 0440	517	690	.45 0613	.54 9387	310	.05	9827	23
5880	38	4893	516	693	5070	4930	307	.07	9824	22
5940	39	.45 9301	515	695	.45 9481	.54 0519	305	.07	9820	21
6000	40	8.46 3665	514	697	8.46 3849	11.53 6151	303	.05	9.99 9816	20
6060	41	.46 7985	512	700	.46 8172	.53 1828	300	.07	9813	19
6120	42	.47 2263	511	702	.47 2454	.52 7546	298	.07	9809	18
6180	43	.47 6493	510	705	.47 6693	.52 3307	295	.07	9805	17
6240	44	.48 0693	509	707	.48 0892	.51 9108	293	.07	9801	16
6300	45	8.48 4848	507	710	8.48 5050	11.51 4950	290	.05	9.99 9797	15
6360	46	.48 8963	506	713	.48 9170	.51 0830	287	.07	9794	14
6420	47	.49 3040	505	715	.49 3250	.50 6750	285	.07	9790	13
6480	48	.49 7078	503	718	.49 7293	.50 2707	282	.07	9786	12
6540	49	.50 1080	502	720	.50 1298	.49 8702	280	.07	9782	11
6600	50	8.50 5045	501	723	8.50 5267	11.49 4733	277	.07	9.99 9778	10
6660	51	.50 8974	499	726	.50 9200	.49 0800	274	.08	9774	9
6720	52	.51 2867	498	729	.51 3098	.48 6902	271	.07	9769	8
6780	53	.51 6726	497	731	.51 6961	.48 3039	269	.07	9765	7
6840	54	.52 0551	495	734	.52 0790	.47 9210	266	.07	9761	6
6900	55	8.52 4343	494	737	8.52 4586	11.47 5414	263	.07	9.99 9757	5
6960	56	.52 8102	492	740	.52 8349	.47 1651	260	.08	9753	4
7020	57	.53 1828	491	743	.53 2080	.46 7920	257	.07	9748	3
7080	58	5523	490	745	5779	4221	255	.07	9744	2
7140	59	.53 9186	488	748	.53 9447	.46 0553	252	.08	9740	1
7200	60	8.54 2819	487	751	8.54 3084	11.45 6916	249		9.99 9735	0
			4.685				15.314			
″	′	Cosine.	S.*	T.*	Cotang.	Tang.	C.*	D. 1″.	Sine.	′

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* For use of S, T, and C see pape 490.

2° TABLE XII.—*Continued.* **177°**

′	Sine.	D. 1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**8.54 2819**	60.05	**9.99 9735**	.07	**8.54 3084**	60.12	**11.45 6916**	**60**
1	6422	59.55	9731	.08	.54 6691	59.62	.45 3309	59
2	.54 9995	59.07	9726	.07	.55 0268	59.15	.44 9732	58
3	.55 3539	58.58	9722	.08	3817	58.65	6183	57
4	.55 7054	58.10	9717	.07	.55 7336	58.20	.44 2664	56
5	**8.56 0540**	57.65	**9.99 9713**	.08	**8.56 0828**	57.72	**11.43 9172**	**55**
6	3999	57.20	9708	.07	4291	57.27	5709	54
7	.56 7431	56.75	9704	.08	.56 7727	56.83	.43 2273	53
8	.57 0836	56.30	9699	.08	.57 1137	56.38	.42 8863	52
9	4214	55.87	9694	.08	4520	55.95	5480	51
10	**8.57 7566**	55.43	**9.99 9689**	.07	**8.57 7877**	55.52	**11.42 2123**	**50**
11	.58 0892	55.02	9685	.08	.58 1208	55.10	.41 8792	49
12	4193	54.60	9680	.08	4514	54.68	5486	48
13	.58 7469	54.20	9675	.08	.58 7795	54.27	.41 2205	47
14	.59 0721	53.78	9670	.08	.59 1051	53.87	.40 8949	46
15	**8.59 3948**	53.40	**9.99 9665**	.08	**8.59 4283**	53.48	**11.40 5717**	**45**
16	.59 7152	53.00	9660	.08	.59 7492	53.08	.40 2508	44
17	.60 0332	52.62	9655	.08	.60 0677	52.70	.39 9323	43
18	3489	52.23	9650	.08	3839	52.32	6161	42
19	6623	51.85	9645	.08	.60 6978	51.93	.39 3022	41
20	**8.60 9734**	51.48	**9.99 9640**	.08	**8.61 0094**	51.58	**11.38 9906**	**40**
21	.61 2823	51.13	9635	.10	3189	51.22	6811	39
22	5891	50.77	9629	.08	6262	50.85	3738	38
23	.61 8937	50.42	9624	.08	.61 9313	50.50	.38 0687	37
24	.62 1962	50.05	9619	.08	.62 2343	50.15	.37 7657	36
25	**8.62 4965**	49.72	**9.99 9614**	.10	**8.62 5352**	49.80	**11.37 4648**	**35**
26	.62 7948	49.38	9608	.08	.62 8340	49.47	.37 1660	34
27	.63 0911	49.05	9603	.10	.63 1308	49.13	.36 8692	33
28	3854	48.70	9597	.08	4256	48.80	5744	32
29	6776	48.40	9592	.10	.63 7184	48.48	.36 2816	31
30	**8.63 9680**	48.05	**9.99 9586**	.08	**8.64 0093**	48.15	**11.35 9907**	**30**
31	.64 2563	47.75	9581	.10	2982	47.85	7018	29
32	5428	47.43	9575	.08	5853	47.52	4147	28
33	.64 8274	47.13	9570	.10	.64 8704	47.22	.35 1296	27
34	.65 1102	46.82	9564	.10	.65 1537	46.92	.34 8463	26
35	**8.65 3911**	46.52	**9.99 9558**	.08	**8.65 4352**	46.62	**11.34 5648**	**25**
36	6702	46.22	9553	.10	7149	46.32	2851	24
37	.65 9475	45.92	9547	.10	.65 9928	46.02	.34 0072	23
38	.66 2230	45.63	9541	.10	.66 2689	45.73	.33 7311	22
39	4968	45.35	9535	.10	5433	45.45	4567	21
40	**8.66 7689**	45.07	**9.99 9529**	.08	**8.66 8160**	45.17	**11.33 1840**	**20**
41	.67 0393	44.78	9524	.10	.67 0870	44.88	.32 9130	19
42	3080	44.52	9518	.10	3563	44.60	6437	18
43	5751	44.23	9512	.10	6239	44.35	3761	17
44	.67 8405	43.97	9506	.10	.67 8900	44.07	.32 1100	16
45	**8.68 1043**	43.70	**9.99 9500**	.12	**8.68 1544**	43.80	**11.31 8456**	**15**
46	3665	43.45	9493	.10	4172	43.53	5828	14
47	6272	43.18	9487	.10	6784	43.28	3216	13
48	.68 8863	42.92	9481	.10	.68 9381	43.03	.31 0619	12
49	.69 1438	42.67	9475	.10	.69 1963	42.77	.30 8037	11
50	**8.69 3998**	42.42	**9.99 9469**	.10	**8.69 4529**	42.53	**11.30 5471**	**10**
51	6543	42.17	9463	.12	7081	42.27	2919	9
52	.69 9073	41.93	9456	.10	.69 9617	42.03	.30 0383	8
53	.70 1589	41.68	9450	.12	.70 2139	41.78	.29 7861	7
54	4090	41.45	9443	.10	4646	41.57	5354	6
55	**8.70 6577**	41.20	**9.99 9437**	.10	**8.70 7140**	41.30	**11.29 2860**	**5**
56	.70 9049	40.97	9431	.12	.70 9618	41.08	.29 0382	4
57	.71 1507	40.75	9424	.10	.71 2083	40.85	.28 7917	3
58	3952	40.52	9418	.12	4534	40.63	5466	2
59	6383	40.28	9411	.12	6972	40.40	3028	1
60	**8.71 8800**		**9.99 9404**		**8.71 9396**		**11.28 0604**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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3° TABLE XII.—*Continued.* 176°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	8.71 8800	40.07	9.99 9404	.10	8.71 9396	40.17	11.28 0604	60
1	.72 1204	39.85	9398	.12	.72 1806	39.97	.27 8194	59
2	3595	39.62	9391	.12	4204	39.73	5796	58
3	5972	39.42	9384	.10	6588	39.52	3412	57
4	.72 8337	39.18	9378	.12	.72 8959	39.30	.27 1041	56
5	8.73 0688	38.98	9.99 9371	.12	8.73 1317	39.10	11.26 8683	55
6	3027	38.78	9364	.12	3663	38.88	6337	54
7	5354	38.55	9357	.12	5996	38.68	4004	53
8	7667	38.37	9350	.12	.73 8317	38.48	.26 1683	52
9	.73 9969	38.17	9343	.12	.74 0626	38.27	.25 9374	51
10	8.74 2259	37.95	9.99 9336	.12	8.74 2922	38.08	11.25 7078	50
11	4536	37.77	9329	.12	5207	37.87	4793	49
12	6802	37.55	9322	.12	7479	37.68	2521	48
13	.74 9055	37.37	9315	.12	.74 9740	37.48	.25 0260	47
14	.75 1297	37.18	9308	.12	.75 1989	37.30	.24 8011	46
15	8.75 3528	36.98	9.99 9301	.12	8.75 4227	37.10	11.24 5773	45
16	5747	36.80	9294	.12	6453	36.92	3547	44
17	.75 7955	36.60	9287	.13	.75 8668	36.73	.24 1332	43
18	.76 0151	36.43	9279	.12	.76 0872	36.55	.23 9128	42
19	2337	36.23	9272	.12	3065	36.35	6935	41
20	8.76 4511	36.07	9.99 9265	.13	8.76 5246	36.18	11.23 4754	40
21	6675	35.88	9257	.12	7417	36.02	2583	39
22	.76 8828	35.70	9250	.13	.76 9578	35.82	.23 0422	38
23	.77 0970	35.52	9242	.12	.77 1727	35.65	.22 8273	37
24	3101	35.37	9235	.13	3866	35.48	6134	36
25	8.77 5223	35.17	9.99 9227	.12	8.77 5995	35.32	11.22 4005	35
26	7333	35.02	9220	.13	.77 8114	35.13	.22 1886	34
27	.77 9434	34.83	9212	.12	.78 0222	34.97	.21 9778	33
28	.78 1524	34.68	9205	.13	2320	34.80	7680	32
29	3605	34.50	9197	.13	4408	34.63	5592	31
30	8.78 5675	34.35	9.99 9189	.13	8.78 6486	34.47	11.21 3514	30
31	7736	34.18	9181	.12	.78 8554	34.32	.21 1446	29
32	.78 9787	34.02	9174	.13	.79 0613	34.15	.20 9387	28
33	.79 1828	33.85	9166	.13	2662	33.98	7338	27
34	3859	33.70	9158	.13	4701	33.83	5299	26
35	8.79 5881	33.55	9.99 9150	.13	8.79 6731	33.68	11.20 3269	25
36	7894	33.38	9142	.13	.79 8752	33.52	.20 1248	24
37	.79 9897	33.25	9134	.13	.80 0763	33.37	.19 9237	23
38	.80 1892	33.07	9126	.13	2765	33.22	7235	22
39	3876	32.93	9118	.13	4758	33.07	5242	21
40	8.80 5852	32.78	9.99 9110	.13	8.80 6742	32.92	11.19 3258	20
41	7819	32.63	9102	.13	.80 8717	32.77	.19 1283	19
42	.80 9777	32.48	9094	.13	.81 0683	32.63	.18 9317	18
43	.81 1726	32.35	9086	.15	2641	32.47	7359	17
44	3667	32.20	9077	.13	4589	32.33	5411	16
45	8.81 5599	32.05	9.99 9069	.13	8.81 6529	32.20	11.18 3471	15
46	7522	31.90	9061	.13	.81 8461	32.05	.18 1539	14
47	.81 9436	31.78	9053	.15	.82 0384	31.90	.17 9616	13
48	.82 1343	31.62	9044	.13	2298	31.78	7702	12
49	3240	31.50	9036	.15	4205	31.63	5795	11
50	8.82 5130	31.35	9.99 9027	.13	8.82 6103	31.48	11.17 3897	10
51	7011	31.22	9019	.15	7992	31.37	2008	9
52	.82 8884	31.08	9010	.13	.82 9874	31.23	.17 0126	8
53	.83 0749	30.97	9002	.15	.83 1748	31.08	.16 8252	7
54	2607	30.82	8993	.15	3613	30.97	6387	6
55	8.83 4456	30.68	9.99 8984	.13	8.83 5471	30.83	11.16 4529	5
56	6297	30.55	8976	.15	7321	30.70	2679	4
57	8130	30.43	8967	.15	.83 9163	30.58	.16 0837	3
58	.83 9956	30.30	8958	.13	.84 0998	30.45	.15 9002	2
59	.84 1774	30.18	8950	.15	2825	30.32	7175	1
60	8.84 3585		9.99 8941		8.84 4644		11.15 5356	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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4° TABLE XII.—*Continued.* 175°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**8.84 3585**	30.03	**9.99 8941**	.15	**8.84 4644**	30.18	**11.15 5356**	**60**
1	5387	29.93	8932	.15	6455	30.08	3545	59
2	7183	29.80	8923	.15	.84 8260	29.95	.15 1740	58
3	.84 8971	29.67	8914	.15	.85 0057	29.82	.14 9943	57
4	.85 0751	29.57	8905	.15	1846	29.70	8154	56
5	**8.85 2525**	29.43	**9.99 8896**	.15	**8.85 3628**	29.58	**11.14 6372**	**55**
6	4291	29.30	8887	.15	5403	29.47	4597	54
7	6049	29.20	8878	.15	7171	29.35	2829	53
8	7801	29.08	8869	.15	.85 8932	29.23	.14 1068	52
9	.85 9546	28.95	8860	.15	.86 0686	29.12	.13 9314	51
10	**8.86 1283**	28.85	**9.99 8851**	.17	**8.86 2433**	29.00	**11.13 7567**	**50**
11	3014	28.73	8841	.15	4173	28.88	5827	49
12	4738	28.62	8832	.15	5906	28.77	4094	48
13	6455	28.50	8823	.17	7632	28.65	2368	47
14	8165	28.38	8813	.15	.86 9351	28.55	.13 0649	46
15	**8.86 9868**	28.28	**9.99 8804**	.15	**8.87 1064**	28.43	**11.12 8936**	**45**
16	.87 1565	28.17	8795	.17	2770	28.32	7230	44
17	3255	28.05	8785	.15	4469	28.22	5531	43
18	4938	27.95	8776	.17	6162	28.12	3838	42
19	6615	27.83	8766	.15	7849	28.00	2151	41
20	**8.87 8285**	27.73	**9.99 8757**	.17	**8.87 9529**	27.88	**11.12 0471**	**40**
21	.87 9949	27.63	8747	.15	.88 1202	27.78	.11 8798	39
22	.88 1607	27.52	8738	.17	2869	27.68	7131	38
23	3258	27.42	8728	.17	4530	27.58	5470	37
24	4903	27.32	8718	.17	6185	27.47	3815	36
25	**8.88 6542**	27.20	**9.99 8708**	.15	**8.88 7833**	27.38	**11.11 2167**	**35**
26	8174	27.12	8699	.17	.88 9476	27.27	.11 0524	34
27	.88 9801	27.00	8689	.17	.89 1112	27.17	.10 8888	33
28	.89 1421	26.90	8679	.17	2742	27.07	7258	32
29	3035	26.80	8669	.17	4366	26.97	5634	31
30	**8.89 4643**	26.72	**9.99 8659**	.17	**8.89 5984**	26.87	**11.10 4016**	**30**
31	6246	26.60	8649	.17	7596	26.78	2404	29
32	7842	26.50	8639	.17	.89 9203	26.67	.10 0797	28
33	.89 9432	26.42	8629	.17	.90 0803	26.58	.09 9197	27
34	.90 1017	26.32	8619	.17	2398	26.48	7602	26
35	**8.90 2596**	26.22	**9.99 8609**	.17	**8.90 3987**	26.38	**11.09 6013**	**25**
36	4169	26.12	8599	.17	5570	26.28	4430	24
37	5736	26.02	8589	.18	7147	26.20	2853	23
38	7297	25.93	8578	.17	.90 8719	26.10	.09 1281	22
39	.90 8853	25.85	8568	.17	.91 0285	26.02	.08 9715	21
40	**8.91 0404**	25.75	**9.99 8558**	.17	**8.91 1846**	25.92	**11.08 8154**	**20**
41	1949	25.65	8548	.18	3401	25.83	6599	19
42	3488	25.57	8537	.17	4951	25.73	5049	18
43	5022	25.47	8527	.18	6495	25.63	3505	17
44	6550	25.38	8516	.17	8034	25.57	1966	16
45	**8.91 8073**	25.30	**9.99 8506**	.18	**8.91 9563**	25.47	**11.08 0432**	**15**
46	.91 9591	25.20	8495	.17	.92 1096	25.38	.07 8904	14
47	.92 1103	25.12	8485	.18	2619	25.28	7381	13
48	2610	25.03	8474	.17	4136	25.22	5864	12
49	4112	24.95	8464	.18	5649	25.12	4351	11
50	**8.92 5609**	24.85	**9.99 8453**	.18	**8.92 7156**	25.03	**11.07 2844**	**10**
51	7100	24.78	8442	.18	.92 8658	24.95	.07 1342	9
52	.92 8587	24.68	8431	.17	.93 0155	24.87	.06 9845	8
53	.93 0068	24.60	8421	.18	1647	24.78	8353	7
54	1544	24.52	8410	.18	3134	24.70	6866	6
55	**8.93 3015**	24.43	**9.99 8399**	.18	**8.93 4616**	24.62	**11.06 5384**	**5**
56	4481	24.35	8388	.18	6093	24.53	3907	4
57	5942	24.27	8377	.18	7565	24.45	2435	3
58	7398	24.20	8366	.18	.93 9032	24.37	.06 0968	2
59	.93 8850	24.10	8355	.18	.94 0494	24.30	.05 9506	1
60	**8.94 0296**		**9.99 8344**		**8.94 1952**		**11.05 8048**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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5° TABLE XII.—*Continued.* 174°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**8.94 0296**	24.03	**9.99 8344**	.18	**8.94 1952**	24.20	**11.05 8048**	**60**
1	1738	23.93	8333	.18	3404	24.13	6596	59
2	3174	23.87	8322	.18	4852	24.05	5148	58
3	4606	23.80	8311	.18	6295	23.98	3705	57
4	6034	23.70	8300	.18	7734	23.90	2266	56
5	**8.94 7456**	23.63	**9.99 8289**	.20	**8.94 9168**	23.82	**11.05 0832**	**55**
6	.94 8874	23.55	8277	.18	.95 0597	23.73	.04 9403	54
7	.95 0287	23.48	8266	.18	2021	23.67	7979	53
8	1696	23.40	8255	.20	3441	23.58	6559	52
9	3100	23.32	8243	.18	4856	23.52	5144	51
10	**8.95 4499**	23.25	**9.99 8232**	.20	**8.95 6267**	23.45	**11.04 3733**	**50**
11	5894	23.17	8220	.18	7674	23.35	2326	49
12	7284	23.10	8209	.20	.95 9075	23.30	.04 0925	48
13	.95 8670	23.03	8197	.18	.96 0473	23.22	.03 9527	47
14	.96 0052	22.95	8186	.20	1866	23.15	8134	46
15	**8.96 1429**	22.87	**9.99 8174**	.18	**8.96 3255**	23.07	**11.03 6745**	**45**
16	2801	22.82	8163	.20	4639	23.00	5361	44
17	4170	22.73	8151	.20	6019	22.92	3981	43
18	5534	22.65	8139	.18	7394	22.87	2606	42
19	6893	22.60	8128	.20	.96 8766	22.78	.03 1234	41
20	**8.96 8249**	22.52	**9.99 8116**	.20	**8.97 0133**	22.72	**11.02 9867**	**40**
21	.96 9600	22.45	8104	.20	1496	22.65	8504	39
22	.97 0947	22.37	8092	.20	2855	22.57	7145	38
23	2289	22.32	8080	.20	4209	22.52	5791	37
24	3628	22.23	8068	.20	5560	22.43	4440	36
25	**8.97 4962**	22.18	**9.99 8056**	.20	**8.97 6906**	22.37	**11.02 3094**	**35**
26	6293	22.10	8044	.20	8248	22.30	1752	34
27	7619	22.03	8032	.20	.97 9586	22.25	.02 0414	33
28	.97 8941	21.97	8020	.20	.98 0921	22.17	.01 9079	32
29	.98 0259	21.90	8008	.20	2251	22.10	7749	31
30	**8.98 1573**	21.83	**9.99 7996**	.20	**8.98 3577**	22.03	**11.01 6423**	**30**
31	2883	21.77	7984	.20	4899	21.97	5101	29
32	4189	21.72	7972	.22	6217	21.92	3783	28
33	5491	21.63	7959	.20	7532	21.83	2468	27
34	6789	21.57	7947	.20	.98 8842	21.78	.01 1158	26
35	**8.98 8083**	21.52	**9.99 7935**	.22	**8.99 0149**	21.70	**11.00 9851**	**25**
36	.98 9374	21.43	7922	.20	1451	21.65	8549	24
37	.99 0660	21.38	7910	.22	2750	21.58	7250	23
38	1943	21.32	7897	.20	4045	21.53	5955	22
39	3222	21.25	7885	.22	5337	21.45	4663	21
40	**8.99 4497**	21.18	**9.99 7872**	.20	**8.99 6624**	21.40	**11.00 3376**	**20**
41	5763	21.13	7860	.22	7908	21.33	2092	19
42	7036	21.05	7847	.20	8.99 9188	21.28	11.00 0812	18
43	8299	21.02	7835	.22	9.00 0465	21.22	10.99 9535	17
44	8.99 9560	20.93	7822	.22	1738	21.15	8262	16
45	**9.00 0816**	20.88	**9.99 7809**	.20	**9.00 3007**	21.08	**10.99 6993**	**15**
46	2069	20.82	7797	.22	4272	21.03	5728	14
47	3318	20.75	7784	.22	5534	20.97	4466	13
48	4563	20.70	7771	.22	6792	20.92	3208	12
49	5805	20.65	7758	.22	8047	20.85	1953	11
50	**9.00 7044**	20.57	**9.99 7745**	.22	**9.00 9298**	20.80	**10.99 0702**	**10**
51	8278	20.53	7732	.22	.01 0546	20.73	.98 9454	9
52	.00 9510	20.45	7719	.22	1790	20.68	8210	8
53	.01 0737	20.42	7706	.22	3031	20.62	6969	7
54	1962	20.33	7693	.22	4268	20.57	5732	6
55	**9.01 3182**	20.30	**9.99 7680**	.22	**9.01 5502**	20.50	**10.98 4498**	**5**
56	4400	20.22	7667	.22	6732	20.45	3268	4
57	5613	20.18	7654	.22	7959	20.40	2041	3
58	6824	20.12	7641	.22	.01 9183	20.33	.98 0817	2
59	8031	20.07	7628	.23	.02 0403	20.28	.97 9597	1
60	**9.01 9235**		**9.99 7614**		**9.02 1620**		**10.97 8380**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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6° TABLE XII.—*Continued.* 173°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.01 9235	20.00	9.99 7614	.22	9.02 1620	20.23	10.97 8380	60
1	.02 0435	19.95	7601	.22	2834	20.17	7166	59
2	1632	19.88	7588	.23	4044	20.12	5956	58
3	2825	19.85	7574	.22	5251	20.07	4749	57
4	4016	19.78	7561	.23	6455	20.00	3545	56
5	9.02 5203	19.72	9.99 7547	.22	9.02 7655	19.95	10.97 2345	55
6	6386	19.68	7534	.23	.02 8852	19.90	.97 1148	54
7	7567	19.62	7520	.22	.03 0046	19.85	.96 9954	53
8	8744	19.57	7507	.23	1237	19.80	8763	52
9	.02 9918	19.52	7493	.22	2425	19.73	7575	51
10	9.03 1089	19.47	9.99 7480	.23	9.03 3609	19.70	10.96 6391	50
11	2257	19.40	7466	.23	4791	19.63	5209	49
12	3421	19.35	7452	.22	5969	19.58	4031	48
13	4582	19.32	7439	.23	7144	19.53	2856	47
14	5741	19.25	7425	.23	8316	19.48	1684	46
15	9.03 6896	19.20	9.99 7411	.23	9.03 9485	19.43	10.96 0515	45
16	8048	19.15	7397	.23	.04 0651	19.37	.95 9349	44
17	.03 9197	19.08	7383	.23	1813	19.33	8187	43
18	.04 0342	19.05	7369	.23	2973	19.28	7027	42
19	1485	19.00	7355	.23	4130	19.23	5870	41
20	9.04 2625	18.95	9.99 7341	.23	9.04 5284	19.17	10.95 4716	40
21	3762	18.88	7327	.23	6434	19.13	3566	39
22	4895	18.85	7313	.23	7582	19.08	2418	38
23	6026	18.80	7299	.23	8727	19.03	1273	37
24	7154	18.75	7285	.23	.04 9869	18.98	.95 0131	36
25	9.04 8279	18.68	9.99 7271	.23	9.05 1003	18.93	10.94 8992	35
26	.04 9400	18.65	7257	.25	2144	18.88	7856	34
27	.05 0519	18.60	7242	.23	3277	18.83	6723	33
28	1635	18.57	7228	.23	4407	18.80	5593	32
29	2749	18.50	7214	.25	5535	18.73	4465	31
30	9.05 3859	18.45	9.99 7199	.23	9.05 6659	18.70	10.94 3341	30
31	4966	18.42	7185	.25	7781	18.65	2219	29
32	6071	18.35	7170	.23	.05 8900	18.60	.94 1100	28
33	7172	18.32	7156	.25	.06 0016	18.57	.93 9984	27
34	8271	18.27	7141	.23	1130	18.50	8870	26
35	9.05 9367	18.22	9.99 7127	.25	9.06 2240	18.47	10.93 7760	25
36	.06 0460	18.18	7112	.23	3348	18.42	6652	24
37	1551	18.13	7098	.25	4453	18.38	5547	23
38	2639	18.08	7083	.25	5556	18.32	4444	22
39	3724	18.03	7068	.25	6655	18.28	3345	21
40	9.06 4806	17.98	9.99 7053	.23	9.06 7752	18.25	10.93 2248	20
41	5885	17.95	7039	.25	8846	18.20	1154	19
42	6962	17.90	7024	.25	.06 9938	18.15	.93 0062	18
43	8036	17.85	7009	.25	.07 1027	18.10	.92 8973	17
44	.06 9107	17.82	6994	.25	2113	18.07	7887	16
45	9.07 0176	17.77	9.99 6979	.25	9.07 3197	18.02	10.92 6803	15
46	1242	17.73	6964	.25	4278	17.97	5722	14
47	2306	17.67	6949	.25	5356	17.93	4644	13
48	3366	17.63	6934	.25	6432	17.88	3568	12
49	4424	17.60	6919	.25	7505	17.85	2495	11
50	9.07 5480	17.55	9.99 6904	.25	9.07 8576	17.80	10.92 1424	10
51	6533	17.50	6889	.25	.07 9644	17.77	.92 0356	9
52	7583	17.47	6874	.27	.08 0710	17.72	.91 9290	8
53	8631	17.42	6858	.25	1773	17.67	8227	7
54	.07 9676	17.38	6843	.27	2833	17.63	7167	6
55	9.08 0719	17.33	9.99 6828	.27	9.08 3891	17.60	10.91 6109	5
56	1759	17.30	6812	.25	4947	17.55	5053	4
57	2797	17.25	6797	.25	6000	17.50	4000	3
58	3832	17.20	6782	.27	7050	17.47	2950	2
59	4864	17.17	6766	.25	8098	17.43	1902	1
60	9.08 5894		9.99 6751		9.08 9144		10.91 0856	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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7° TABLE XII.—*Continued.* 172°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.08 5894**	17.13	**9.99 6751**	.27	**9.08 9144**	17.38	**10.91 0856**	**60**
1	6922	17.08	6735	.25	.09 0187	17.35	.90 9813	59
2	7947	17.05	6720	.27	1228	17.30	8772	58
3	8970	17.00	6704	.27	2266	17.27	7734	57
4	.08 9990	16.97	6688	.25	3302	17.23	6698	56
5	**9.09 1008**	16.93	**9.99 6673**	.27	**9.09 4336**	17.18	**10.90 5664**	**55**
6	2024	16.88	6657	.27	5367	17.13	4633	54
7	3037	16.83	6641	.27	6395	17.12	3605	53
8	4047	16.82	6625	.25	7422	17.07	2578	52
9	5056	16.77	6610	.27	8446	17.03	1554	51
10	**9.09 6062**	16.72	**9.99 6594**	.27	**9.09 9468**	16.98	**10.90 0532**	**50**
11	7065	16.68	6578	.27	.10 0487	16.95	.89 9513	49
12	8066	16.65	6562	.27	1504	16.92	8496	48
13	.09 9065	16.62	6546	.27	2519	16.88	7481	47
14	.10 0062	16.57	6530	.27	3532	16.83	6468	46
15	**9.10 1056**	16.53	**9.99 6514**	.27	**9.10 4542**	16.80	**10.89 5458**	**45**
16	2048	16.48	6498	.27	5550	16.77	4450	44
17	3037	16.47	6482	.28	6556	16.72	3444	43
18	4025	16.42	6465	.27	7559	16.68	2441	42
19	5010	16.37	6449	.27	8560	16.65	1440	41
20	**9.10 5992**	16.35	**9.99 6433**	.27	**9.10 9559**	16.62	**10.89 0441**	**40**
21	6973	16.30	6417	.28	.11 0556	16.58	.88 9444	39
22	7951	16.27	6400	.27	1551	16.53	8449	38
23	8927	16.23	6384	.27	2543	16.50	7457	37
24	.10 9901	16.20	6368	.28	3533	16.47	6467	36
25	**9.11 0873**	16.15	**9.99 6351**	.27	**9.11 4521**	16.43	**10.88 5479**	**35**
26	1842	16.12	6335	.28	5507	16.40	4493	34
27	2809	16.08	6318	.27	6491	16.35	3509	33
28	3774	16.05	6302	.28	7472	16.33	2528	32
29	4737	16.02	6285	.27	8452	16.28	1548	31
30	**9.11 5698**	15.97	**9.99 6269**	.28	**9.11 9429**	16.25	**10.88 0571**	**30**
31	6656	15.95	6252	.28	.12 0404	16.22	.87 9596	29
32	7613	15.90	6235	.27	1377	16.18	8623	28
33	8567	15.87	6219	.28	2348	16.15	7652	27
34	.11 9519	15.83	6202	.28	3317	16.12	6683	26
35	**9.12 0469**	15.80	**9.99 6185**	.28	**9.12 4284**	16.08	**10.87 5716**	**25**
36	1417	15.75	6168	.28	5249	16.03	4751	24
37	2362	15.73	6151	.28	6211	16.02	3789	23
38	3306	15.70	6134	.28	7172	15.97	2828	22
39	4248	15.65	6117	.28	8130	15.95	1870	21
40	**9.12 5187**	15.63	**9.99 6100**	.28	**9.12 9087**	15.90	**10.87 0913**	**20**
41	6125	15.58	6083	.28	.13 0041	15.88	.86 9959	19
42	7060	15.55	6066	.28	0994	15.83	9006	18
43	7993	15.53	6049	.28	1944	15.82	8056	17
44	8925	15.48	6032	.28	2893	15.77	7107	16
45	**9.12 9854**	15.45	**9.99 6015**	.28	**9.13 3839**	15.75	**10.86 6161**	**15**
46	.13 0781	15.42	5998	.30	4784	15.70	5216	14
47	1706	15.40	5980	.28	5726	15.68	4274	13
48	2630	15.35	5963	.28	6667	15.63	3333	12
49	3551	15.32	5946	.30	7605	15.62	2395	11
50	**9.13 4470**	15.28	**9.99 5928**	.28	**9.13 8542**	15.57	**10.86 1458**	**10**
51	5387	15.27	5911	.28	.13 9476	15.55	.86 0524	9
52	6303	15.22	5894	.30	.14 0409	15.52	.85 9591	8
53	7216	15.20	5876	.28	1340	15.48	8660	7
54	8128	15.15	5859	.30	2269	15.45	7731	6
55	**9.13 9037**	15.12	**9.99 5841**	.30	**9.14 3196**	15.42	**10.85 6804**	**5**
56	.13 9944	15.10	5823	.28	4121	15.38	5879	4
57	.14 0850	15.07	5806	.30	5044	15.37	4956	3
58	1754	15.02	5788	.28	5966	15.32	4034	2
59	2655	15.00	5771	.30	6885	15.30	3115	1
60	**9.14 3555**		**9.99 5753**		**9.14 7803**		**10.85 2197**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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8° TABLE XII.—*Continued.* 171°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.14 3555**	14.97	**9.99 5753**	.30	**9.14 7803**	15.25	**10.85 2197**	**60**
1	4453	14.93	5735	.30	8718	15.23	1282	59
2	5349	14.90	5717	.30	.14 9632	15.20	.85 0368	58
3	6243	14.88	5699	.30	.15 0544	15.17	.84 9456	57
4	7136	14.83	5681	.28	1454	15.15	8546	56
5	**9.14 8026**	14.82	**9.99 5664**	.30	**9.15 2363**	15.10	**10.84 7637**	**55**
6	8915	14.78	5646	.30	3269	15.08	6731	54
7	.14 9802	14.73	5628	.30	4174	15.05	5826	53
8	.15 0686	14.72	5610	.32	5077	15.02	4923	52
9	1569	14.70	5591	.30	5978	14.98	4022	51
10	**9.15 2451**	14.65	**9.99 5573**	.30	**9.15 6877**	14.97	**10.84 3123**	**50**
11	3330	14.63	5555	.30	7775	14.93	2225	49
12	4208	14.58	5537	.30	8671	14.90	1329	48
13	5083	14.57	5519	.30	.15 9565	14.87	.84 0435	47
14	5957	14.55	5501	.32	.16 0457	14.83	.83 9543	46
15	**9.15 6830**	14.50	**9.99 5482**	.30	**9.16 1347**	14.82	**10.83 8653**	**45**
16	7700	14.48	5464	.30	2236	14.78	7764	44
17	8569	14.43	5446	.32	3123	14.75	6877	43
18	.15 9435	14.43	5427	.30	4008	14.73	5992	42
19	.16 0301	14.38	5409	.32	4892	14.70	5108	41
20	**9.16 1164**	14.35	**9.99 5390**	.30	**9.16 5774**	14.67	**10.83 4226**	**40**
21	2025	14.33	5372	.32	6654	14.63	3346	39
22	2885	14.30	5353	.32	7532	14.62	2468	38
23	3743	14.28	5334	.30	8409	14.58	1591	37
24	4600	14.23	5316	.32	.16 9284	14.55	.83 0716	36
25	**9.16 5454**	14.22	**9.99 5297**	.32	**9.17 0157**	14.53	**10.82 9843**	**35**
26	6307	14.20	5278	.30	1029	14.50	8971	34
27	7159	14.15	5260	.32	1899	14.47	8101	33
28	8008	14.13	5241	.32	2767	14.45	7233	32
29	8856	14.10	5222	.32	3634	14.42	6366	31
30	**9.16 9702**	14.08	**9.99 5203**	.32	**9.17 4499**	14.38	**10.82 5501**	**30**
31	.17 0547	14.03	5184	.32	5362	14.37	4638	29
32	1389	14.02	5165	.32	6224	14.33	3776	28
33	2230	14.00	5146	.32	7084	14.30	2916	27
34	3070	13.97	5127	.32	7942	14.28	2058	26
35	**9.17 3908**	13.93	**9.99 5108**	.32	**9.17 8799**	14.27	**10.82 1201**	**25**
36	4744	13.90	5089	.32	.17 9655	14.22	.82 0345	24
37	5578	13.88	5070	.32	.18 0508	14.20	.81 9492	23
38	6411	13.85	5051	.32	1360	14.18	8640	22
39	7242	13.83	5032	.32	2211	14.13	7789	21
40	**9.17 8072**	13.80	**9.99 5013**	.33	**9.18 3059**	14.13	**10.81 6941**	**20**
41	8900	13.77	4993	.32	3907	14.08	6093	19
42	.17 9726	13.75	4974	.32	4752	14.08	5248	18
43	.18 0551	13.72	4955	.33	5597	14.03	4403	17
44	1374	13.70	4935	.32	6439	14.02	3561	16
45	**9.18 2196**	13.67	**9.99 4916**	.33	**9.18 7280**	14.00	**10.81 2720**	**15**
46	3016	13.63	4896	.32	8120	13.97	1880	14
47	3834	13.62	4877	.33	8958	13.93	1042	13
48	4651	13.58	4857	.32	.18 9794	13.92	.81 0206	12
49	5466	13.57	4838	.33	.19 0629	13.88	.80 9371	11
50	**9.18 6280**	13.53	**9.99 4818**	.33	**9.19 1462**	13.87	**10.80 8538**	**10**
51	7092	13.52	4798	.32	2294	13.83	7706	9
52	7903	13.48	4779	.33	3124	13.82	6876	8
53	8712	13.45	4759	.33	3953	13.78	6047	7
54	.18 9519	13.43	4739	.32	4780	13.77	5220	6
55	**9.19 0325**	13.42	**9.99 4720**	.33	**9.19 5606**	13.73	**10.80 4394**	**5**
56	1130	13.38	4700	.33	6430	13.72	3570	4
57	1933	13.35	4680	.33	7253	13.68	2747	3
58	2734	13.33	4660	.33	8074	13.67	1926	2
59	3534	13.30	4640	.33	8894	13.65	1106	1
60	**9.19 4332**		**9.99 4620**		**9.19 9713**		**10.80 0287**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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9° TABLE XII.—*Continued.* 170°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.19 4332**	13.28	**9.99 4620**	.33	**9.19 9713**	13.60	**10.80 0287**	**60**
1	5129	13.27	4600	.33	.20 0529	13.60	.79 9471	59
2	5925	13.23	4580	.33	1345	13.57	8655	58
3	6719	13.20	4560	.33	2159	13.53	7841	57
4	7511	13.18	4540	.35	2971	13.52	7029	56
5	**9.19 8302**	13.15	**9.99 4519**	.33	**9.20 3782**	13.50	**10.79 6218**	**55**
6	9091	13.13	4499	.33	4592	13.47	5408	54
7	.19 9879	13.12	4479	.33	5400	13.45	4600	53
8	.20 0666	13.08	4459	.35	6207	13.43	3793	52
9	1451	13.05	4438	.33	7013	13.40	2987	51
10	**9.20 2234**	13.05	**9.99 4418**	.33	**9.20 7817**	13.37	**10.79 2183**	**50**
11	3017	13.00	4398	.35	8619	13.35	1381	49
12	3797	13.00	4377	.33	.20 9420	13.33	.79 0580	48
13	4577	12.95	4357	.35	.21 0220	13.30	.78 9780	47
14	5354	12.95	4336	.33	1018	13.28	8982	46
15	**9.20 1631**	12.92	**9.99 4316**	.35	**9.21 1815**	13.27	**10.78 8185**	**45**
16	6906	12.88	4295	.35	2611	13.23	7389	44
17	7679	12.88	4274	.33	3405	13.22	6595	43
18	8452	12.83	4254	.35	4198	13.18	5802	42
19	9222	12.83	4233	.35	4989	13.18	5011	41
20	**9.20 9992**	12.80	**9.99 4212**	.35	**9.21 5780**	13.13	**10.78 4220**	**40**
21	.21 0760	12.77	4191	.33	6568	13.13	3432	39
22	1526	12.75	4171	.35	7356	13.10	2644	38
23	2291	12.73	4150	.35	8142	13.07	1858	37
24	3055	12.72	4129	.35	8926	13.07	1074	36
25	**9.21 3818**	12.68	**9.99 4108**	.35	**9.21 9710**	13.03	**10.78 0290**	**35**
26	4579	12.65	4087	.35	.22 0492	13.00	.77 9508	34
27	5338	12.65	4066	.35	1272	13.00	8728	33
28	6097	12.62	4045	.35	2052	12.97	7948	32
29	6854	12.58	4024	.35	2830	12.95	7170	31
30	**9.21 7609**	12.57	**9.99 4003**	.35	**9.22 3607**	12.92	**10.77 6393**	**30**
31	8363	12.55	3982	.37	4382	12.90	5618	29
32	9116	12.53	3960	.35	5156	12.88	4844	28
33	.21 9868	12.50	3939	.35	5929	12.85	4071	27
34	.22 0618	12.48	3918	.35	6700	12.85	3300	26
35	**9.22 1367**	12.47	**9.99 3897**	.37	**9.22 7471**	12.80	**10.77 2529**	**25**
36	2115	12.43	3875	.35	8239	12.80	1761	24
37	2861	12.42	3854	.37	9007	12.77	0993	23
38	3606	12.38	3832	.35	.22 9773	12.77	.77 0227	22
39	4349	12.38	3811	.37	.23 0539	12.72	.76 9461	21
40	**9.22 5092**	12.35	**9.99 3739**	.35	**9.23 1302**	12.72	**10.76 8698**	**20**
41	5833	12.33	3768	.37	2065	12.68	7935	19
42	6573	12.30	3746	.35	2826	12.67	7174	18
43	7311	12.28	3725	.37	3586	12.65	6414	17
44	8048	12.27	3703	.37	4345	12.63	5655	16
45	**9.22 8784**	12.23	**9.99 3681**	.35	**9.23 5103**	12.60	**10.76 4897**	**15**
46	.22 9518	12.23	3660	.37	5859	12.58	4141	14
47	.23 0252	12.20	3638	.37	6614	12.57	3386	13
48	0984	12.18	3616	.37	7368	12.53	2632	12
49	1715	12.15	3594	.37	8120	12.53	1880	11
50	**9.23 2444**	12.13	**9.99 3572**	.37	**9.23 8872**	12.50	**10.76 1128**	**10**
51	3172	12.12	3550	.37	.23 9622	12.48	.76 0378	9
52	3899	12.10	3528	.37	.24 0371	12.45	.75 9629	8
53	4625	12.07	3506	.37	1118	12.45	8882	7
54	5349	12.07	3484	.37	1865	12.42	8135	6
55	**9.23 6073**	12.03	**9.99 3462**	.37	**9.24 2610**	12.40	**10.75 7390**	**5**
56	6795	12.00	3440	.37	3354	12.38	6646	4
57	7515	12.00	3418	.37	4097	12.37	5903	3
58	8235	11.97	3396	.37	4839	12.33	5161	2
59	8953	11.95	3374	.38	5579	12.33	4421	1
60	**9.23 9670**		**9.99 3351**		**9.24 6319**		**10.75 3681**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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10° TABLE XII.—*Continued.* 169°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.23 9670**	11.93	**9.99 3351**	.37	**9.24 6319**	12.30	**10.75 3681**	**60**
1	.24 0386	11.92	3329	.37	7057	12.28	2943	59
2	1101	11.88	3307	.38	7794	12.27	2206	58
3	1814	11.87	3284	.37	8530	12.23	1470	57
4	2526	11.85	3262	.37	9264	12.23	0736	56
5	**9.24 3237**	11.83	**9.99 3240**	.38	**9.24 9998**	12.20	**10.75 0002**	**55**
6	3947	11.82	3217	.37	.25 0730	12.18	.74 9270	54
7	4656	11.78	3195	.38	1461	12.17	8539	53
8	5363	11.77	3172	.38	2191	12.15	7809	52
9	6069	11.77	3149	.37	2920	12.13	7080	51
10	**9.24 6775**	11.72	**9.99 3127**	.38	**9.25 3648**	12.10	**10.74 6352**	**50**
11	7478	11.72	3104	.38	4374	12.10	5626	49
12	8181	11.70	3081	.37	5100	12.07	4900	48
13	8883	11.67	3059	.38	5824	12.05	4176	47
14	.24 9583	11.65	3036	.38	6547	12.03	3453	46
15	**9.25 0282**	11.63	**9.99 3013**	.38	**9.25 7269**	12.02	**10.74 2731**	**45**
16	0980	11.62	2990	.38	7990	12.00	2010	44
17	1677	11.60	2967	.38	8710	11.98	1290	43
18	2373	11.57	2944	.38	.25 9429	11.95	.74 0571	42
19	3067	11.57	2921	.38	.26 0146	11.95	.73 9854	41
20	**9.25 3761**	11.53	**9.99 2898**	.38	**9.26 0863**	11.92	**10.73 9137**	**40**
21	4453	11.52	2875	.38	1578	11.90	8422	39
22	5144	11.50	2852	.38	2292	11.88	7708	38
23	5834	11.48	2829	.38	3005	11.87	6995	37
24	6523	11.47	2806	.38	3717	11.85	6283	36
25	**9.25 7211**	11.45	**9.99 2783**	.40	**9.26 4428**	11.83	**10.73 5572**	**35**
26	7898	11.42	2759	.38	5138	11.82	4862	34
27	8583	11.42	2736	.38	5847	11.80	4153	33
28	9268	11.38	2713	.38	6555	11.77	3445	32
29	.25 9951	11.37	2690	.40	7261	11.77	2739	31
30	**9.26 0633**	11.35	**9.99 2666**	.38	**9.26 7967**	11.73	**10.73 2033**	**30**
31	1314	11.33	2643	.40	8671	11.73	1329	29
32	1994	11.32	2619	.38	.26 9375	11.70	.73 0625	28
33	2673	11.30	2596	.40	.27 0077	11.70	.72 9923	27
34	3351	11.27	2572	.38	0779	11.67	9221	26
35	**9.26 4027**	11.27	**9.99 2549**	.40	**9.27 1479**	11.65	**10.72 8521**	**25**
36	4703	11.23	2525	.40	2178	11.63	7822	24
37	5377	11.23	2501	.38	2876	11.62	7124	23
38	6051	11.20	2478	.40	3573	11.60	6427	22
39	6723	11.20	2454	.40	4269	11.58	5731	21
40	**9.26 7395**	11.17	**9.99 2430**	.40	**9.27 4964**	11.57	**10.72 5036**	**20**
41	8065	11.15	2406	.40	5658	11.55	4342	19
42	8734	11.13	2382	.38	6351	11.53	3649	18
43	.26 9402	11.12	2359	.40	7043	11.52	2957	17
44	.27 0069	11.10	2335	.40	7734	11.50	2266	16
45	**9.27 0735**	11.08	**9.99 2311**	.40	**9.27 8424**	11.48	**10.72 1576**	**15**
46	1400	11.07	2287	.40	9113	11.47	0887	14
47	2064	11.03	2263	.40	.27 9801	11.45	.72 0199	13
48	2726	11.03	2239	.42	.28 0488	11.43	.71 9512	12
49	3388	11.02	2214	.40	1174	11.40	8826	11
50	**9.27 4049**	10.98	**9.99 2190**	.40	**9.28 1858**	11.40	**10.71 8142**	**10**
51	4708	10.98	2166	.40	2542	11.38	7458	9
52	5367	10.97	2142	.40	3225	11.37	6775	8
53	6025	10.93	2118	.42	3907	11.35	6093	7
54	6681	10.93	2093	.40	4588	11.33	5412	6
55	**9.27 7337**	10.90	**9.99 2069**	.42	**9.28 5268**	11.32	**10.71 4732**	**5**
56	7991	10.90	2044	.40	5947	11.28	4053	4
57	8645	10.87	2020	.40	6624	11.28	3376	3
58	9297	10.85	1996	.42	7301	11.27	2699	2
59	.27 9948	10.85	1971	.40	7977	11.25	2023	1
60	**9.28 0599**		**9.99 1947**		**9.28 8652**		**10.71 1348**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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11° **Table XII.**—*Continued.* 168°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.28 0599**	10.82	**9.99 1947**	.42	**9.28 8652**	11.23	**10.71 1348**	**60**
1	1248	10.82	1922	.42	9326	11.22	0674	59
2	1897	10.78	1897	.40	.28 9999	11.20	.71 0001	58
3	2544	10.77	1873	.42	.29 0671	11.18	.70 9329	57
4	3190	10.77	1848	.42	1342	11.18	8658	56
5	**9.28 3836**	10.73	**9.99 1823**	.40	**9.29 2013**	11.15	**10.70 7987**	**55**
6	4480	10.73	1799	.42	2682	11.13	7318	54
7	5124	10.70	1774	.42	3350	11.12	6650	53
8	5766	10.70	1749	.42	4017	11.12	5983	52
9	6408	10.67	1724	.42	4684	11.08	5316	51
10	**9.28 7048**	10.67	**9.99 1699**	.42	**9.29 5349**	11.07	**10.70 4651**	**50**
11	7688	10.63	1674	.42	6013	11.07	3987	49
12	8326	10.63	1649	.42	6677	11.03	3323	48
13	8964	10.60	1624	.42	7339	11.03	2661	47
14	.28 9600	10.60	1599	.42	8001	11.02	1999	46
15	**9.29 0236**	10.57	**9.99 1574**	.42	**9.29 8662**	11.00	**10.70 1338**	**45**
16	0870	10.57	1549	.42	9322	10.97	0678	44
17	1504	10.55	1524	.43	.29 9980	10.97	.70 0020	43
18	2137	10.52	1498	.42	.30 0638	10.95	.69 9362	42
19	2768	10.52	1473	.42	1295	10.93	8705	41
20	**9.29 3399**	10.50	**9.99 1448**	.43	**9.30 1951**	10.93	**10.69 8049**	**40**
21	4029	10.48	1422	.42	2607	10.90	7393	39
22	4658	10.47	1397	.42	3261	10.88	6739	38
23	5286	10.45	1372	.43	3914	10.88	6086	37
24	5913	10.43	1346	.42	4567	10.85	5433	36
25	**9.29 6539**	10.42	**9.99 1321**	.43	**9.30 5218**	10.85	**10.69 4782**	**35**
26	7164	10.40	1295	.42	5869	10.83	4131	34
27	7788	10.40	1270	.43	6519	10.82	3481	33
28	8412	10.37	1244	.43	7168	10.80	2832	32
29	9034	10.35	1218	.42	7816	10.78	2184	31
30	**9.29 9655**	10.35	**9.99 1193**	.43	**9.30 8463**	10.77	**10.69 1537**	**30**
31	.30 0276	10.32	1167	.43	9109	10.75	0891	29
32	0895	10.32	1141	.43	.30 9754	10.75	.69 0246	28
33	1514	10.30	1115	.42	.31 0399	10.72	.68 9601	27
34	2132	10.27	1090	.43	1042	10.72	8958	26
35	**9.30 2748**	10.27	**9.99 1064**	.43	**9.31 1685**	10.70	**10.68 8315**	**25**
36	3364	10.25	1038	.43	2327	10.68	7673	24
37	3979	10.23	1012	.43	2968	10.67	7032	23
38	4593	10.23	0986	.43	3608	10.65	6392	22
39	5207	10.20	0960	.43	4247	10.63	5753	21
40	**9.30 5819**	10.18	**9.99 0934**	.43	**9.31 4885**	10.63	**10.68 5115**	**20**
41	6430	10.18	0908	.43	5523	10.60	4477	19
42	7041	10.15	0882	.45	6159	10.60	3841	18
43	7650	10.15	0855	.43	6795	10.58	3205	17
44	8259	10.13	0829	.43	7430	10.57	2570	16
45	**9.30 8867**	10.12	**9.99 0803**	.43	**9.31 8064**	10.55	**10.68 1936**	**15**
46	.30 9474	10.10	0777	.45	8697	10.55	1303	14
47	.31 0080	10.08	0750	.43	9330	10.52	0670	13
48	0685	10.07	0724	.45	.31 9961	10.52	.68 0039	12
49	1289	10.07	0697	.43	.32 0592	10.50	.67 9408	11
50	**9.31 1893**	10.03	**9.99 0671**	.43	**9.32 1222**	10.48	**10.67 8778**	**10**
51	2495	10.03	0645	.45	1851	10.47	8149	9
52	3097	10.02	0618	.45	2479	10.45	7521	8
53	3698	9.98	0591	.43	3106	10.45	6894	7
54	4297	10.00	0565	.45	3733	10.42	6267	6
55	**9.31 4897**	9.97	**9.99 0538**	.45	**9.32 4358**	10.42	**10.67 5642**	**5**
56	5495	9.95	0511	.43	4983	10.40	5017	4
57	6092	9.95	0485	.45	5607	10.40	4393	3
58	6689	9.92	0458	.45	6231	10.37	3769	2
59	7284	9.92	0431	.45	6853	10.37	3147	1
60	**9.31 7879**		**9.99 0404**		**9.32 7475**		**10.67 2525**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

101° 78°

12° TABLE XII.—*Continued.* 167°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.31 7879**	9.90	**9.99 0404**	.43	**9.32 7475**	10.33	**10.67 2525**	**60**
1	8473	9.88	0378	.45	8095	10.33	1905	59
2	9066	9.87	0351	.45	8715	10.32	1285	58
3	.31 9658	9.85	0324	.45	9334	10.32	0666	57
4	.32 0249	9.85	0297	.45	.32 9953	10.28	.67 0047	56
5	**9.32 0840**	9.83	**9.99 0270**	.45	**9.33 0570**	10.28	**10.66 9430**	**55**
6	1430	9.82	0243	.47	1187	10.27	8813	54
7	2019	9.80	0215	.45	1803	10.25	8197	53
8	2607	9.78	0188	.45	2418	10.25	7582	52
9	3194	9.77	0161	.45	3033	10.22	6967	51
10	**9.32 3780**	9.77	**9.99 0134**	.45	**9.33 3646**	10.22	**10.66 6354**	**50**
11	4366	9.73	0107	.47	4259	10.20	5741	49
12	4950	9.73	0079	.45	4871	10.18	5129	48
13	5534	9.72	0052	.45	5482	10.18	4518	47
14	6117	9.72	.99 0025	.47	6093	10.15	3907	46
15	**9.32 6700**	9.68	**9.98 9997**	.45	**9.33 6702**	10.15	**10.66 3298**	**45**
16	7281	9.68	9970	.47	7311	10.13	2689	44
17	7862	9.67	9942	.45	7919	10.13	2081	43
18	8442	9.65	9915	.47	8527	10.10	1473	42
19	9021	9.63	9887	.45	9133	10.10	0867	41
20	**9.32 9599**	9.62	**9.98 9860**	.47	**9.33 9739**	10.08	**10.66 0261**	**40**
21	.33 0176	9.62	9832	.47	.34 0344	10.07	.65 9656	39
22	0753	9.60	9804	.45	0948	10.07	9052	38
23	1329	9.57	9777	.47	1552	10.05	8448	37
24	1903	9.58	9749	.47	2155	10.03	7845	36
25	**9.33 2478**	9.55	**9.98 9721**	.47	**9.34 2757**	10.02	**10.65 7243**	**35**
26	3051	9.55	9693	.47	3358	10.00	6642	34
27	3624	9.52	9665	.47	3958	10.00	6042	33
28	4195	9.53	9637	.45	4558	9.98	5442	32
29	4767	9.50	9610	.47	5157	9.97	4843	31
30	**9.33 5337**	9.48	**9.98 9582**	.48	**9.34 5755**	9.97	**10.65 4245**	**30**
31	5906	9.48	9553	.47	6353	9.93	3647	29
32	6475	9.47	9525	.47	6949	9.93	3051	28
33	7043	9.45	9497	.47	7545	9.93	2455	27
34	7610	9.43	9469	.47	8141	9.90	1859	26
35	**9.33 8176**	9.43	**9.98 9441**	.47	**9.34 8735**	9.90	**10.65 1265**	**25**
36	8742	9.42	9413	.47	9329	9.88	0671	24
37	9307	9.40	9385	.48	.34 9922	9.87	.65 0078	23
38	.33 9871	9.38	9356	.47	.35 0514	9.87	.64 9486	22
39	.34 0434	9.37	9328	.47	1106	9.85	8894	21
40	**9.34 0996**	9.37	**9.98 9300**	.48	**9.35 1697**	9.83	**10.64 8303**	**20**
41	1558	9.35	9271	.47	2287	9.82	7713	19
42	2119	9.33	9243	.48	2876	9.82	7124	18
43	2679	9.33	9214	.47	3465	9.80	6535	17
44	3239	9.30	9186	.48	4053	9.78	5947	16
45	**9.34 3797**	9.30	**9.98 9157**	.48	**9.35 4640**	9.78	**10.64 5360**	**15**
46	4355	9.28	9128	.47	5227	9.77	4773	14
47	4912	9.28	9100	.48	5813	9.75	4187	13
48	5469	9.25	9071	.48	6398	9.73	3602	12
49	6024	9.25	9042	.47	6982	9.73	3018	11
50	**9.34 6579**	9.25	**9.98 9014**	.48	**9.35 7566**	9.72	**10.64 2434**	**10**
51	7134	9.22	8985	.48	8149	9.70	1851	9
52	7687	9.22	8956	.48	8731	9.70	1269	8
53	8240	9.20	8927	.48	9313	9.67	0687	7
54	8792	9.18	8898	.48	.35 9893	9.68	.64 0107	6
55	**9.34 9343**	9.17	**9.98 8869**	.48	**9.36 0474**	9.65	**10.63 9526**	**5**
56	.34 9893	9.17	8840	.48	1053	9.65	8947	4
57	.35 0443	9.15	8811	.48	1632	9.63	8368	3
58	0992	9.13	8782	.48	2210	9.62	7790	2
59	1540	9.13	8753	.48	2787	9.62	7213	1
60	**9.35 2088**		**9.98 8724**		**9.36 3364**		**10.63 6636**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

102° 77°

13° TABLE XII.—*Continued.* 166°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.35 2088	9.12	9.98 8724	.48	9.36 3364	9.60	10.63 6636	60
1	2635	9.10	8695	.48	3940	9.58	6060	59
2	3181	9.08	8666	.50	4515	9.58	5485	58
3	3726	9.08	8636	.48	5090	9.57	4910	57
4	4271	9.07	8607	.48	5664	9.55	4336	56
5	9.35 4815	9.05	9.98 8578	.50	9.36 6237	9.55	10.63 3763	55
6	5358	9.05	8548	.48	6810	9.53	3190	54
7	5901	9.03	8519	.50	7382	9.52	2618	53
8	6443	9.02	8489	.48	7953	9.52	2047	52
9	6984	9.00	8460	.50	8524	9.50	1476	51
10	9.35 7524	9.00	9.98 8430	.48	9.36 9094	9.48	10.63 0906	50
11	8064	8.98	8401	.50	.36 9663	9.48	.63 0337	49
12	8603	8.97	8371	.48	.37 0232	9.45	.62 9768	48
13	9141	8.95	8342	.50	0799	9.47	9201	47
14	.35 9678	8.95	8312	.50	1367	9.43	8633	46
15	9.36 0215	8.95	9.98 8282	.50	9.37 1933	9.43	10.62 8067	45
16	0752	8.92	8252	.48	2499	9.42	7501	44
17	1287	8.92	8223	.50	3064	9.42	6936	43
18	1822	8.90	8193	.50	3629	9.40	6371	42
19	2356	8.88	8163	.50	4193	9.38	5807	41
20	9.36 2889	8.88	9.98 8133	.50	9.37 4756	9.38	10.62 5244	40
21	3422	8.87	8103	.50	5319	9.37	4681	39
22	3954	8.85	8073	.50	5881	9.35	4119	38
23	4485	8.85	8043	.50	6442	9.35	3558	37
24	5016	8.83	8013	.50	7003	9.33	2997	36
25	9.36 5546	8.82	9.98 7983	.50	9.37 7563	9.32	10.62 2437	35
26	6075	8.82	7953	.52	8122	9.32	1878	34
27	6604	8.78	7922	.50	8681	9.30	1319	33
28	7131	8.80	7892	.50	9239	9.30	0761	32
29	7659	8.77	7862	.50	.37 9797	9.28	.62 0203	31
30	9.36 8185	8.77	9.98 7832	.52	9.38 0354	9.27	10.61 9646	30
31	8711	8.75	7801	.50	0910	9.27	9090	29
32	9236	8.75	7771	.52	1466	9.23	8534	28
33	.36 9761	8.72	7740	.50	2020	9.25	7980	27
34	.37 0285	8.72	7710	.52	2575	9.23	7425	26
35	9.37 0808	8.70	9.98 7679	.50	9.38 3129	9.22	10.61 6871	25
36	1330	8.70	7649	.52	3682	9.20	6318	24
37	1852	8.68	7618	.50	4234	9.20	5766	23
38	2373	8.68	7588	.52	4786	9.18	5214	22
39	2894	8.67	7557	.52	5337	9.18	4663	21
40	9.37 3414	8.65	9.98 7526	.50	9.38 5888	9.17	10.61 4112	20
41	3933	8.65	7496	.52	6438	9.15	3562	19
42	4452	8.63	7465	.52	6987	9.15	3013	18
43	4970	8.62	7434	.52	7536	9.13	2464	17
44	5487	8.60	7403	.52	8084	9.12	1916	16
45	9.37 6003	8.60	9.98 7372	.52	9.38 8631	9.12	10.61 1369	15
46	6519	8.60	7341	.52	9178	9.10	0822	14
47	7035	8.57	7310	.52	.38 9724	9.10	.61 0276	13
48	7549	8.57	7279	.52	.39 0270	9.08	.60 9730	12
49	8063	8.57	7248	.52	0815	9.08	9185	11
50	9.37 8577	8.53	9.98 7217	.52	9.39 1360	9.05	10.60 8640	10
51	9089	8.53	7186	.52	1903	9.07	8097	9
52	.37 9601	8.53	7155	.52	2447	9.03	7553	8
53	.38 0113	8.52	7124	.53	2989	9.03	7011	7
54	0624	8.50	7092	.52	3531	9.03	6469	6
55	9.38 1134	8.48	9.98 7061	.52	9.39 4073	9.02	10.60 5927	5
56	1643	8.48	7030	.53	4614	9.00	5386	4
57	2152	8.48	6998	.52	5154	9.00	4846	3
58	2661	8.45	6967	.52	5694	8.98	4306	2
59	3168	8.45	6936	.53	6233	8.97	3767	1
60	9.38 3675		9.98 6904		9.39 6771		10.60 3229	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

103° 76°

14° TABLE XII.—*Continued.* 165°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.38 3675**	8.45	**9.98 6904**	.52	**9.39 6771**	8.97	**10.60 3229**	**60**
1	4182	8.42	6873	.53	7309	8.95	2691	59
2	4687	8.42	6841	.53	7846	8.95	2154	58
3	5192	8.42	6809	.52	8383	8.93	1617	57
4	5697	8.40	6778	.53	8919	8.93	1081	56
5	**9.38 6201**	8.38	**9.98 6746**	.53	**9.39 9455**	8.92	**10.60 0545**	**55**
6	6704	8.38	6714	.52	.39 9990	8.90	.60 0010	54
7	7207	8.37	6683	.53	.40 0524	8.90	.59 9476	53
8	7709	8.35	6651	.53	1058	8.88	8942	52
9	8210	8.35	6619	.53	1591	8.88	8409	51
10	**9.38 8711**	8.33	**9.98 6587**	.53	**9.40 2124**	8.87	**10.59 7876**	**50**
11	9211	8.33	6555	.53	2656	8.85	7344	49
12	.38 9711	8.32	6523	.53	3187	8.85	6813	48
13	.39 0210	8.30	6491	.53	3718	8.85	6282	47
14	0708	8.30	6459	.53	4249	8.82	5751	46
15	**9.39 1206**	8.28	**9.98 6427**	.53	**9.40 4778**	8.83	**10.59 5222**	**45**
16	1703	8.27	6395	.53	5308	8.80	4692	44
17	2199	8.27	6363	.53	5836	8.80	4164	43
18	2695	8.27	6331	.53	6364	8.80	3636	42
19	3191	8.23	6299	.55	6892	8.78	3108	41
20	**9.39 3685**	8.23	**9.98 6266**	.53	**9.40 7419**	8.77	**10.59 2581**	**40**
21	4179	8.23	6234	.53	7945	8.77	2055	39
22	4673	8.22	6202	.55	8471	8.75	1529	38
23	5166	8.20	6169	.53	8996	8.75	1004	37
24	5658	8.20	6137	.55	.40 9521	8.73	.59 0479	36
25	**9.39 6150**	8.18	**9.98 6104**	.53	**9.41 0045**	8.73	**10.58 9955**	**35**
26	6641	8.18	6072	.55	0569	8.72	9431	34
27	7132	8.15	6039	.53	1092	8.72	8908	33
28	7621	8.17	6007	.55	1615	8.70	8385	32
29	8111	8.15	5974	.53	2137	8.68	7863	31
30	**9.39 8600**	8.13	**9.98 5942**	.55	**9.41 2658**	8.68	**10.58 7342**	**30**
31	9088	8.12	5909	.55	3179	8.67	6821	29
32	.39 9575	8.12	5876	.55	3699	8.67	6301	28
33	.40 0062	8.12	5843	.53	4219	8.65	5781	27
34	0549	8.10	5811	.55	4738	8.65	5262	26
35	**9.40 1035**	8.08	**9.98 5778**	.55	**9.41 5257**	8.63	**10.58 4743**	**25**
36	1520	8.08	5745	.55	5775	8.63	4225	24
37	2005	8.07	5712	.55	6293	8.62	3707	23
38	2489	8.05	5679	.55	6810	8.60	3190	22
39	2972	8.05	5646	.55	7326	8.60	2674	21
40	**9.40 3455**	8.05	**9.98 5613**	.55	**9.41 7842**	8.60	**10.58 2158**	**20**
41	3938	8.03	5580	.55	8358	8.53	1642	19
42	4420	8.02	5547	.55	8873	8.57	1127	18
43	4901	8.02	5514	.57	9387	8.57	0613	17
44	5382	8.00	5480	.55	.41 9901	8.57	.58 0099	16
45	**9.40 5862**	7.98	**9.98 5447**	.55	**9.42 0415**	8.55	**10.57 9585**	**15**
46	6341	7.98	5414	.55	0927	8.55	9073	14
47	6820	7.98	5381	.57	1440	8.53	8560	13
48	7299	7.97	5347	.55	1952	8.52	8048	12
49	7777	7.95	5314	.57	2463	8.52	7537	11
50	**9.40 8254**	7.95	**9.98 5280**	.55	**9.42 2974**	8.50	**10.57 7026**	**10**
51	8731	7.93	5247	.57	3484	8.48	6516	9
52	9207	7.93	5213	.55	3993	8.50	6007	8
53	.40 9682	7.92	5180	.57	4503	8.47	5497	7
54	.41 0157	7.92	5146	.55	5011	8.47	4989	6
55	**9.41 0632**	7.90	**9.98 5113**	.57	**9.42 5519**	8.47	**10.57 4481**	**5**
56	1106	7.88	5079	.57	6027	8.45	3973	4
57	1579	7.88	5045	.57	6534	8.45	3466	3
58	2052	7.87	5011	.55	7041	8.43	2959	2
59	2524	7.87	4978	.57	7547	8.42	2453	1
60	**9.41 2996**		**9.98 4944**		**9.42 8052**		**10.57 1948**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

104° 75°

15° TABLE XII.—*Continued.* 164°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.41 2996**	7.85	**9.98 4944**	.57	**9.42 8052**	8.43	**10.57 1948**	**60**
1	3467	7.85	4910	.57	8558	8.40	1442	59
2	3938	7.83	4876	.57	9062	8.40	0938	58
3	4408	7.83	4842	.57	.42 9566	8.40	.57 0434	57
4	4878	7.82	4808	.57	.43 0070	8.38	.56 9930	56
5	**9.41 5347**	7.80	**9.98 4774**	.57	**9.43 0573**	8.37	**10.56 9427**	**55**
6	5815	7.80	4740	.57	1075	8.37	8925	54
7	6283	7.80	4706	.57	1577	8.37	8423	53
8	6751	7.77	4672	.57	2079	8.35	7921	52
9	7217	7.78	4638	.58	2580	8.33	7420	51
10	**9.41 7684**	7.77	**9.98 4603**	.57	**9.43 3080**	8.33	**10.56 6920**	**50**
11	8150	7.75	4569	.57	3580	8.33	6420	49
12	8615	7.73	4535	.58	4080	8.32	5920	48
13	9079	7.75	4500	.57	4579	8.32	5421	47
14	.41 9544	7.72	4466	.57	5078	8.30	4922	46
15	**9.42 0007**	7.72	**9.98 4432**	.58	**9.43 5576**	8.28	**10.56 4424**	**45**
16	0470	7.72	4397	.57	6073	8.28	3927	44
17	0933	7.70	4363	.58	6570	8.28	3430	43
18	1395	7.70	4328	.57	7067	8.27	2933	42
19	1857	7.68	4294	.58	7563	8.27	2437	41
20	**9.42 2318**	7.67	**9.98 4259**	.58	**9.43 8059**	8.25	**10.56 1941**	**40**
21	2778	7.67	4224	.57	8554	8.23	1446	39
22	3238	7.65	4190	.58	9048	8.25	0952	38
23	3697	7.65	4155	.58	.43 9543	8.22	.56 0457	37
24	4156	7.65	4120	.58	.44 0036	8.22	.55 9964	36
25	**9.42 4615**	7.63	**9.98 4085**	.58	**9.44 0529**	8.22	**10.55 9471**	**35**
26	5073	7.62	4050	.58	1022	8.20	8978	34
27	5530	7.62	4015	.57	1514	8.20	8486	33
28	5987	7.60	3981	.58	2006	8.18	7994	32
29	6443	7.60	3946	.58	2497	8.18	7503	31
30	**9.42 6899**	7.58	**9.98 3911**	.60	**9.44 2988**	8.18	**10.55 7012**	**30**
31	7354	7.58	3875	.58	3479	8.15	6521	29
32	7809	7.57	3840	.58	3968	8.17	6032	28
33	8263	7.57	3805	.58	4458	8.15	5542	27
34	8717	7.55	3770	.58	4947	8.13	5053	26
35	**9.42 9170**	7.55	**9.98 3735**	.58	**9.44 5435**	8.13	**10.55 4565**	**25**
36	.42 9623	7.53	3700	.60	5923	8.13	4077	24
37	.43 0075	7.53	3664	.58	6411	8.12	3589	23
38	0527	7.52	3629	.58	6898	8.10	3102	22
39	0978	7.52	3594	.60	7384	8.10	2616	21
40	**9.43 1429**	7.50	**9.98 3558**	.58	**9.44 7870**	8.10	**10.55 2130**	**20**
41	1879	7.50	3523	.60	8356	8.08	1644	19
42	2329	7.48	3487	.58	8841	8.08	1159	18
43	2778	7.47	3452	.60	9326	8.07	0674	17
44	3226	7.48	3416	.58	.44 9810	8.07	.55 0190	16
45	**9.43 3675**	7.45	**9.98 3381**	.60	**9.45 0294**	8.05	**10.54 9706**	**15**
46	4122	7.45	3345	.60	0777	8.05	9223	14
47	4569	7.45	3309	.60	1260	8.05	8740	13
48	5016	7.43	3273	.58	1743	8.03	8257	12
49	5462	7.43	3238	.60	2225	8.02	7775	11
50	**9.43 5908**	7.42	**9.98 3202**	.60	**9.45 2706**	8.02	**10.54 7294**	**10**
51	6353	7.42	3166	.60	3187	8.02	6813	9
52	6798	7.40	3130	.60	3668	8.00	6332	8
53	7242	7.40	3094	.60	4148	8.00	5852	7
54	7686	7.38	3058	.60	4628	7.98	5372	6
55	**9.43 8129**	7.38	**9.98 3022**	.60	**9.45 5107**	7.98	**10.54 4893**	**5**
56	8572	7.37	2986	.60	5586	7.97	4414	4
57	9014	7.37	2950	.60	6064	7.97	3936	3
58	9456	7.35	2914	.60	6542	7.95	3458	2
59	.43 9897	7.35	2878	.60	7019	7.95	2981	1
60	**9.44 0338**		**9.98 2843**		**9.45 7496**		**10.54 2404**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

105° 74°

16° TABLE XII.—*Continued.* 163°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.44 0338**	7.33	**9.98 2842**	.62	**9.45 7496**	7.95	**10.54 2504**	**60**
1	0778	7.33	2805	.60	7973	7.93	2027	59
2	1218	7.33	2769	.60	8449	7.93	1551	58
3	1658	7.30	2733	.62	8925	7.92	1075	57
4	2096	7.32	2696	.60	9400	7.92	0600	56
5	**9.44 2535**	7.30	**9.98 2660**	.60	**9.45 9875**	7.90	**10.54 0125**	**55**
6	2973	7.28	2624	.62	.46 0349	7.90	.53 9651	54
7	3410	7.28	2587	.60	0823	7.90	9177	53
8	3847	7.28	2551	.62	1297	7.88	8703	52
9	4284	7.27	2514	.62	1770	7.87	8230	51
10	**9.44 4720**	7.25	**9.98 2477**	.60	**9.46 2242**	7.88	**10.53 7758**	**50**
11	5155	7.25	2441	.62	2715	7.85	7285	49
12	5590	7.25	2404	.62	3186	7.87	6814	48
13	6025	7.23	2367	.60	3658	7.83	6342	47
14	6459	7.23	2331	.62	4128	7.85	5872	46
15	**9.44 6893**	7.22	**9.98 2294**	.62	**9.46 4599**	7.83	**10.53 5401**	**45**
16	7326	7.22	2257	.62	5069	7.83	4931	44
17	7759	7.20	2220	.62	5539	7.82	4461	43
18	8191	7.20	2183	.62	6008	7.82	3992	42
19	8623	7.18	2146	.62	6477	7.80	3523	41
20	**9.44 9054**	7.18	**9.98 2109**	.62	**9.46 6945**	7.80	**10.53 3055**	**40**
21	9485	7.17	2072	.62	7413	7.78	2587	39
22	.44 9915	7.17	2035	.62	7880	7.78	2120	38
23	.45 0345	7.17	1998	.62	8347	7.78	1653	37
24	0775	7.15	1961	.62	8814	7.77	1186	36
25	**9.45 1204**	7.13	**9.98 1924**	.63	**9.46 9280**	7.77	**10.53 0720**	**35**
26	1632	7.13	1886	.62	.46 9746	7.75	.53 0254	34
27	2060	7.13	1849	.62	.47 0211	7.75	.52 9789	33
28	2488	7.12	1812	.63	0676	7.75	9324	32
29	2915	7.12	1774	.62	1141	7.73	8859	31
30	**9.45 3342**	7.10	**9.98 1737**	.62	**9.47 1605**	7.73	**10.52 8395**	**30**
31	3768	7.10	1700	.63	2069	7.72	7931	29
32	4194	7.08	1662	.62	2532	7.72	7468	28
33	4619	7.08	1625	.63	2995	7.70	7005	27
34	5044	7.08	1587	.63	3457	7.70	6543	26
35	**9.45 5469**	7.07	**9.98 1549**	.62	**9.47 3919**	7.70	**10.52 6081**	**25**
36	5893	7.05	1512	.63	4381	7.68	5619	24
37	6316	7.05	1474	.63	4842	7.68	5158	23
38	6739	7.05	1436	.62	5303	7.67	4697	22
39	7162	7.03	1399	.63	5763	7.67	4237	21
40	**9.45 7584**	7.03	**9.98 1361**	.63	**9.47 6223**	7.67	**10.52 3777**	**20**
41	8006	7.02	1323	.63	6683	7.65	3317	19
42	8427	7.02	1285	.63	7142	7.65	2858	18
43	8848	7.00	1247	.63	7601	7.63	2399	17
44	9268	7.00	1209	.63	8059	7.63	1941	16
45	**9.45 9688**	7.00	**9.98 1171**	.63	**9.47 8517**	7.63	**10.52 1483**	**15**
46	.46 0108	6.98	1133	.63	8975	7.62	1025	14
47	0527	6.98	1095	.63	9432	7.62	0568	13
48	0946	6.97	1057	.63	.47 9889	7.60	.52 0111	12
49	1364	6.97	1019	.63	.48 0345	7.60	.51 9655	11
50	**9.46 1782**	6.95	**9.98 0981**	.65	**9.48 0801**	7.60	**10.51 9199**	**10**
51	2199	6.95	0942	.63	1257	7.58	8743	9
52	2616	6.93	0904	.63	1712	7.58	8288	8
53	3032	6.93	0866	.65	2167	7.57	7833	7
54	3448	6.93	0827	.63	2621	7.57	7379	6
55	**9.46 3864**	6.92	**9.98 0789**	.65	**9.48 3075**	7.57	**10.51 6925**	**5**
56	4279	6.92	0750	.63	3529	7.55	6471	4
57	4694	6.90	0712	.65	3982	7.55	6018	3
58	5108	6.90	0673	.63	4435	7.53	5565	2
59	5522	6.88	0635	.65	4887	7.53	5113	1
60	**9.46 5935**		**9.98 0596**		**9.48 5339**		**10.51 4661**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

106° 73°

17° TABLE XII.—*Continued.* 162°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.46 5935**	6.88	**9.98 0596**	.63	**9.48 5339**	7.53	**10.51 4661**	**60**
1	6348	6.88	0558	.65	5791	7.52	4209	59
2	6761	6.87	0519	.65	6242	7.52	3758	58
3	7173	6.87	0480	.63	6693	7.50	3307	57
4	7585	6.85	0442	.65	7143	7.50	2857	56
5	**9.46 7996**	6.85	**9.98 0403**	.65	**9.48 7593**	7.50	**10.51 2407**	**55**
6	8407	6.83	0364	.65	8043	7.48	1957	54
7	8817	6.83	0325	.65	8492	7.48	1508	53
8	9227	6.83	0286	.65	8941	7.48	1059	52
9	.46 9637	6.82	0247	.65	9390	7.47	0610	51
10	**9.47 0046**	6.82	**9.98 0208**	.65	**9.48 9838**	7.47	**10.51 0162**	**50**
11	0455	6.80	0169	.65	.49 0286	7.45	.50 9714	49
12	0863	6.80	0130	.65	0733	7.45	9267	48
13	1271	6.80	0091	.65	1180	7.45	8820	47
14	1679	6.78	0052	.67	1627	7.43	8373	46
15	**9.47 2086**	6.77	**9.98 0012**	.65	**9.49 2073**	7.43	**10.50 7927**	**45**
16	2492	6.77	.97 9973	.65	2519	7.43	7481	44
17	2898	6.77	9934	.65	2965	7.42	7035	43
18	3304	6.77	9895	.67	3410	7.40	6590	42
19	3710	6.75	9855	.65	3854	7.42	6146	41
20	**9.47 4115**	6.73	**9.97 9816**	.67	**9.49 4299**	7.40	**10.50 5701**	**40**
21	4519	6.73	9776	.65	4743	7.38	5257	39
22	4923	6.73	9737	.67	5186	7.40	4814	38
23	5327	6.72	9697	.65	5630	7.38	4370	37
24	5730	6.72	9658	.67	6073	7.37	3927	36
25	**9.47 6133**	6.72	**9.97 9618**	.65	**9.49 6515**	7.37	**10.50 3485**	**35**
26	6536	6.70	9579	.67	6957	7.37	3043	34
27	6938	6.70	9539	.67	7399	7.37	2601	33
28	7340	6.68	9499	.67	7841	7.35	2159	32
29	7741	6.68	9459	.65	8282	7.33	1718	31
30	**9.47 8142**	6.67	**9.97 9420**	.67	**9.49 8722**	7.35	**10.50 1278**	**30**
31	8542	6.67	9380	.67	9163	7.33	0837	29
32	8942	6.67	9340	.67	.49 9603	7.32	.50 0397	28
33	9342	6.65	9300	.67	.50 0042	7.32	.49 9958	27
34	.47 9741	6.65	9260	.67	0481	7.32	9519	26
35	**9.48 0140**	6.65	**9.97 9220**	.67	**9.50 0920**	7.32	**10.49 9080**	**25**
36	0539	6.63	9180	.67	1359	7.30	8641	24
37	0937	6.62	9140	.67	1797	7.30	8203	23
38	1334	6.62	9100	.68	2235	7.28	7765	22
39	1731	6.62	9059	.67	2672	7.28	7328	21
40	**9.48 2128**	6.62	**9.97 9019**	.67	**9.50 3109**	7.28	**10.49 6891**	**20**
41	2525	6.60	8979	.67	3546	7.27	6454	19
42	2921	6.58	8939	.68	3982	7.27	6018	18
43	3316	6.60	8898	.67	4418	7.27	5582	17
44	3712	6.58	8858	.68	4854	7.25	5146	16
45	**9.48 4107**	6.57	**9.97 8817**	.67	**9.50 5289**	7.25	**10.49 4711**	**15**
46	4501	6.57	8777	.67	5724	7.25	4276	14
47	4895	6.57	8737	.68	6159	7.23	3841	13
48	5289	6.55	8696	.68	6593	7.23	3407	12
49	5682	6.55	8655	.67	7027	7.23	2973	11
50	**9.48 6075**	6.53	**9.97 8615**	.68	**9.50 7460**	7.22	**10.49 2540**	**10**
51	6467	6.55	8574	.68	7893	7.22	2107	9
52	6860	6.52	8533	.67	8326	7.22	1674	8
53	7251	6.53	8493	.68	8759	7.20	1241	7
54	7643	6.52	8452	.68	9191	7.18	0809	6
55	**9.48 8034**	6.50	**9.97 8411**	.68	**9.50 9622**	7.20	**10.49 0378**	**5**
56	8424	6.50	8370	.68	.51 0054	7.18	.48 9946	4
57	8814	6.50	8329	.68	0485	7.18	9515	3
58	9204	6.48	8288	.68	0916	7.17	9084	2
59	9593	6.48	8247	.68	1346	7.17	8654	1
60	**9.48 9982**		**9.97 8206**		**9.51 1776**		**10.48 8224**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

107° 72°

18° TABLE XII.—*Continued.* **161°**

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.48 9982**	6.48	**9.97 8206**	.68	**9.51 1776**	7.17	**10.48 8224**	**60**
1	.49 0371	6.47	8165	.68	2206	7.15	7794	59
2	0759	6.47	8124	.68	2635	7.15	7365	58
3	1147	6.47	8083	.68	3064	7.15	6936	57
4	1535	6.45	8042	.68	3493	7.13	6507	56
5	**9.49 1922**	6.43	**9.97 8001**	.70	**9.51 3921**	7.13	**10.48 6079**	**55**
6	2308	6.45	7959	.68	4349	7.13	5651	54
7	2695	6.43	7918	.68	4777	7.12	5223	53
8	3081	6.42	7877	.70	5204	7.12	4796	52
9	3466	6.42	7835	.68	5631	7.10	4369	51
10	**9.49 3851**	6.42	**9.97 7794**	.70	**9.51 6057**	7.12	**10.48 3934**	**50**
11	4236	6.42	7752	.68	6484	7.10	3516	49
12	4621	6.40	7711	.70	6910	7.08	3090	48
13	5005	6.38	7669	.68	7335	7.10	2665	47
14	5388	6.40	7628	.70	7761	7.08	2239	46
15	**9.49 5772**	6.37	**9.97 7586**	.70	**9.51 8186**	7.07	**10.48 1814**	**45**
16	6154	6.38	7544	.68	8610	7.07	1390	44
17	6537	6.37	7503	.70	9034	7.07	0966	43
18	6919	6.37	7461	.70	9458	7.07	0542	42
19	7301	6.35	7419	.70	.51 9882	7.05	.48 0118	41
20	**9.49 7682**	6.35	**9.97 7377**	.70	**9.52 0305**	7.05	**10.47 9695**	**40**
21	8064	6.33	7335	.70	0728	7.05	9272	39
22	8444	6.35	7293	.70	1151	7.03	8849	38
23	8825	6.32	7251	.70	1573	7.03	8427	37
24	9204	6.33	7209	.70	1995	7.03	8005	36
25	**9.49 9584**	6.32	**9.97 7167**	.70	**9.52 2417**	7.02	**10.47 7583**	**35**
26	.49 9963	6.32	7125	.70	2833	7.02	7162	34
27	.50 0342	6.32	7083	.70	3259	7.02	6741	33
28	0721	6.30	7041	.70	3680	7.00	6320	32
29	1099	6.28	6999	.70	4100	7.00	5900	31
30	**9.50 1476**	6.30	**9.97 6957**	.72	**9.52 4520**	7.00	**10.47 5480**	**30**
31	1854	6.28	6914	.70	4940	6.98	5060	29
32	2231	6.27	6872	.70	5359	6.98	4641	28
33	2607	6.28	6830	.72	5778	6.98	4222	27
34	2984	6.27	6787	.70	6197	6.97	3803	26
35	**9.50 3360**	6.25	**9.97 6745**	.72	**9.52 6615**	6.97	**10.47 3385**	**25**
36	3735	6.25	6702	.70	7033	6.97	2967	24
37	4110	6.25	6660	.72	7451	6.95	2549	23
38	4485	6.25	6617	.72	7863	6.95	2132	22
39	4860	6.23	6574	.70	8285	6.95	1715	21
40	**9.50 5234**	6.23	**9.97 6532**	.72	**9.52 8702**	6.95	**10.47 1298**	**20**
41	5608	6.22	6489	.72	9119	6.93	0881	19
42	5981	6.22	6446	.70	9535	6.93	0465	18
43	6354	6.22	6404	.72	.52 9951	6.92	.47 0049	17
44	6727	6.20	6361	.72	.53 0366	6.92	.46 9634	16
45	**9.50 7099**	6.20	**9.97 6318**	.72	**9.53 0781**	6.92	**10.46 9219**	**15**
46	7471	6.20	6275	.72	1196	6.92	8804	14
47	7843	6.18	6232	.72	1611	6.90	8389	13
48	8214	6.18	6189	.72	2025	6.90	7975	12
49	8585	6.18	6146	.72	2439	6.90	7561	11
50	**9.50 8956**	6.17	**9.97 6103**	.72	**9.53 2853**	6.88	**10.46 7147**	**10**
51	9326	6.17	6060	.72	3266	6.88	6734	9
52	.50 9696	6.15	6017	.72	3679	6.88	6321	8
53	.51 0065	6.15	5974	.73	4092	6.87	5908	7
54	0434	6.15	5930	.72	4504	6.87	5496	6
55	**9.51 0803**	6.15	**9.97 5887**	.72	**9.53 4916**	6.87	**10.46 5084**	**5**
56	1172	6.13	5844	.73	5328	6.85	4672	4
57	1540	6.12	5800	.72	5739	6.85	4261	3
58	1907	6.13	5757	.72	6150	6.85	3850	2
59	2275	6.12	5714	.73	6561	6.85	3439	1
60	**9.51 2642**		**9.97 5670**		**9.53 6972**		**10.46 3028**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

108° **71°**

19° Table XII.—*Continued.* 160°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.51 2642**	6.12	**9.97 5670**	.72	**9.53 6972**	6.83	**10.46 3028**	**60**
1	3009	6.10	5627	.73	7382	6.83	2618	59
2	3375	6.10	5583	.73	7792	6.83	2208	58
3	3741	6.10	5539	.72	8202	6.82	1798	57
4	4107	6.08	5496	.73	8611	6.82	1389	56
5	**9.51 4472**	6.08	**9.97 5452**	.73	**9.53 9020**	6.82	**10.46 0980**	**55**
6	4837	6.08	5408	.72	9429	6.80	0571	54
7	5202	6.07	5365	.73	.53 9837	6.80	.46 0163	53
8	5566	6.07	5321	.73	.54 0245	6.80	.45 9755	52
9	5930	6.07	5277	.73	0653	6.80	9347	51
10	**9.51 6294**	6.05	**9.97 5233**	.73	**9.54 1061**	6.78	**10 45 8939**	**50**
11	6657	6.05	5189	.73	1468	6.78	8532	49
12	7020	6.03	5145	.73	1875	6.77	8125	48
13	7382	6.05	5101	.73	2281	6.78	7719	47
14	7745	6.03	5057	.73	2688	6.77	7312	46
15	**9.51 8107**	6.02	**9.97 5013**	.73	**9.54 3094**	6.75	**10.45 6906**	**45**
16	8468	6.02	4969	.73	3499	6.77	6501	44
17	8829	6.02	4925	.75	3905	6.75	6095	43
18	9190	6.02	4880	.73	4310	6.75	5690	42
19	9551	6.00	4836	.73	4715	6.73	5285	41
20	**9.51 9911**	6.00	**9.97 4792**	.73	**9.54 5119**	6.75	**10.45 4881**	**40**
21	.52 0271	6.00	4748	.75	5524	6.73	4476	39
22	0631	5.98	4703	.73	5928	6.72	4072	38
23	0990	5.98	4659	.75	6331	6.73	3669	37
24	1349	5.97	4614	.73	6735	6.72	3265	36
25	**9.52 1707**	5.98	**9.97 4570**	.75	**9.54 7138**	6.70	**10.45 2862**	**35**
26	2066	5.97	4525	.73	7540	6.72	2460	34
27	2424	5.95	4481	.75	7943	6.70	2057	33
28	2781	5.95	4436	.75	8345	6.70	1655	32
29	3138	5.95	4391	.73	8747	6.70	1253	31
30	**9.52 3495**	5.95	**9.97 4347**	.75	**9.54 9149**	6.68	**10.45 0851**	**30**
31	3852	5.93	4302	.75	9550	6.68	0450	29
32	4208	5.93	4257	.75	.54 9951	6.68	.45 0049	28
33	4564	5.93	4212	.75	.55 0352	6.67	.44 9648	27
34	4920	5.92	4167	.75	0752	6.68	9248	26
35	**9.52 5275**	5.92	**9.97 4122**	.75	**9.55 1153**	6.65	**10.44 8847**	**25**
36	5630	5.90	4077	.75	1552	6.67	8448	24
37	5984	5.92	4032	.75	1952	6.65	8048	23
38	6339	5.90	3987	.75	2351	6.65	7649	22
39	6693	5.88	3942	.75	2750	6.65	7250	21
40	**9.52 7046**	5.90	**9.97 3897**	.75	**9.55 3149**	6.65	**10.44 6851**	**20**
41	7400	5.88	3352	.75	3548	6.63	6452	19
42	7753	5.87	3807	.77	3946	6.63	6054	18
43	8105	5.88	3761	.75	4344	6.62	5656	17
44	8458	5.87	3716	.75	4741	6.63	5259	16
45	**9.52 8810**	5.85	**9.97 3671**	.77	**9.55 5139**	6.62	**10.44 4861**	**15**
46	9161	5.87	3625	.75	5536	6.62	4464	14
47	9513	5.85	3580	.75	5933	6.60	4067	13
48	.52 9864	5.85	3535	.77	6329	6.60	3671	12
49	.53 0215	5.83	3489	.75	6725	6.60	3275	11
50	**9.53 0565**	5.83	**9.97 3444**	.77	**9.55 7121**	6.60	**10.44 2879**	**10**
51	0915	5.83	3398	.77	7517	6.60	2483	9
52	1265	5.82	3352	.75	7913	6.58	2087	8
53	1614	5.82	3307	.77	8308	6.58	1692	7
54	1963	5.82	3261	.77	8703	6.57	1297	6
55	**9.53 2312**	5.82	**9.97 3215**	.77	**9.55 9097**	6.57	**10.44 0903**	**5**
56	2661	5.80	3169	.75	9491	6.57	0509	4
57	3009	5.80	3124	.77	.55 9885	6.57	.44 0115	3
58	3357	5.78	3078	.77	.56 0279	6.57	.43 9721	2
59	3704	5.80	3032	.77	0673	6.55	9327	1
60	**9.53 4052**		**9.97 2986**		**9.56 1066**		**10.43 8934**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

109° 70°

20° TABLE XII.—*Continued.* 159°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.53 4052	5.78	9.97 2986	.77	9.56 1066	6.55	10.43 8934	60
1	4399	5.77	2940	.77	1459	6.53	8541	59
2	4745	5.78	2894	.77	1851	6.55	8149	58
3	5092	5.77	2848	.77	2244	6.53	7756	57
4	5438	5.75	2802	.78	2636	6.53	7364	56
5	9.53 5783	5.77	9.97 2755	.77	9.56 3028	6.52	10.43 6972	55
6	6129	5.75	2709	.77	3419	6.53	6581	54
7	6474	5.73	2663	.77	3811	6.52	6189	53
8	6818	5.75	2617	.78	4202	6.52	5798	52
9	7163	5.73	2570	.77	4593	6.50	5407	51
10	9.53 7507	5.73	9.97 2524	.77	9.56 4983	6.50	10.43 5017	50
11	7851	5.72	2478	.78	5373	6.50	4627	49
12	8194	5.73	2431	.77	5763	6.50	4237	48
13	8538	5.70	2385	.78	6153	6.48	3847	47
14	8880	5.72	2338	.78	6542	6.50	3458	46
15	9.53 9223	5.70	9.97 2291	.77	9.56 6932	6.47	10.43 3068	45
16	9565	5.70	2245	.78	7320	6.48	2680	44
17	.53 9907	5.68	2198	.78	7709	6.48	2291	43
18	.54 0249	5.68	2151	.77	8098	6.47	1902	42
19	0590	5.68	2105	.78	8486	6.45	1514	41
20	9.54 0931	5.68	9.97 2058	.78	9.56 8873	6.47	10.43 1127	40
21	1272	5.68	2011	.78	9261	6.45	0739	39
22	1613	5.67	1964	.78	.56 9648	6.45	.43 0352	38
23	1953	5.67	1917	.78	.57 0035	6.45	.42 9965	37
24	2293	5.65	1870	.78	0422	6.45	9578	36
25	9.54 2632	5.65	9.97 1823	.78	9.57 0809	6.43	10.42 9191	35
26	2971	5.65	1776	.78	1195	6.43	8805	34
27	3310	5.65	1729	.78	1581	6.43	8419	33
28	3649	5.63	1682	.78	1967	6.42	8033	32
29	3987	5.63	1635	.78	2352	6.43	7648	31
30	9.54 4325	5.63	9.97 1588	.80	9.57 2738	6.42	10.42 7262	30
31	4663	5.62	1540	.78	3123	6.40	6877	29
32	5000	5.63	1493	.78	3507	6.42	6493	28
33	5338	5.60	1446	.80	3892	6.40	6108	27
34	5674	5.62	1398	.78	4276	6.40	5724	26
35	9.54 6011	5.60	9.97 1351	.80	9.57 4660	6.40	10.42 5340	25
36	6347	5.60	1303	.78	5044	6.38	4956	24
37	6683	5.60	1256	.80	5427	6.38	4573	23
38	7019	5.58	1208	.78	5810	6.38	4190	22
39	7354	5.58	1161	.80	6193	6.38	3807	21
40	9.54 7689	5.58	9.97 1113	.78	9.57 6576	6.38	10.42 3424	20
41	8024	5.58	1066	.80	6959	6.37	3041	19
42	8359	5.57	1018	.80	7341	6.37	2659	18
43	8693	5.57	0970	.80	7723	6.35	2277	17
44	9027	5.55	0922	.80	8104	6.37	1896	16
45	9.54 9360	5.55	9.97 0874	.78	9.57 8486	6.35	10.42 1514	15
46	.54 9693	5.55	0827	.80	8867	6.35	1133	14
47	.55 0026	5.55	0779	.80	9248	6.35	0752	13
48	0359	5.55	0731	.80	.57 9629	6.33	.42 0371	12
49	0692	5.53	0683	.80	.58 0009	6.33	.41 9991	11
50	9.55 1024	5.53	9.97 0635	.82	9.58 0389	6.33	10.41 9611	10
51	1356	5.52	0586	.80	0769	6.33	9231	9
52	1687	5.52	0538	.80	1149	6.32	8851	8
53	2018	5.52	0490	.80	1528	6.32	8472	7
54	2349	5.52	0442	.80	1907	6.32	8093	6
55	9.55 2680	5.50	9.97 0394	.82	9.58 2286	6.32	10.41 7714	5
56	3010	5.52	0345	.80	2665	6.32	7335	4
57	3341	5.48	0297	.80	3044	6.30	6956	3
58	3670	5.50	0249	.82	3422	6.30	6578	2
59	4000	5.48	0200	.80	3800	6.28	6200	1
60	9.55 4329		9.97 0152		9.58 4177		10.41 5823	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

110° 69°

21° TABLE XII.—*Continued.* 158°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.55 4329	5.48	9.97 0152	.82	9.58 4177	6.30	10.41 5823	60
1	4658	5.48	0103	.80	4555	6.28	5445	59
2	4987	5.47	0055	.82	4932	6.28	5068	58
3	5315	5.47	.97 0006	.82	5309	6.28	4691	57
4	5643	5.47	.96 9957	.80	5686	6.27	4314	56
5	9.55 5971	5.47	9.96 9909	.82	9.58 6062	6.28	10.41 3938	55
6	6299	5.45	9860	.82	6439	6.27	3561	54
7	6626	5.45	9811	.82	6815	6.25	3185	53
8	6953	5.45	9762	.80	7190	6.27	2810	52
9	7280	5.43	9714	.82	7566	6.25	2434	51
10	9.55 7606	5.43	9.96 9665	.82	9.58 7941	6.25	10.41 2059	50
11	7932	5.43	9616	.82	8316	6.25	1684	49
12	8258	5.42	9567	.82	8691	6.25	1309	48
13	8583	5.43	9518	.82	9066	6.23	0934	47
14	8909	5.42	9469	.82	9440	6.23	0560	46
15	9.55 9234	5.40	9.96 9420	.83	9.58 9814	6.23	10.41 0186	45
16	9558	5.42	9370	.82	.59 0188	6.23	.40 9812	44
17	.55 9883	5.40	9321	.82	0562	6.22	9438	43
18	.56 0207	5.40	9272	.82	0935	6.22	9065	42
19	0531	5.40	9223	.83	1308	6.22	8692	41
20	9.56 0855	5.38	9.96 9173	.82	9.59 1681	6.22	10.40 8319	40
21	1178	5.38	9124	.82	2054	6.20	7946	39
22	1501	5.38	9075	.83	2426	6.22	7574	38
23	1824	5.37	9025	.82	2799	6.20	7201	37
24	2146	5.37	8976	.83	3171	6.18	6829	36
25	9.56 2468	5.37	9.96 8926	.82	9.59 3542	6.20	10.40 6458	35
26	2790	5.37	8877	.83	3914	6.18	6086	34
27	3112	5.35	8827	.83	4285	6.18	5715	33
28	3433	5.37	8777	.82	4656	6.18	5344	32
29	3755	5.33	8728	.83	5027	6.18	4973	31
30	9.56 4075	5.35	9.96 8678	.83	9.59 5398	6.17	10.40 4602	30
31	4396	5.33	8628	.83	5768	6.17	4232	29
32	4716	5.33	8578	.83	6138	6.17	3862	28
33	5036	5.33	8528	.82	6508	6.17	3492	27
34	5356	5.33	8479	.83	6878	6.15	3122	26
35	9.56 5676	5.32	9.96 8429	.83	9.59 7247	6.15	10.40 2753	25
36	5995	5.32	8379	.83	7616	6.15	2384	24
37	6314	5.30	8329	.85	7985	6.15	2015	23
38	6632	5.32	8278	.83	8354	6.13	1646	22
39	6951	5.30	8228	.83	8722	6.15	1278	21
40	9.56 7269	5.30	9.96 8178	.83	9.59 9091	6.13	10.40 0909	20
41	7587	5.28	8128	.83	9459	6.13	0541	19
42	7904	5.30	8078	.85	.59 9827	6.12	.40 0173	18
43	8222	5.28	8027	.83	.60 0194	6.13	.39 9806	17
44	8539	5.28	7977	.83	0562	6.12	9438	16
45	9.56 8856	5.27	9.96 7927	.85	9.60 0929	6.12	10.39 9071	15
46	9172	5.27	7876	.83	1296	6.12	8704	14
47	9488	5.27	7826	.85	1663	6.10	8337	13
48	.56 9804	5.27	7775	.83	2029	6.10	7971	12
49	.57 0120	5.25	7725	.85	2395	6.10	7605	11
50	9.57 0435	5.27	9.96 7674	.83	9.60 2761	6.10	10.39 7239	10
51	0751	5.25	7624	.85	3127	6.10	6873	9
52	1066	5.23	7573	.85	3493	6.08	6507	8
53	1380	5.25	7522	.85	3858	6.08	6142	7
54	1695	5.23	7471	.83	4223	6.08	5777	6
55	9.57 2009	5.23	9.96 7421	.85	9.60 4588	6.08	10.39 5412	5
56	2323	5.22	7370	.85	4953	6.07	5047	4
57	2636	5.23	7319	.85	5317	6.08	4683	3
58	2950	5.22	7268	.85	5682	6.07	4318	2
59	3263	5.20	7217	.85	6046	6.07	3954	1
60	9.57 3575		9.96 7166		9.60 6410		10.39 3590	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

111° 68°

22° TABLE XII.—*Continued.* 157°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.57 3575**	5.22	**9.96 7166**	.85	**9.60 6410**	6.05	**10.39 3590**	**60**
1	3888	5.20	7115	.85	6773	6.07	3227	59
2	4200	5.20	7064	.85	7137	6.05	2863	58
3	4512	5.20	7013	.87	7500	6.05	2500	57
4	4824	5.20	6961	.85	7863	6.03	2137	56
5	**9.57 5136**	5.18	**9.96 6910**	.85	**9.60 8225**	6.05	**10.39 1775**	**55**
6	5447	5.18	6859	.85	8588	6.03	1412	54
7	5758	5.18	6808	.87	8950	6.03	1050	53
8	6069	5.17	6756	.85	9312	6.03	0688	52
9	6379	5.17	6705	.87	.60 9674	6.03	.39 0326	51
10	**9.57 6689**	5.17	**9.96 6653**	.85	**9.61 0036**	6.02	**10.38 9964**	**50**
11	6999	5.17	6602	.87	0397	6.03	9603	49
12	7309	5.15	6550	.85	0759	6.02	9241	48
13	7618	5.15	6499	.87	1120	6.00	8880	47
14	7927	5.15	6447	.87	1480	6.02	8520	46
15	**9.57 8236**	5.15	**9.96 6395**	.85	**9.61 1841**	6.00	**10.38 8159**	**45**
16	8545	5.13	6344	.87	2201	6.00	7799	44
17	8853	5.15	6292	.87	2561	6.00	7439	43
18	9162	5.13	6240	.87	2921	6.00	7079	42
19	9470	5.12	6188	.87	3281	6.00	6719	41
20	**9.57 9777**	5.13	**9.96 6136**	.85	**9.61 3641**	5.98	**10.38 6359**	**40**
21	.58 0085	5.12	6085	.87	4000	5.98	6000	39
22	0392	5.12	6033	.87	4359	5.98	5641	38
23	0699	5.10	5981	.87	4718	5.98	5282	37
24	1005	5.12	5929	.88	5077	5.97	4923	36
25	**9.58 1312**	5.10	**9.96 5876**	.87	**9.61 5435**	5.97	**10.38 4565**	**35**
26	1618	5.10	5824	.87	5793	5.97	4207	34
27	1924	5.08	5772	.87	6151	5.97	3849	33
28	2229	5.10	5720	.87	6509	5.97	3491	32
29	2535	5.08	5668	.88	6867	5.95	3133	31
30	**9.58 2840**	5.08	**9.96 5615**	.87	**9.61 7224**	5.97	**10.38 2776**	**30**
31	3145	5.07	5563	.87	7582	5.95	2418	29
32	3449	5.08	5511	.88	7939	5.93	2061	28
33	3754	5.07	5458	.87	8295	5.95	1705	27
34	4058	5.05	5406	.88	8652	5.93	1348	26
35	**9.58 4361**	5.07	**9.96 5353**	.87	**9.61 9008**	5.93	**10.38 0992**	**25**
36	4665	5.05	5301	.88	9364	5.93	0636	24
37	4968	5.07	5248	.88	.61 9720	5.93	.38 0280	23
38	5272	5.03	5195	.87	.62 0076	5.93	.37 9924	22
39	5574	5.05	5143	.88	0432	5.92	9568	21
40	**9.58 5877**	5.03	**9.96 5090**	.88	**9.62 0787**	5.92	**10.37 9213**	**20**
41	6179	5.05	5037	.88	1142	5.92	8858	19
42	6482	5.02	4984	.88	1497	5.92	8503	18
43	6783	5.03	4931	.87	1852	5.92	8148	17
44	7085	5.02	4879	.88	2207	5.90	7793	16
45	**9.58 7386**	5.03	**9.96 4826**	.88	**9.62 2561**	5.90	**10.37 7439**	**15**
46	7688	5.02	4773	.88	2915	5.90	7085	14
47	7989	5.00	4720	.90	3269	5.90	6731	13
48	8289	5.02	4666	.88	3623	5.88	6377	12
49	8590	5.00	4613	.88	3976	5.90	6024	11
50	**9.58 8890**	5.00	**9.96 4560**	.88	**9.62 4330**	5.88	**10.37 5670**	**10**
51	9190	4.98	4507	.88	4683	5.88	5317	9
52	9489	5.00	4454	.90	5036	5.87	4964	8
53	.58 9789	4.98	4400	.88	5388	5.88	4612	7
54	.59 0088	4.98	4347	.88	5741	5.87	4259	6
55	**9.59 0387**	4.98	**9.96 4294**	.90	**9.62 6093**	5.87	**10.37 3907**	**5**
56	0686	4.97	4240	.88	6445	5.87	3555	4
57	0984	4.97	4187	.90	6797	5.87	3203	3
58	1282	4.97	4133	.88	7149	5.87	2851	2
59	1580	4.97	4080	.90	7501	5.85	2499	1
60	**9.59 1878**		**9.96 4026**		**9.62 7852**		**10.37 2148**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

112° 67°

23° TABLE XII.—*Continued.* 156°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.59 1878	4.97	9.96 4026	.90	9.62 7825	5.85	10.37 2148	60
1	2176	4.95	3972	.88	8203	5.85	1797	59
2	2473	4.95	3919	.90	8554	5.85	1446	58
3	2770	4.95	3865	.90	8905	5.83	1095	57
4	3067	4.93	3811	.90	9255	5.85	0745	56
5	9.59 3363	4.93	9.96 3757	.88	9.62 9606	5.83	10.37 0394	55
6	3659	4.93	3704	.90	.62 9956	5.83	.37 0044	54
7	3955	4.93	3650	.90	.63 0306	5.83	.36 9694	53
8	4251	4.93	3596	.90	0656	5.82	9344	52
9	4547	4.92	3542	.90	1005	5.83	8995	51
10	9.59 4842	4.92	9.96 3488	.90	9.63 1355	5.82	10.36 8645	50
11	5137	4.92	3434	.92	1704	5.82	8296	49
12	5432	4.92	3379	.90	2053	5.82	7947	48
13	5727	4.90	3325	.90	2402	5.80	7598	47
14	6021	4.90	3271	.90	2750	5.82	7250	46
15	9.59 6315	4.90	9.96 3217	.90	9.63 3099	5.80	10.36 6901	45
16	6609	4.90	3163	.92	3447	5.80	6553	44
17	6903	4.88	3108	.90	3795	5.80	6205	43
18	7196	4.90	3054	.92	4143	5.78	5857	42
19	7490	4.88	2999	.90	4490	5.80	5510	41
20	9.59 7783	4.87	9.96 2945	.92	9.63 4838	5.78	10.36 5162	40
21	8075	4.88	2890	.90	5185	5.78	4815	39
22	8368	4.87	2836	.92	5532	5.78	4468	38
23	8660	4.87	2781	.90	5879	5.78	4121	37
24	8952	4.87	2727	.92	6226	5.77	3774	36
25	9.59 9244	4.87	9.96 2672	.92	9.63 6572	5.78	10.36 3428	35
26	9536	4.85	2617	.92	6919	5.77	3081	34
27	.59 9827	4.85	2562	.90	7265	5.77	2735	33
28	.60 0118	4.85	2508	.92	7611	5.75	2389	32
29	0409	4.85	2453	.92	7956	5.77	2044	31
30	9.60 0700	4.83	9.96 2398	.92	9.63 8302	5.75	10.36 1698	30
31	0990	4.83	2343	.92	8647	5.75	1353	29
32	1280	4.83	2288	.92	8992	5.75	1008	28
33	1570	4.83	2233	.92	9337	5.75	0663	27
34	1860	4.83	2178	.92	.63 9682	5.75	.36 0318	26
35	9.60 2150	4.82	9.96 2123	.93	9.64 0027	5.73	10.35 9973	25
36	2439	4.82	2067	.92	0371	5.75	9629	24
37	2728	4.82	2012	.92	0716	5.73	9284	23
38	3017	4.80	1957	.92	1060	5.73	8940	22
39	3305	4.82	1902	.93	1404	5.72	8596	21
40	9.60 3594	4.80	9.96 1846	.92	9.64 1747	5.73	10.35 8253	20
41	3882	4.80	1791	.93	2091	5.72	7909	19
42	4170	4.78	1735	.92	2434	5.72	7566	18
43	4457	4.80	1680	.93	2777	5.72	7223	17
44	4745	4.78	1624	.92	3120	5.72	6880	16
45	9.60 5032	4.78	9.96 1569	.93	9.64 3463	5.72	10.35 6537	15
46	5319	4.78	1513	.92	3806	5.70	6194	14
47	5606	4.77	1458	.93	4148	5.70	5852	13
48	5892	4.78	1402	.93	4490	5.70	5510	12
49	6179	4.77	1346	.93	4832	5.70	5168	11
50	9.60 6465	4.77	9.96 1290	.92	9.64 5174	5.70	10.35 4826	10
51	6751	4.75	1235	.93	5516	5.68	4484	9
52	7036	4.77	1179	.93	5857	5.70	4143	8
53	7322	4.75	1123	.93	6199	5.68	3801	7
54	7607	4.75	1067	.93	6540	5.68	3460	6
55	9.60 7892	4.75	9.96 1011	.93	9.64 6881	5.68	10.35 3119	5
56	8177	4.73	0955	.93	7222	5.67	2778	4
57	8461	4.73	0899	.93	7562	5.68	2438	3
58	8745	4.73	0843	.95	7903	5.67	2097	2
59	9029	4.73	0786	.93	8243	5.67	1757	1
60	9.60 9313		9.96 0730		9.64 8583		10.35 1417	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

113° 66°

24° TABLE XII.—*Continued.* **155°**

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.60 9313	4.73	9.96 0730	.93	9.64 8583	5.67	10.35 1417	60
1	9597	4.72	0674	.93	8923	5.67	1077	59
2	.60 9880	4.73	0618	.95	9263	5.65	0737	58
3	.61 0164	4.72	0561	.93	9602	5.67	0398	57
4	0447	4.70	0505	.95	.64 9942	5.65	.35 0058	56
5	9.61 0729	4.72	9.96 0448	.93	9.65 0281	5.65	10.34 9719	55
6	1012	4.70	0392	.95	0620	5.65	9380	54
7	1294	4.70	0335	.93	0959	5.63	9041	53
8	1576	4.70	0279	.95	1297	5.65	8703	52
9	1858	4.70	0222	.95	1636	5.63	8364	51
10	9.61 2140	4.68	9.96 0165	.93	9.65 1974	5.63	10.34 8026	50
11	2421	4.68	0109	.95	2312	5.63	7688	49
12	2702	4.68	.96 0052	.95	2650	5.63	7350	48
13	2983	4.68	.95 9995	.95	2988	5.63	7012	47
14	3264	4.68	9938	.93	3326	5.62	6674	46
15	9.61 3545	4.67	9.95 9882	.95	9.65 3663	5.62	10.34 6337	45
16	3825	4.67	9825	.95	4000	5.62	6000	44
17	4105	4.67	9768	.95	4337	5.62	5663	43
18	4385	4.67	9711	.95	4674	5.62	5326	42
19	4665	4.65	9654	.97	5011	5.62	4989	41
20	9.61 4944	4.65	9.95 9596	.95	9.65 5348	5.60	10.34 4652	40
21	5223	4.65	9539	.95	5684	5.60	4316	39
22	5502	4.65	9482	.95	6020	5.60	3980	38
23	5781	4.65	9425	.95	6356	5.60	3644	37
24	6060	4.63	9368	.97	6692	5.60	3308	36
25	9.61 6338	4.63	9.95 9310	.95	9.65 7028	5.60	10.34 2972	35
26	6616	4.63	9253	.97	7364	5.58	2636	34
27	6894	4.63	9195	.95	7699	5.58	2301	33
28	7172	4.63	9138	.97	8034	5.58	1966	32
29	7450	4.62	9080	.95	8369	5.58	1631	31
30	9.61 7727	4.62	9.95 9023	.97	9.65 8704	5.58	10.34 1296	30
31	8004	4.62	8965	.95	9039	5.57	0961	29
32	8281	4.62	8908	.97	9373	5.58	0627	28
33	8558	4.60	8850	.97	.65 9708	5.57	.34 0292	27
34	8834	4.60	8792	.97	.66 0042	5.57	.33 9958	26
35	9.61 9110	4.60	9.95 8734	.95	9.66 0376	5.57	10.33 9624	25
36	9386	4.60	8677	.97	0710	5.55	9290	24
37	9662	4.60	8619	.97	1043	5.57	8957	23
38	.61 9938	4.58	8561	.97	1377	5.55	8623	22
39	.62 0213	4.58	8503	.97	1710	5.55	8290	21
40	9.62 0488	4.58	9.95 8445	.97	9.66 2043	5.55	10.33 7957	20
41	0763	4.58	8387	.97	2376	5.55	7624	19
42	1038	4.58	8329	.97	2709	5.55	7291	18
43	1313	4.57	8271	.97	3042	5.55	6958	17
44	1587	4.57	8213	.98	3375	5.53	6625	16
45	9.62 1861	4.57	9.95 8154	.97	9.66 3707	5.53	10.33 6293	15
46	2135	4.57	8096	.97	4039	5.53	5961	14
47	2409	4.55	8038	.98	4371	5.53	5629	13
48	2682	4.57	7979	.97	4703	5.53	5297	12
49	2956	4.55	7921	.97	5035	5.52	4965	11
50	9.62 3229	4.55	9.95 7863	.98	9.66 5366	5.53	10.33 4634	10
51	3502	4.53	7804	.97	5698	5.52	4302	9
52	3774	4.55	7746	.98	6029	5.52	3971	8
53	4047	4.53	7687	.98	6360	5.52	3640	7
54	4319	4.53	7628	.97	6691	5.50	3309	6
55	9.62 4591	4.53	9.95 7570	.98	9.66 7021	5.52	10.33 2979	5
56	4863	4.53	7511	.98	7352	5.50	2648	4
57	5135	4.52	7452	.98	7682	5.52	2318	3
58	5406	4.52	7393	.97	8013	5.50	1987	2
59	5677	4.52	7335	.98	8343	5.50	1657	1
60	9.62 5948		9.95 7276		9.66 8673		10.33 1327	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

114° **65°**

25° TABLE XII.—*Continued.* 154°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.62 5948	4.52	9.95 7276	.98	9.66 8673	5.48	10.33 1327	60
1	6219	4.52	7217	.98	9002	5.50	0998	59
2	6490	4.50	7158	.98	9332	5.48	0668	58
3	6760	4.50	7099	.98	9661	5.50	0339	57
4	7030	4.50	7040	.98	.66 9991	5.48	.33 0009	56
5	9.62 7300	4.50	9.95 6981	1.00	9.67 0320	5.48	10.32 9680	55
6	7570	4.50	6921	.98	0649	5.47	9351	54
7	7840	4.48	6862	.98	0977	5.48	9023	53
8	8109	4.48	6803	.98	1306	5.48	8694	52
9	8378	4.48	6744	1.00	1635	5.47	8365	51
10	9.62 8647	4.48	9.95 6684	.98	9.67 1963	5.47	10.32 8037	50
11	8916	4.48	6625	.98	2291	5.47	7709	49
12	9185	4.47	6566	1.00	2619	5.47	7381	48
13	9453	4.47	6506	.98	2947	5.45	7053	47
14	9721	4.47	6447	1.00	3274	5.47	6726	46
15	9.62 9989	4.47	9.95 6387	1.00	9.67 3602	5.45	10.32 6398	45
16	.63 0257	4.45	6327	.98	3929	5.47	6071	44
17	0524	4.47	6268	1.00	4257	5.45	5743	43
18	0792	4.45	6208	1.00	4584	5.45	5416	42
19	1059	4.45	6148	.98	4911	5.43	5089	41
20	9.63 1326	4.45	9.95 6089	1.00	9.67 5237	5.45	10.32 4763	40
21	1593	4.43	6029	1.00	5564	5.43	4436	39
22	1859	4.43	5969	1.00	5890	5.45	4110	38
23	2125	4.45	5909	1.00	6217	5.43	3783	37
24	2392	4.43	5849	1.00	6543	5.43	3457	36
25	9.63 2658	4.42	9.95 5789	1.00	9.67 6869	5.42	10.32 3131	35
26	2923	4.43	5729	1.00	7194	5.43	2806	34
27	3189	4.42	5669	1.00	7520	5.43	2480	33
28	3454	4.42	5609	.98	7846	5.42	2154	32
29	3719	4.42	5548	1.00	8171	5.42	1829	31
30	9.63 3984	4.42	9.95 5488	1.00	9.67 8496	5.42	10.32 1504	30
31	4249	4.42	5428	1.00	8821	5.42	1179	29
32	4514	4.40	5368	1.02	9146	5.42	0854	28
33	4778	4.40	5307	1.00	9471	5.40	0529	27
34	5042	4.40	5247	1.02	.67 9795	5.42	.32 0205	26
35	9.63 5306	4.40	9.95 5186	1.00	9.68 0120	5.40	10.31 9880	25
36	5570	4.40	5126	1.02	0444	5.40	9556	24
37	5834	4.38	5065	1.00	0768	5.40	9232	23
38	6097	4.38	5005	1.02	1092	5.40	8908	22
39	6360	4.38	4944	1.02	1416	5.40	8584	21
40	9.63 6623	4.38	9.95 4883	1.00	9.68 1740	5.38	10.31 8260	20
41	6886	4.37	4823	1.02	2063	5.40	7937	19
42	7148	4.38	4762	1.02	2387	5.38	7613	18
43	7411	4.37	4701	1.02	2710	5.38	7290	17
44	7673	4.37	4640	1.02	3033	5.38	6967	16
45	9.63 7935	4.37	9.95 4579	1.02	9.68 3356	5.38	10.31 6644	15
46	8197	4.35	4518	1.02	3679	5.37	6321	14
47	8458	4.37	4457	1.02	4001	5.38	5999	13
48	8720	4.35	4396	1.02	4324	5.37	5676	12
49	8981	4.35	4335	1.02	4646	5.37	5354	11
50	9.63 9242	4.35	9.95 4274	1.02	9.68 4968	5.37	10.31 5032	10
51	9503	4.35	4213	1.02	5290	5.37	4710	9
52	.63 9764	4.33	4152	1.03	5612	5.37	4388	8
53	.64 0024	4.33	4090	1.02	5934	5.35	4066	7
54	0284	4.33	4029	1.02	6255	5.37	3745	6
55	9.64 0544	4.33	9.95 3968	1.03	9.68 6577	5.35	10.31 3423	5
56	0804	4.33	3906	1.02	6898	5.35	3102	4
57	1064	4.33	3845	1.03	7219	5.35	2781	3
58	1324	4.32	3783	1.02	7540	5.35	2460	2
59	1583	4.32	3722	1.03	7861	5.35	2139	1
60	9.64 1842		9.95 3660		9.68 8182		10.31 1818	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

115° 64°

26° TABLE XII.—*Continued.* 153°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.64 1842	4.32	9.95 3660	1.02	9.68 8182	5.33	10.31 1818	60
1	2101	4.32	3599	1.03	8502	5.32	1498	59
2	2360	4.30	3537	1.03	8823	5.33	1177	58
3	2618	4.32	3475	1.03	9143	5.33	0857	57
4	2877	4.30	3413	1.02	9463	5.33	0537	56
5	9.64 3135	4.30	9.95 3352	1.03	9.68 9783	5.33	10.31 0217	55
6	3393	4.28	3290	1.03	.69 0103	5.33	.30 9897	54
7	3650	4.30	3228	1.03	0423	5.32	9577	53
8	3908	4.28	3166	1.03	0742	5.33	9258	52
9	4165	4.30	3104	1.03	1062	5.32	8938	51
10	9.64 4423	4.28	9.95 3042	1.03	9.69 1381	5.32	10.30 8619	50
11	4680	4.27	2980	1.03	1700	5.32	8300	49
12	4936	4.28	2918	1.05	2019	5.32	7981	48
13	5193	4.28	2855	1.03	2338	5.30	7662	47
14	5450	4.27	2793	1.03	2656	5.32	7344	46
15	9.64 5706	4.27	9.95 2731	1.03	9.69 2975	5.30	10.30 7025	45
16	5962	4.27	2669	1.05	3293	5.32	6707	44
17	6218	4.27	2606	1.03	3612	5.30	6388	43
18	6474	4.25	2544	1.05	3930	5.30	6070	42
19	6729	4.25	2481	1.03	4228	5.30	5752	41
20	9.64 6984	4.27	9.95 2419	1.05	9.69 4566	5.28	10.30 5434	40
21	7240	4.23	2356	1.03	4883	5.30	5117	39
22	7494	4.25	2294	1.05	5201	5.28	4799	38
23	7749	4.25	2231	1.05	5518	5.30	4482	37
24	8004	4.23	2168	1.03	5836	5.28	4164	36
25	9.64 8258	4.23	9.95 2106	1.05	9.69 6153	5.28	10.30 3847	35
26	8512	4.23	2043	1.05	6470	5.28	3530	34
27	8766	4.23	1980	1.05	6787	5.27	3213	33
28	9020	4.23	1917	1.05	7103	5.28	2897	32
29	9274	4.22	1854	1.05	7420	5.27	2580	31
30	9.64 9527	4.23	9.95 1791	1.05	9.69 7736	5.28	10.30 2264	30
31	.64 9781	4.22	1728	1.05	8053	5.27	1947	29
32	.65 0034	4.22	1665	1.05	8369	5.27	1631	28
33	0287	4.20	1602	1.05	8685	5.27	1315	27
34	0539	4.22	1539	1.05	9001	5.25	0999	26
35	9.65 0792	4.20	9.95 1476	1.07	9.69 9316	5.27	10.30 0684	25
36	1044	4.22	1412	1.05	9632	5.25	0368	24
37	1297	4.20	1349	1.05	.69 9947	5.27	.30 0053	23
38	1549	4.18	1286	1.07	.70 0263	5.25	.29 9737	22
39	1800	4.20	1222	1.05	0578	5.25	9422	21
40	9.65 2052	4.20	9.95 1159	1.05	9.70 0893	5.25	10.29 9107	20
41	2304	4.18	1096	1.07	1208	5.25	8792	19
42	2555	4.18	1032	1.07	1523	5.23	8477	18
43	2806	4.18	0968	1.05	1837	5.25	8163	17
44	3057	4.18	0905	1.07	2152	5.23	7848	16
45	9.65 3308	4.17	9.95 0841	1.05	9.70 2466	5.25	10.29 7534	15
46	3558	4.17	0778	1.07	2781	5.23	7219	14
47	3808	4.18	0714	1.07	3095	5.23	6905	13
48	4059	4.17	0650	1.07	3409	5.22	6591	12
49	4309	4.15	0586	1.07	3722	5.23	6278	11
50	9.65 4558	4.17	9.95 0522	1.07	9.70 4036	5.23	10.29 5964	10
51	4808	4.17	0458	1.07	4350	5.22	5650	9
52	5058	4.15	0394	1.07	4663	5.22	5337	8
53	5307	4.15	0330	1.07	4976	5.23	5024	7
54	5556	4.15	0266	1.07	5290	5.22	4710	6
55	9.65 5805	4.15	9.95 0202	1.07	9.70 5603	5.22	10.29 4397	5
56	6054	4.13	0138	1.07	5916	5.20	4084	4
57	6302	4.15	0074	1.07	6228	5.22	3772	3
58	6551	4.13	.95 0010	1.08	6541	5.22	3459	2
59	6799	4.13	.94 9945	1.07	6854	5.20	3146	1
60	9.65 7047		9.94 9881		9.70 7166		10.29 2834	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

116° 63°

27° TABLE XII.—*Continued.* 152°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.65 7047**	4.13	**9.94 9881**	1.08	**9.70 7166**	5.20	**10.29 2834**	**60**
1	7295	4.12	9816	1.07	7478	5.20	2522	59
2	7542	4.13	9752	1.07	7790	5.20	2210	58
3	7790	4.12	9688	1.08	8102	5.20	1898	57
4	8037	4.12	9623	1.08	8414	5.20	1586	56
5	**9.65 8284**	4.12	**9.94 9558**	1.07	**9.70 8726**	5.18	**10.29 1274**	**55**
6	8531	4.12	9494	1.08	9037	5.20	0963	54
7	8778	4.12	9429	1.08	9349	5.18	0651	53
8	9025	4.10	9364	1.07	9660	5.18	0340	52
9	9271	4.10	9300	1.08	9971	5.18	.29 0029	51
10	**9.65 9517**	4.10	**9.94 9235**	1.08	**9.71 0282**	5.18	**10.28 9718**	**50**
11	.65 9763	4.10	9170	1.08	0593	5.18	9407	49
12	.66 0009	4.10	9105	1.08	0904	5.18	9096	48
13	0255	4.10	9040	1.08	1215	5.17	8785	47
14	0501	4.08	8975	1.08	.70 1525	5.18	8475	46
15	**9.66 0746**	4.08	**9.94 8910**	1.08	**9.71 1836**	5.17	**10.28 8164**	**45**
16	0991	4.08	8845	1.08	2146	5.17	7854	44
17	1236	4.08	8780	1.08	2456	5.17	7544	43
18	1481	4.08	8715	1.08	2766	5.17	7234	42
19	1726	4.07	8650	1.10	3076	5.17	6924	41
20	**9.66 1970**	4.07	**9.94 8584**	1.08	**9.71 3386**	5.17	**10.28 6614**	**40**
21	2214	4.08	8519	1.08	3696	5.15	6304	39
22	2459	4.07	8454	1.10	4005	5.15	5995	38
23	2703	4.05	8388	1.08	4314	5.17	5686	37
24	2946	4.07	8323	1.10	4624	5.15	5376	36
25	**9.66 3190**	4.05	**9.94 8257**	1.08	**9.71 4933**	5.15	**10.28 5067**	**35**
26	3433	4.07	8192	1.10	5242	5.15	4758	34
27	3677	4.05	8126	1.10	5551	5.15	4449	33
28	3920	4.05	8060	1.08	5860	5.13	4140	32
29	4163	4.05	7995	1.10	6168	5.15	3832	31
30	**9.66 4406**	4.03	**9.94 7929**	1.10	**9.71 6477**	5.13	**10.28 3523**	**30**
31	4648	4.05	7863	1.10	6785	5.13	3215	29
32	4891	4.03	7797	1.10	7093	5.13	2907	28
33	5133	4.03	7731	1.10	7401	5.13	2599	27
34	5375	4.03	7665	1.08	7709	5.13	2291	26
35	**9.66 5617**	4.03	**9.94 7600**	1.12	**9.71 8017**	5.13	**10.28 1983**	**25**
36	5859	4.02	7533	1.10	8325	5.13	1675	24
37	6100	4.03	7467	1.10	8633	5.12	1367	23
38	6342	4.02	7401	1.10	8940	5.13	1060	22
39	6583	4.02	7335	1.10	9248	5.12	0752	21
40	**9.66 6824**	4.02	**9.94 7269**	1.10	**9.71 9555**	5.12	**10.28 0445**	**20**
41	7065	4.00	7203	1.12	.71 9862	5.12	.28 0138	19
42	7305	4.02	7136	1.10	.72 0169	5.12	.27 9831	18
43	7546	4.00	7070	1.10	0476	5.12	9524	17
44	7786	4.02	7004	1.12	0783	5.10	9217	16
45	**9.66 8027**	4.00	**9.94 6937**	1.10	**9.72 1089**	5.12	**10.27 8911**	**15**
46	8267	3.98	6871	1.12	1396	5.10	8604	14
47	8506	4.00	6804	1.10	1702	5.12	8298	13
48	8746	4.00	6738	1.12	2009	5.10	7991	12
49	8986	3.98	6671	1.12	2315	5.10	7685	11
50	**9.66 9225**	3.98	**9.94 6604**	1.10	**9.72 2621**	5.10	**10.27 7379**	**10**
51	9464	3.98	6538	1.12	2927	5.08	7073	9
52	9703	3.98	6471	1.12	3232	5.10	6768	8
53	.66 9942	3.98	6404	1.12	3538	5.10	6462	7
54	.67 0181	3.97	6337	1.12	3844	5.08	6156	6
55	**9.67 0419**	3.98	**9.94 6270**	1.12	**9.72 4149**	5.08	**10.27 5851**	**5**
56	0658	3.97	6203	1.12	4454	5.10	5546	4
57	0896	3.97	6136	1.12	4760	5.08	5240	3
58	1134	3.97	6069	1.12	5065	5.08	4935	2
59	1372	3.95	6002	1.12	5370	5.07	4630	1
60	**9.67 1609**		**9.94 5935**		**9.72 5674**		**10.27 4326**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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28° TABLE XII.—*Continued.* 151°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.67 1609**	3.97	**9.94 5935**	1.12	**9.72 5674**	5.08	**10.27 4326**	**60**
1	1847	3.95	5868	1.13	5979	5.08	4021	59
2	2084	3.95	5800	1.12	6284	5.07	3716	58
3	2321	3.95	5733	1.12	6588	5.07	3412	57
4	2558	3.95	5666	1.13	6892	5.05	3108	56
5	**9.67 2795**	3.95	**9.94 5598**	1.12	**9.72 7197**	5.07	**10.27 2803**	**55**
6	3032	3.93	5531	1.12	7501	5.07	2499	54
7	3268	3.95	5464	1.13	7805	5.07	2195	53
8	3505	3.93	5396	1.13	8109	5.05	1891	52
9	3741	3.93	5328	1.12	8412	5.07	1588	51
10	**9.67 3977**	3.93	**9.94 5261**	1.13	**9.72 8716**	5.07	**10.27 1284**	**50**
11	4213	3.92	5193	1.13	9020	5.05	0980	49
12	4448	3.93	5125	1.12	9323	5.05	0677	48
13	4684	3.92	5058	1.13	9626	5.05	0374	47
14	4919	3.93	4990	1.13	.72 9929	5.07	.27 0071	46
15	**9.67 5155**	3.92	**9.94 4922**	1.13	**9.73 0233**	5.03	**10.26 9767**	**45**
16	5390	3.90	4854	1.13	0535	5.05	9465	44
17	5624	3.92	4786	1.13	0838	5.05	9162	43
18	5859	3.92	4718	1.13	1141	5.05	8859	42
19	6094	3.90	4650	1.13	1444	5.03	8556	41
20	**9.67 6328**	3.90	**9.94 4582**	1.13	**9.73 1746**	5.03	**10.26 8254**	**40**
21	6562	3.90	4514	1.13	2048	5.05	7952	39
22	6796	3.90	4446	1.15	2351	5.03	7649	38
23	7030	3.90	4377	1.13	2653	5.03	7347	37
24	7264	3.90	4309	1.13	2955	5.03	7045	36
25	**9.67 7498**	3.88	**9.94 4241**	1.15	**9.73 3257**	5.02	**10.26 6743**	**35**
26	7731	3.88	4172	1.13	3558	5.03	6442	34
27	7964	3.88	4104	1.13	3860	5.03	6140	33
28	8197	3.88	4036	1.15	4162	5.02	5838	32
29	8430	3.88	3967	1.13	4463	5.02	5537	31
30	**9.67 8663**	3.87	**9.94 3899**	1.15	**9.73 4764**	5.03	**10.26 5236**	**30**
31	8895	3.88	3830	1.15	5066	5.02	4934	29
32	9128	3.87	3761	1.13	5367	5.02	4633	28
33	9360	3.87	3693	1.15	5668	5.02	4332	27
34	9592	3.87	3624	1.15	5969	5.00	4031	26
35	**9.67 9824**	3.87	**9.94 3555**	1.15	**9.73 6269**	5.02	**10.26 3731**	**25**
36	.68 0056	3.87	3486	1.15	6570	5.00	3430	24
37	0288	3.85	3417	1.15	6870	5.02	3130	23
38	0519	3.85	3348	1.15	7171	5.00	2829	22
39	0750	3.87	3279	1.15	7471	5.00	2529	21
40	**9.68 0982**	3.85	**9.94 3210**	1.15	**9.73 7771**	5.00	**10.26 2229**	**20**
41	1213	3.83	3141	1.15	8071	5.00	1929	19
42	1443	3.85	3072	1.15	8371	5.00	1629	18
43	1674	3.85	3003	1.15	8671	5.00	1329	17
44	1905	3.83	2934	1.17	8971	5.00	1029	16
45	**9.68 2135**	3.83	**9.94 2864**	1.15	**9.73 9271**	4.98	**10.26 0729**	**15**
46	2365	3.83	2795	1.15	9570	5.00	0430	14
47	2595	3.83	2726	1.17	.73 9870	4.98	.26 0130	13
48	2825	3.83	2656	1.15	.74 0169	4.98	.25 9831	12
49	3055	3.82	2587	1.17	0468	4.98	9532	11
50	**9.68 3284**	3.83	**9.94 2517**	1.15	**9.74 0767**	4.98	**10.25 9233**	**10**
51	3514	3.82	2448	1.17	1066	4.98	8934	9
52	3743	3.82	2378	1.17	1365	4.98	8635	8
53	3972	3.82	2308	1.15	1664	4.97	8336	7
54	4201	3.82	2239	1.17	1962	4.98	8038	6
55	**9.68 4430**	3.80	**9.94 2169**	1.17	**9.74 2261**	4.97	**10.25 7739**	**5**
56	4658	3.82	2099	1.17	2559	4.98	7441	4
57	4887	3.80	2029	1.17	2858	4.97	7142	3
58	5115	3.80	1959	1.17	3156	4.97	6844	2
59	5343	3.80	1889	1.17	3454	4.97	6546	1
60	**9.68 5571**		**9.94 1819**		**9.74 3752**		**10.25 6248**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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TABLE XII.—*Continued.*

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′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.68 5571**	3.80	**9.94 1819**	1.17	**9.74 3752**	4.97	**10.25 6248**	**60**
1	5799	3.80	1749	1.17	4050	4.97	5950	59
2	6027	3.78	1679	1.17	4348	4.95	5652	58
3	6254	3.80	1609	1.17	4645	4.97	5355	57
4	6482	3.78	1539	1.17	4943	4.95	5057	56
5	**9.68 6709**	3.78	**9.94 1469**	1.18	**9.74 5240**	4.97	**10.25 4760**	**55**
6	6936	3.78	1398	1.17	5538	4.95	4462	54
7	7163	3.77	1328	1.17	5835	4.95	4165	53
8	7389	3.78	1258	1.18	6132	4.95	3868	52
9	7616	3.78	1187	1.17	6429	4.95	3571	51
10	**9.68 7843**	3.77	**9.94 1117**	1.18	**9.74 6726**	4.95	**10.25 3274**	**50**
11	8069	3.77	1046	1.18	7023	4.93	2977	49
12	8295	3.77	0975	1.17	7319	4.95	2681	48
13	8521	3.77	0905	1.18	7616	4.95	2384	47
14	8747	3.75	0834	1.18	7913	4.93	2087	46
15	**9.68 8972**	3.77	**9.94 0763**	1.17	**9.74 8209**	4.93	**10.25 1791**	**45**
16	9198	3.75	0693	1.18	8505	4.93	1495	44
17	9423	3.75	0622	1.18	8801	4.93	1199	43
18	9648	3.75	0551	1.18	9097	4.93	0903	42
19	.68 9873	3.75	0480	1.18	9393	4.93	0607	41
20	**9.69 0098**	3.75	**9.94 0409**	1.18	**9.74 9689**	4.93	**10.25 0311**	**40**
21	0323	3.75	0338	1.18	.74 9985	4.93	.25 0015	39
22	0548	3.73	0267	1.18	.75 0281	4.92	.24 9719	38
23	0772	3.73	0196	1.18	0576	4.93	9424	37
24	0996	3.73	0125	1.18	0872	4.92	9128	36
25	**9.69 1220**	3.73	**9.94 0054**	1.20	**9.75 1167**	4.92	**10.24 8833**	**35**
26	1444	3.73	.93 9982	1.18	1462	4.92	8538	34
27	1668	3.73	9911	1.18	1757	4.92	8243	33
28	1892	3.72	9840	1.20	2052	4.92	7948	32
29	2115	3.73	9768	1.18	2347	4.92	7653	31
30	**9.69 2339**	3.72	**9.93 9697**	1.20	**9.75 2642**	4.92	**10.24 7358**	**30**
31	2562	3.72	9625	1.18	2937	4.90	7063	29
32	2785	3.72	9554	1.20	3231	4.92	6769	28
33	3008	3.72	9482	1.20	3526	4.90	6474	27
34	3231	3.70	9410	1.18	3820	4.92	6180	26
35	**9.69 3453**	3.72	**9.93 9339**	1.20	**9.75 4115**	4.90	**10.24 5885**	**25**
36	3676	3.70	9267	1.20	4409	4.90	5591	24
37	3898	3.70	9195	1.20	4703	4.90	5297	23
38	4120	3.70	9123	1.18	4997	4.90	5003	22
39	4342	3.70	9052	1.20	5291	4.90	4709	21
40	**9.69 4564**	3.70	**9.93 8980**	1.20	**9.75 5585**	4.88	**10.24 4415**	**20**
41	4786	3.68	8908	1.20	5378	4.90	4122	19
42	5007	3.70	8836	1.22	6172	4.88	3828	18
43	5229	3.68	8763	1.20	6465	4.90	3535	17
44	5450	3.68	8691	1.20	6759	4.88	3241	16
45	**9.69 5671**	3.68	**9.93 8619**	1.20	**9.75 7052**	4.88	**10.24 2948**	**15**
46	5892	3.68	8547	1.20	7345	4.88	2655	14
47	6113	3.68	8475	1.22	7638	4.88	2362	13
48	6334	3.67	8402	1.20	7931	4.88	2069	12
49	6554	3.68	8330	1.20	8224	4.88	1776	11
50	**9.69 6775**	3.67	**9.93 8258**	1.22	**9.75 8517**	4.88	**10.24 1483**	**10**
51	6995	3.67	8185	1.20	8810	4.87	1190	9
52	7215	3.67	8113	1.22	9102	4.88	0898	8
53	7435	3.65	8040	1.22	9395	4.87	0605	7
54	7654	3.67	7967	1.20	9687	4.87	0313	6
55	**9.69 7874**	3.67	**9.93 7895**	1.22	**9.75 9979**	4.88	**10.24 0021**	**5**
56	8094	3.65	7822	1.22	.76 0272	4.87	.23 9728	4
57	8313	3.65	7749	1.22	0564	4.87	9436	3
58	8532	3.65	7676	1.20	0856	4.87	9144	2
59	8751	3.65	7604	1.22	1148	4.85	8852	1
60	**9.69 8970**		**9.93 7531**		**9.76 1439**		**10.23 8561**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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30° TABLE XII.—*Continued.* 149°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.69 8970**	3.65	**9.93 7531**	1.22	**9.76 1439**	4.87	**10.23 8561**	**60**
1	9189	3.63	7458	1.22	1731	4.87	8269	59
2	9407	3.65	7385	1.22	2023	4.85	7977	58
3	9626	3.63	7312	1.23	2314	4.87	7686	57
4	.69 9844	3.63	7238	1.22	2606	4.85	7394	56
5	**9.70 0062**	3.63	**9.93 7165**	1.22	**9.76 2897**	4.85	**10.23 7103**	**55**
6	0280	3.63	7092	1.22	3188	4.85	6812	54
7	0498	3.63	7019	1.22	3479	4.85	6521	53
8	0716	3.62	6946	1.23	3770	4.85	6230	52
9	0933	3.63	6872	1.22	4061	4.85	5939	51
10	**9.70 1151**	3.62	**9.93 6799**	1.23	**9.76 4352**	4.85	**10.23 5648**	**50**
11	1368	3.62	6725	1.22	4643	4.83	5357	49
12	1585	3.62	6652	1.23	4933	4.85	5067	48
13	1802	3.62	6578	1.22	5221	4.83	4776	47
14	2019	3.62	6505	1.23	5514	4.85	4486	46
15	**9.70 2236**	3.60	**9.93 6431**	1.23	**9.76 5805**	4.83	**10.23 4195**	**45**
16	2452	3.62	6357	1.22	6095	4.83	3905	44
17	2669	3.60	6284	1.23	6385	4.83	3615	43
18	2885	3.60	6210	1.23	6675	4.83	3325	42
19	3101	3.60	6136	1.23	6965	4.83	3035	41
20	**9.70 3317**	3.60	**9.93 6062**	1.23	**9.76 7255**	4.83	**10.23 2745**	**40**
21	3533	3.60	5988	1.23	7545	4.82	2455	39
22	3749	3.58	5914	1.23	7834	4.83	2166	38
23	3964	3.58	5840	1.23	8124	4.83	1876	37
24	4179	3.60	5766	1.23	8414	4.82	1586	36
25	**9.70 4395**	3.58	**9.93 5692**	1.23	**9.76 8703**	4.82	**10.23 1297**	**35**
26	4610	3.58	5618	1.25	8992	4.82	1008	34
27	4825	3.58	5543	1.23	9281	4.83	0719	33
28	5040	3.57	5469	1.23	9571	4.82	0429	32
29	5254	3.58	5395	1.25	.76 9860	4.80	.23 0140	31
30	**9.70 5469**	3.57	**9.93 5320**	1.23	**9.77 0143**	4.82	**10.22 9852**	**30**
31	5683	3.58	5246	1.25	0437	4.82	9563	29
32	5898	3.57	5171	1.23	0726	4.82	9274	28
33	6112	3.57	5097	1.25	1015	4.80	8985	27
34	6326	3.55	5022	1.23	1303	4.82	8697	26
35	**9.70 6539**	3.57	**9.93 4948**	1.25	**9.77 1592**	4.80	**10.22 8408**	**25**
36	6753	3.57	4873	1.25	1880	4.80	8120	24
37	6967	3.55	4798	1.25	2168	4.82	7832	23
38	7180	3.55	4723	1.23	2457	4.80	7543	22
39	7393	3.55	4649	1.25	2745	4.80	7255	21
40	**9.70 7606**	3.55	**9.93 4574**	1.25	**9.77 3033**	4.80	**10.22 6967**	**20**
41	7819	3.55	4499	1.25	3321	4.78	6679	19
42	8032	3.55	4424	1.25	3608	4.80	6392	18
43	8245	3.55	4349	1.25	3896	4.80	6104	17
44	8458	3.53	4274	1.25	4184	4.78	5816	16
45	**9.70 8670**	3.53	**9.93 4199**	1.27	**9.77 4471**	4.80	**10.22 5529**	**15**
46	8882	3.53	4123	1.25	4759	4.78	5241	14
47	9094	3.53	4048	1.25	5046	4.78	4954	13
48	9306	3.53	3973	1.25	5333	4.80	4667	12
49	9518	3.53	3898	1.27	5621	4.78	4379	11
50	**9.70 9730**	3.52	**9.93 3822**	1.25	**9.77 5908**	4.78	**10.22 4092**	**10**
51	.70 9941	3.53	3747	1.27	6195	4.78	3805	9
52	.71 0153	3.52	3671	1.25	6482	4.77	3518	8
53	0364	3.52	3596	1.27	6768	4.78	3232	7
54	0575	3.52	3520	1.25	7055	4.78	2945	6
55	**9.71 0786**	3.52	**9.93 3445**	1.27	**9.77 7342**	4.77	**10.22 2658**	**5**
56	0997	3.52	3369	1.27	7628	4.78	2372	4
57	1208	3.52	3293	1.27	7915	4.77	2085	3
58	1419	3.50	3217	1.27	8201	4.78	1799	2
59	1629	3.50	3141	1.25	8488	4.77	1512	1
60	**9.71 1839**		**9.93 3066**		**9.77 8774**		**10.22 1226**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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31° TABLE XII.—*Continued.* 148°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.71 1839**	3.52	**9.93 3066**	1.27	**9.77 8774**	4.77	**10.22 1226**	**60**
1	2050	3.50	2990	1.27	9060	4.77	0940	59
2	2260	3.48	2914	1.27	9346	4.77	0654	58
3	2469	3.50	2838	1.27	9632	4.77	0368	57
4	2679	3.50	2762	1.28	.77 9918	4.75	.22 0082	56
5	**9.71 2889**	3.48	**9.93 2685**	1.27	**9.78 0203**	4.77	**10.21 9797**	**55**
6	3098	3.50	2609	1.27	0489	4.77	9511	54
7	3308	3.48	2533	1.27	0775	4.75	9225	53
8	3517	3.48	2457	1.28	1060	4.77	8940	52
9	3726	3.48	2380	1.27	1346	4.75	8654	51
10	**9.71 3935**	3.48	**9.93 2304**	1.27	**9.78 1631**	4.75	**10.21 8369**	**50**
11	4144	3.47	2228	1.28	1916	4.75	8084	49
12	4352	3.48	2151	1.27	2201	4.75	7799	48
13	4561	3.47	2075	1.28	2486	4.75	7514	47
14	4769	3.48	1998	1.28	2771	4.75	7229	46
15	**9.71 4978**	3.47	**9.93 1921**	1.27	**9.78 3056**	4.75	**10.21 6944**	**45**
16	5186	3.47	1845	1.28	3341	4.75	6659	44
17	5394	3.47	1768	1.28	3626	4.73	6374	43
18	5602	3.45	1691	1.28	3910	4.75	6090	42
19	5809	3.47	1614	1.28	4195	4.73	5805	41
20	**9.71 6017**	3.45	**9.93 1537**	1.28	**9.78 4479**	4.75	**10.21 5521**	**40**
21	6224	3.47	1460	1.28	4764	4.73	5236	39
22	6432	3.45	1383	1.28	5048	4.73	4952	38
23	6639	3.45	1306	1.28	5332	4.73	4668	37
24	6846	3.45	1229	1.28	5616	4.73	4384	36
25	**9.71 7053**	3.43	**9.93 1152**	1.28	**9.78 5900**	4.73	**10.21 4100**	**35**
26	7259	3.45	1075	1.28	6184	4.73	3816	34
27	7466	3.45	0998	1.28	6468	4.73	3532	33
28	7673	3.43	0921	1.30	6752	4.73	3248	32
29	7879	3.43	0843	1.28	7036	4.72	2964	31
30	**9.71 8085**	3.43	**9.93 0766**	1.30	**9.78 7319**	4.73	**10.21 2681**	**30**
31	8291	3.43	0688	1.28	7603	4.72	2397	29
32	8497	3.43	0611	1.30	7886	4.73	2114	28
33	8703	3.43	0533	1.28	8170	4.72	1830	27
34	8909	3.42	0456	1.30	8453	4.72	1547	26
35	**9.71 9114**	3.43	**9.93 0378**	1.30	**9.78 8736**	4.72	**10.21 1264**	**25**
36	9320	3.42	0300	1.28	9019	4.72	0981	24
37	9525	3.42	0223	1.30	9302	4.72	0698	23
38	9730	3.42	0145	1.30	9585	4.72	0415	22
39	.71 9935	3.42	.93 0067	1.30	.78 9863	4.72	.21 0132	21
40	**9.72 0140**	3.42	**9.92 9989**	1.30	**9.79 0151**	4.72	**10.20 9849**	**20**
41	0345	3.40	9911	1.30	0434	4.70	9566	19
42	0549	3.42	9833	1.30	0716	4.72	9284	18
43	0754	3.40	9755	1.30	0999	4.70	9001	17
44	0958	3.40	9677	1.30	1281	4.70	8719	16
45	**9.72 1162**	3.40	**9.92 9599**	1.30	**9.79 1563**	4.72	**10.20 8437**	**15**
46	1366	3.40	9521	1.32	1846	4.70	8154	14
47	1570	3.40	9442	1.30	2128	4.70	7872	13
48	1774	3.40	9364	1.30	2410	4.70	7590	12
49	1978	3.38	9286	1.32	2692	4.70	7308	11
50	**9.72 2181**	3.40	**9.92 9207**	1.30	**9.79 2974**	4.70	**10.20 7026**	**10**
51	2385	3.38	9129	1.32	3256	4.70	6744	9
52	2588	3.38	9050	1.30	3538	4.68	6462	8
53	2791	3.38	8972	1.32	3819	4.70	6181	7
54	2994	3.38	8893	1.30	4101	4.70	5899	6
55	**9.72 3197**	3.38	**9.92 8815**	1.32	**9.79 4383**	4.68	**10.20 5617**	**5**
56	3400	3.38	8736	1.32	4664	4.70	5336	4
57	3603	3.37	8657	1.32	4946	4.68	5054	3
58	3805	3.37	8578	1.32	5227	4.68	4773	2
59	4007	3.38	8499	1.32	5508	4.68	4492	1
60	**9.72 4210**		**9.92 8420**		**9.79 5789**		**10.20 4211**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

121° 58°

32° TABLE XII.—*Continued.* 147°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.72 4210**	3.37	**9.92 8420**	1.30	**9.79 5789**	4.68	**10.20 4211**	**60**
1	4412	3.37	8342	1.32	6070	4.68	3930	59
2	4614	3.37	8263	1.33	6351	4.68	3649	58
3	4816	3.35	8183	1.32	6632	4.68	3368	57
4	5017	3.37	8104	1.32	6913	4.68	3087	56
5	**9.72 5219**	3.35	**9.92 8025**	1.32	**9.79 7194**	4.67	**10.20 2806**	**55**
6	5420	3.37	7946	1.32	7474	4.68	2526	54
7	5622	3.35	7867	1.33	7755	4.68	2245	53
8	5823	3.35	7787	1.32	8036	4.67	1964	52
9	6024	3.35	7708	1.32	8316	4.67	1684	51
10	**9.72 6225**	3.35	**9.92 7629**	1.33	**9.79 8596**	4.68	**10.20 1404**	**50**
11	6426	3.33	7549	1.32	8877	4.67	1123	49
12	6626	3.35	7470	1.33	9157	4.67	0843	48
13	6827	3.33	7390	1.33	9437	4.67	0563	47
14	7027	3.35	7310	1.32	9717	4.67	0283	46
15	**9.72 7228**	3.33	**9.92 7231**	1.33	**9.79 9997**	4.67	**10.20 0003**	**45**
16	7428	3.33	7151	1.33	.80 0277	4.67	.19 9723	44
17	7628	3.33	7071	1.33	0557	4.65	9443	43
18	7828	3.32	6991	1.33	0836	4.67	9164	42
19	8027	3.33	6911	1.33	1116	4.67	8884	41
20	**9.72 8227**	3.33	**9.92 6831**	1.33	**9.80 1396**	4.65	**10.19 8604**	**40**
21	8427	3.32	6751	1.33	1675	4.67	8325	39
22	8626	3.32	6671	1.33	1955	4.65	8045	38
23	8825	3.32	6591	1.33	2234	4.65	7766	37
24	9024	3.32	6511	1.33	2513	4.65	7487	36
25	**9.72 9223**	3.32	**9.92 6431**	1.33	**9.80 2792**	4.67	**10.19 7208**	**35**
26	9422	3.32	6351	1.35	3072	4.65	6928	34
27	9621	3.32	6270	1.33	3351	4.65	6649	33
28	.72 9820	3.30	6190	1.33	3630	4.65	6370	32
29	.73 0018	3.32	6110	1.35	3909	4.63	6091	31
30	**9.73 0217**	3.30	**9.92 6029**	1.33	**9.80 4187**	4.65	**10.19 5813**	**30**
31	0415	3.30	5949	1.35	4466	4.65	5534	29
32	0613	3.30	5868	1.33	4745	4.63	5255	28
33	0811	3.30	5788	1.35	5023	4.65	4977	27
34	1009	3.28	5707	1.35	5302	4.63	4698	26
35	**9.73 1206**	3.30	**9.92 5626**	1.35	**9.80 5580**	4.65	**10.19 4420**	**25**
36	1404	3.30	5545	1.33	5859	4.63	4141	24
37	1602	3.28	5465	1.35	6137	4.63	3863	23
38	1799	3.28	5384	1.35	6415	4.63	3585	22
39	1996	3.28	5303	1.35	6693	4.63	3307	21
40	**9.73 2193**	3.28	**9.92 5222**	1.35	**9.80 6971**	4.63	**10.19 3029**	**20**
41	2390	3.28	5141	1.35	7249	4.63	2751	19
42	2587	3.28	5060	1.35	7527	4.63	2473	18
43	2784	3.27	4979	1.37	7805	4.63	2195	17
44	2980	3.28	4897	1.35	8083	4.63	1917	16
45	**9.73 3177**	3.27	**9.92 4816**	1.35	**9.80 8361**	4.62	**10.19 1639**	**15**
46	3373	3.27	4735	1.35	8638	4.63	1362	14
47	3569	3.27	4654	1.37	8916	4.62	1084	13
48	3765	3.27	4572	1.35	9193	4.63	0807	12
49	3961	3.27	4491	1.37	9471	4.62	0529	11
50	**9.73 4157**	3.27	**9.92 4409**	1.35	**9.80 9748**	4.62	**10.19 0252**	**10**
51	4353	3.27	4328	1.37	.81 0025	4.62	.18 9975	9
52	4549	3.25	4246	1.37	0302	4.63	9698	8
53	4744	3.25	4164	1.35	0580	4.62	9420	7
54	4939	3.27	4083	1.37	0857	4.62	9143	6
55	**9.73 5135**	3.25	**9.92 4001**	1.37	**9.81 1134**	4.60	**10.18 8866**	**5**
56	5330	3.25	3919	1.37	1410	4.62	8590	4
57	5525	3.23	3837	1.37	1687	4.62	8313	3
58	5719	3.25	3755	1.37	1964	4.62	8036	2
59	5914	3.25	3673	1.37	2241	4.60	7759	1
60	**9.73 6109**		**9.92 3591**		**9.81 2517**		**10.18 7483**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

122° 57°

33° **TABLE XII.**—*Continued.* 146°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.73 6109**	3.23	**9.92 3591**	1.37	**9.81 2517**	4.62	**10.18 7483**	**60**
1	6303	3.25	3509	1.37	2794	4.60	7206	59
2	6498	3.23	3427	1.37	3070	4.62	6930	58
3	6692	3.23	3345	1.37	3347	4.60	6653	57
4	6886	3.23	3263	1.37	3623	4.60	6377	56
5	**9.73 7080**	3.23	**9.92 3181**	1.38	**9.81 3899**	4.62	**10.18 6101**	**55**
6	7274	3.22	3098	1.37	4176	4.60	5824	54
7	7467	3.23	3016	1.38	4452	4.60	5548	53
8	7661	3.23	2933	1.37	4728	4.60	5272	52
9	7855	3.22	2851	1.38	5004	4.60	4996	51
10	**9.73 8048**	3.22	**9.92 2768**	1.37	**9.81 5280**	4.58	**10.18 4720**	**50**
11	8241	3.22	2686	1.38	5555	4.60	4445	49
12	8434	3.22	2603	1.38	5831	4.60	4169	48
13	8627	3.22	2520	1.37	6107	4.58	3893	47
14	8820	3.22	2438	1.38	6382	4.60	3618	46
15	**9.73 9013**	3.22	**9.92 2355**	1.38	**9.81 6658**	4.58	**10.18 3342**	**45**
16	9206	3.20	2272	1.38	6933	4.60	3067	44
17	9398	3.20	2189	1.38	7209	4.58	2791	43
18	9590	3.22	2106	1.38	7484	4.58	2516	42
19	9783	3.20	2023	1.38	7759	4.60	2241	41
20	**9.73 9975**	3.20	**9.92 1940**	1.38	**9.81 8035**	4.58	**10.18 1965**	**40**
21	.74 0167	3.20	1857	1.38	8310	4.58	1690	39
22	0359	3.18	1774	1.38	8585	4.58	1415	38
23	0550	3.20	1691	1.40	8860	4.58	1140	37
24	0742	3.20	1607	1.38	9135	4.58	0865	36
25	**9.74 0934**	3.18	**9.92 1524**	1.38	**9.81 9410**	4.57	**10.18 0590**	**35**
26	1125	3.18	1441	1.40	9684	4.58	0316	34
27	1316	3.20	1357	1.38	.81 9959	4.58	.18 0041	33
28	1508	3.18	1274	1.40	.82 0234	4.57	.17 9766	32
29	1699	3.17	1190	1.38	0508	4.58	9492	31
30	**9.74 1889**	3.18	**9.92 1107**	1.40	**9.82 0783**	4.57	**10.17 9217**	**30**
31	2080	3.18	1023	1.40	1057	4.58	8943	29
32	2271	3.18	0939	1.38	1332	4.57	8668	28
33	2462	3.17	0856	1.40	1606	4.57	8394	27
34	2652	3.17	0772	1.40	1880	4.57	8120	26
35	**9.74 2842**	3.18	**9.92 0688**	1.40	**9.82 2154**	4.58	**10.17 7846**	**25**
36	3033	3.17	0604	1.40	2429	4.57	7571	24
37	3223	3.17	0520	1.40	2703	4.57	7297	23
38	3413	3.15	0436	1.40	2977	4.57	7023	22
39	3602	3.17	0352	1.40	3251	4.55	6749	21
40	**9.74 3792**	3.17	**9.92 0268**	1.40	**9.82 3524**	4.57	**10.17 6476**	**20**
41	3982	3.15	0184	1.42	3798	4.57	6202	19
42	4171	3.17	0099	1.40	4072	4.55	5928	18
43	4361	3.15	.92 0015	1.40	4345	4.57	5655	17
44	4550	3.15	.91 9931	1.42	4619	4.57	5381	16
45	**9.74 4739**	3.15	**9.91 9846**	1.40	**9.82 4893**	4.55	**10.17 5107**	**15**
46	4928	3.15	9762	1.42	5166	4.55	4834	14
47	5117	3.15	9677	1.40	5439	4.57	4561	13
48	5306	3.13	9593	1.42	5713	4.55	4287	12
49	5494	3.15	9508	1.40	5986	4.55	4014	11
50	**9.74 5683**	3.13	**9.91 9424**	1.42	**9.82 6254**	4.55	**10.17 3741**	**10**
51	5871	3.15	9339	1.42	6532	4.55	3468	9
52	6060	3.13	9254	1.42	6805	4.55	3195	8
53	6248	3.13	9169	1.40	7078	4.55	2922	7
54	6436	3.13	9085	1.42	7351	4.55	2649	6
55	**9.74 6624**	3.13	**9.91 9000**	1.42	**9.82 7624**	4.55	**10.17 2376**	**5**
56	6812	3.12	8915	1.42	7897	4.55	2103	4
57	6999	3.13	8830	1.42	8170	4.53	1830	3
58	7187	3.12	8745	1.43	8442	4.55	1558	2
59	7374	3.13	8659	1.42	8715	4.53	1285	1
60	**9.74 7562**		**9.91 8574**		**9.82 8987**		**10.17 1013**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

123° 56°

34° TABLE XII.—*Continued.* 145°

′	Sine.	D.1″	Cosine.	D.1″	Tang.	D.1″	Cotang.	′
0	**9.74 7562**	3.12	**9.91 8574**	1.42	**9.82 8987**	4.55	**10.17 1013**	**60**
1	7749	3.12	8489	1.42	9260	4.53	0740	59
2	7936	3.12	8404	1.43	9532	4.55	0468	58
3	8123	3.12	8318	1.42	.82 9805	4.53	.17 0195	57
4	8310	3.12	8233	1.43	.83 0077	4.53	.16 9923	56
5	**9.74 8497**	3.10	**9.91 8147**	1.42	**9.83 0349**	4.53	**10.16 9651**	**55**
6	8683	3.12	8062	1.43	0621	4.53	9379	54
7	8870	3.10	7976	1.42	0893	4.53	9107	53
8	9056	3.12	7891	1.43	1165	4.53	8835	52
9	9243	3.10	7805	1.43	1437	4.53	8563	51
10	**9.74 9429**	3.10	**9.91 7719**	1.42	**9.83 1709**	4.53	**10.16 8291**	**50**
11	9615	3.10	7634	1.43	1981	4.53	8019	49
12	9801	3.10	7548	1.43	2253	4.53	7747	48
13	.74 9987	3.08	7462	1.43	2525	4.52	7475	47
14	.75 0172	3.10	7376	1.43	2796	4.53	7204	46
15	**9.75 0358**	3.08	**9.91 7290**	1.43	**9.83 3068**	4.52	**10.16 6932**	**45**
16	0543	3.10	7204	1.43	3339	4.53	6661	44
17	0729	3.08	7118	1.43	3611	4.52	6389	43
18	0914	3.08	7032	1.43	3882	4.53	6118	42
19	1099	3.08	6946	1.45	4154	4.52	5846	41
20	**9.75 1284**	3.08	**9.91 6859**	1.43	**9.83 4425**	4.52	**10.16 5575**	**40**
21	1469	3.08	6773	1.43	4696	4.52	5304	39
22	1654	3.08	6687	1.45	4967	4.52	5033	38
23	1839	3.07	6600	1.43	5238	4.52	4762	37
24	2023	3.08	6514	1.45	5509	4.52	4491	36
25	**9.75 2208**	3.07	**9.91 6427**	1.43	**9.83 5780**	4.52	**10.16 4220**	**35**
26	2392	3.07	6341	1.45	6051	4.52	3949	34
27	2576	3.07	6254	1.45	6322	4.52	3678	33
28	2760	3.07	6167	1.43	6593	4.52	3407	32
29	2944	3.07	6081	1.45	6864	4.50	3136	31
30	**9.75 3128**	3.07	**9.91 5994**	1.45	**9.83 7134**	4.52	**10.16 2866**	**30**
31	3312	3.05	5907	1.45	7405	4.50	2595	29
32	3495	3.07	5820	1.45	7675	4.52	2325	28
33	3679	3.07	5733	1.45	7946	4.50	2054	27
34	3862	3.07	5646	1.45	8216	4.52	1784	26
35	**9.75 4046**	3.05	**9.91 5559**	1.45	**9.83 8487**	4.50	**10.16 1513**	**25**
36	4229	3.05	5472	1.45	8757	4.50	1243	24
37	4412	3.05	5385	1.47	9027	4.50	0973	23
38	4595	3.05	5297	1.45	9297	4.52	0703	22
39	4778	3.03	5210	1.45	9568	4.50	0432	21
40	**9.75 4960**	3.05	**9.91 5123**	1.47	**9.83 9838**	4.50	**10.16 0162**	**20**
41	5143	3.05	5035	1.45	.84 0108	4.50	.15 9892	19
42	5326	3.03	4948	1.47	0378	4.50	9622	18
43	5508	3.03	4860	1.45	0648	4.48	9352	17
44	5690	3.03	4773	1.47	0917	4.50	9083	16
45	**9.75 5872**	3.03	**9.91 4685**	1.45	**9.84 1187**	4.50	**10.15 8813**	**15**
46	6054	3.03	4598	1.47	1457	4.50	8543	14
47	6236	3.03	4510	1.47	1727	4.48	8273	13
48	6418	3.03	4422	1.47	1996	4.50	8004	12
49	6600	3.03	4334	1.47	2266	4.48	7734	11
50	**9.75 6782**	3.02	**9.91 4246**	1.47	**9.84 2535**	4.50	**10.15 7465**	**10**
51	6963	3.02	4158	1.47	2805	4.48	7195	9
52	7144	3.03	4070	1.47	3074	4.48	6926	8
53	7326	3.02	3982	1.47	3343	4.48	6657	7
54	7507	3.02	3894	1.47	3612	4.50	6388	6
55	**9.75 7688**	3.02	**9.91 3806**	1.47	**9.84 3882**	4.48	**10.15 6118**	**5**
56	7869	3.02	3718	1.47	4151	4.48	5849	4
57	8050	3.00	3630	1.48	4420	4.48	5580	3
58	8230	3.02	3541	1.47	4689	4.48	5311	2
59	8411	3.00	3453	1.47	4958	4.48	5042	1
60	**9.75 8591**		**9.91 3365**		**9.84 5227**		**10.15 4773**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

124° 55°

35° TABLE XII.—*Continued.* 144°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.75 8591**	3.02	**9.91 3365**	1.48	**9.84 5227**	4.48	**10 15 4773**	**60**
1	8772	3.00	3276	1.48	5496	4.47	4504	59
2	8952	3.00	3187	1.47	5764	4.48	4236	58
3	9132	3.00	3099	1.48	6033	4.48	3967	57
4	9312	3.00	3010	1.47	6302	4.47	3698	56
5	**9.75 9492**	3.00	**9.91 2922**	1.48	**9.84 6570**	4.48	**10.15 3430**	**55**
6	9672	3.00	2833	1.48	6839	4.48	3161	54
7	.75 9852	2.98	2744	1.48	7108	4.47	2892	53
8	.76 0031	3.00	2655	1.48	7376	4.47	2624	52
9	0211	2.98	2566	1.48	7644	4.48	2356	51
10	**9.76 0390**	2.98	**9.91 2477**	1.48	**9.84 7913**	4.47	**10.15 2087**	**50**
11	0569	2.98	2388	1.48	8181	4.47	1819	49
12	0748	2.98	2299	1.48	8449	4.47	1551	48
13	0927	2.98	2210	1.48	8717	4.48	1283	47
14	1106	2.98	2121	1.50	8986	4.47	1014	46
15	**9.76 1285**	2.98	**9.91 2031**	1.48	**9.84 9254**	4.47	**10.15 0746**	**45**
16	1464	2.97	1942	1.48	9522	4.47	0478	44
17	1642	2.98	1853	1.50	.84 9790	4.45	.15 0210	43
18	1821	2.97	1763	1.48	.85 0057	4.47	.14 9943	42
19	1999	2.97	1674	1.50	0325	4.47	9675	41
20	**9.76 2177**	2.98	**9.91 1584**	1.48	**9.85 0593**	4.47	**10.14 9407**	**40**
21	2356	2.97	1495	1.50	0861	4.47	9139	39
22	2534	2.97	1405	1.50	1129	4.45	8871	38
23	2712	2.95	1315	1.48	1396	4.47	8604	37
24	2889	2.97	1226	1.50	1664	4.45	8336	36
25	**9.76 3067**	2.97	**9.91 1136**	1.50	**9.85 1931**	4.47	**10.14 8069**	**35**
26	3245	2.95	1046	1.50	2199	4.45	7801	34
27	3422	2.97	0956	1.50	2466	4.45	7534	33
28	3600	2.95	0866	1.50	2733	4.47	7267	32
29	3777	2.95	0776	1.50	3001	4.45	6999	31
30	**9.76 3954**	2.95	**9.91 0686**	1.50	**9.85 3268**	4.45	**10.14 6732**	**30**
31	4131	2.95	0596	1.50	3535	4.45	6465	29
32	4308	2.95	0506	1.52	3802	4.45	6198	28
33	4485	2.95	0415	1.50	4069	4.45	5931	27
34	4662	2.93	0325	1.50	4336	4.45	5664	26
35	**9.76 4838**	2.95	**9.91 0235**	1.52	**9.85 4603**	4.45	**10.14 5397**	**25**
36	5015	2.93	0144	1.50	4870	4.45	5130	24
37	5191	2.93	.91 0054	1.52	5137	4.45	4863	23
38	5367	2.95	.90 9963	1.50	5404	4.45	4596	22
39	5544	2.93	9873	1.52	5671	4.45	4329	21
40	**9.76 5720**	2.93	**9.90 9782**	1.52	**9.85 5938**	4.43	**10.14 4062**	**20**
41	5896	2.93	9691	1.50	6204	4.45	3796	19
42	6072	2.92	9601	1.52	6471	4.43	3529	18
43	6247	2.93	9510	1.52	6737	4.45	3263	17
44	6423	2.92	9419	1.52	7004	4.43	2996	16
45	**9.76 6598**	2.93	**9.90 9328**	1.52	**9.85 7270**	4.45	**10.14 2730**	**15**
46	6774	2.92	9237	1.52	7537	4.43	2463	14
47	6949	2.92	9146	1.52	7803	4.43	2197	13
48	7124	2.93	9055	1.52	8069	4.45	1931	12
49	7300	2.92	8964	1.52	8336	4.43	1664	11
50	**9.76 7475**	2.90	**9.90 8873**	1.53	**9.85 8602**	4.43	**10.14 1398**	**10**
51	7649	2.92	8781	1.52	8868	4.43	1132	9
52	7824	2.92	8690	1.52	9134	4.43	0866	8
53	7999	2.90	8599	1.53	9400	4.43	0600	7
54	8173	2.92	8507	1.52	9666	4.43	0334	6
55	**9.76 8348**	2.90	**9.90 8416**	1.53	**9.85 9932**	4.43	**10.14 0068**	**5**
56	8522	2.92	8324	1.52	.86 0198	4.43	.13 9802	4
57	8697	2.90	8233	1.53	0464	4.43	9536	3
58	8871	2.90	8141	1.53	0730	4.42	9270	2
59	9045	2.90	8049	1.52	0995	4.43	9005	1
60	**9.76 9219**		**9.90 7958**		**9.86 1261**		**10.13 8739**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

125° 54°

36° TABLE XII.—*Continued.* 143°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.76 9219**	2.90	**9.90 7958**	1.53	**9.86 1261**	4.43	**10.13 8739**	**60**
1	9393	2.88	7866	1.53	1527	4.42	8473	59
2	9566	2.90	7774	1.53	1792	4.43	8208	58
3	9740	2.88	7682	1.53	2058	4.42	7942	57
4	.76 9913	2.90	7590	1.53	2323	4.43	7677	56
5	**9.77 0087**	2.88	**9.90 7498**	1.53	**9.86 2589**	4.42	**10.13 7411**	**55**
6	0260	2.88	7406	1.53	2854	4.42	7146	54
7	0433	2.88	7314	1.53	3119	4.43	6881	53
8	0606	2.88	7222	1.55	3385	4.42	6615	52
9	0779	2.88	7129	1.53	3650	4.42	6350	51
10	**9.77 0952**	2.88	**9.90 7037**	1.53	**9.86 3915**	4.42	**10.13 6085**	**50**
11	1125	2.88	6945	1.55	4180	4.42	5820	49
12	1298	2.87	6852	1.53	4445	4.42	5555	48
13	1470	2.88	6760	1.55	4710	4.42	5290	47
14	1643	2.87	6667	1.53	4975	4.42	5025	46
15	**9.77 1815**	2.87	**9.90 6575**	1.55	**9.86 5240**	4.42	**10.13 4760**	**45**
16	1987	2.87	6482	1.55	5505	4.42	4495	44
17	2159	2.87	6389	1.55	5770	4.42	4230	43
18	2331	2.87	6296	1.53	6035	4.42	3965	42
19	2503	2.87	6204	1.55	6300	4.40	3700	41
20	**9.77 2675**	2.87	**9.90 6111**	1.55	**9.86 6564**	4.42	**10.13 3436**	**40**
21	2847	2.85	6018	1.55	6829	4.42	3171	39
22	3018	2.87	5925	1.55	7094	4.40	2906	38
23	3190	2.85	5832	1.55	7358	4.42	2642	37
24	3361	2.87	5739	1.57	7623	4.40	2377	36
25	**9.77 3533**	2.85	**9.90 5645**	1.55	**9.86 7887**	4.42	**10.13 2113**	**35**
26	3704	2.85	5552	1.55	8152	4.40	1848	34
27	3875	2.85	5459	1.55	8416	4.40	1584	33
28	4046	2.85	5366	1.57	8680	4.42	1320	32
29	4217	2.85	5272	1.55	8945	4.40	1055	31
30	**9.77 4383**	2.83	**9.90 5179**	1.57	**9.86 9209**	4.40	**10.13 0791**	**30**
31	4558	2.85	5085	1.55	9473	4.40	0527	29
32	4729	2.83	4992	1.57	.86 9737	4.40	.13 0263	28
33	4899	2.85	4898	1.57	.87 0001	4.40	.12 9999	27
34	5070	2.83	4804	1.55	0265	4.40	9735	26
35	**9.77 5240**	2.83	**9.90 4711**	1.57	**9.87 0529**	4.40	**10.12 9471**	**25**
36	5410	2.83	4617	1.57	0793	4.40	9207	24
37	5580	2.83	4523	1.57	1057	4.40	8943	23
38	5750	2.83	4429	1.57	1321	4.40	8679	22
39	5920	2.83	4335	1.57	1585	4.40	8415	21
40	**9.77 6090**	2.82	**9.90 4241**	1.57	**9.87 1849**	4.38	**10.12 8151**	**20**
41	6259	2.83	4147	1.57	2112	4.40	7888	19
42	6429	2.82	4053	1.57	2376	4 40	7624	18
43	6598	2.83	3959	1.58	3640	4.38	7360	17
44	6768	2.82	3864	1.57	2903	4.40	7097	16
45	**9.77 6937**	2.82	**9.90 3770**	1.57	**9.87 3167**	4.38	**10.12 6833**	**15**
46	7106	2.82	3676	1.58	3430	4.40	6570	14
47	7275	2.82	3581	1.57	3694	4.38	6306	13
48	7444	2.82	3487	1.58	3957	4.38	6043	12
49	7613	2.80	3392	1.57	4220	4.40	5780	11
50	**9.77 7781**	2.82	**9.90 3298**	1.58	**9.87 4484**	4.38	**10.12 5516**	**10**
51	7950	2.82	3203	1.58	4747	4.38	5253	9
52	3119	2.80	3108	1.57	5010	4.38	4990	8
53	8287	2.80	3014	1.58	5273	4.40	4727	7
54	8455	2.82	2919	1.58	5537	4.38	4463	6
55	**9.77 8624**	2.80	**9.90 2824**	1.58	**9.87 5800**	4.38	**10.12 4200**	**5**
56	8792	2.80	2729	1.58	6063	4.38	3937	4
57	8960	2.80	2634	1.58	6326	4.38	3674	3
58	9128	2.78	2539	1.58	6589	4.38	3411	2
59	9295	2.80	2444	1.58	6852	4.37	3148	1
60	**9.77 9463**		**9.90 2349**		**9.87 7114**		**10.12 2886**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

126° 53°

37° TABLE XII.—*Continued.* 142°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.77 9463**	2.80	**9.90 2349**	1.60	**9.87 7114**	4.38	**10.12 2886**	**60**
1	9631	2.78	2253	1.58	7377	4.38	2623	59
2	9798	2.80	2158	1.58	7640	4.38	2360	58
3	.77 9966	2.78	2063	1.60	7903	4.37	2097	57
4	.78 0133	2.78	1967	1.58	8165	4.38	1835	56
5	**9.78 0300**	2.78	**9.90 1872**	1.60	**9.87 8428**	4.38	**10.12 1572**	**55**
6	0467	2.78	1776	1.58	8691	4.37	1309	54
7	0634	2.78	1681	1.60	8953	4.38	1047	53
8	0801	2.78	1585	1.58	9216	4.37	0784	52
9	0968	2.77	1490	1.60	9478	4.38	0522	51
10	**9.78 1134**	2.78	**9.90 1394**	1.60	**9.87 9741**	4.37	**10.12 0259**	**50**
11	1301	2.78	1298	1.60	.88 0003	4.37	.11 9997	49
12	1468	2.77	1202	1.60	0265	4.38	9735	48
13	1634	2.77	1106	1.60	0528	4.37	9472	47
14	1800	2.77	1010	1.60	0790	4.37	9210	46
15	**9.78 1966**	2.77	**9.90 0914**	1.60	**9.88 1052**	4.37	**10.11 8948**	**45**
16	2132	2.77	0818	1.60	1314	4.38	8686	44
17	2298	2.77	0722	1.60	1577	4.37	8423	43
18	2464	2.77	0626	1.62	1839	4.37	8161	42
19	2630	2.77	0529	1.60	2101	4.37	7899	41
20	**9.78 2796**	2.75	**9.90 0433**	1.60	**9.88 2363**	4.37	**10.11 7637**	**40**
21	2961	2.77	0337	1.62	2625	4.37	7375	39
22	3127	2.75	0240	1.60	2887	4.35	7113	38
23	3292	2.77	0144	1.62	3148	4.37	6852	37
24	3458	2.75	.90 0047	1.60	3410	4.37	6590	36
25	**9.78 3623**	2.75	**9.89 9951**	1.62	**9.88 3672**	4.37	**10.11 6328**	**35**
26	3788	2.75	9854	1.62	3934	4.37	6066	34
27	3953	2.75	9757	1.62	4196	4.35	5804	33
28	4118	2.73	9660	1.60	4457	4.37	5543	32
29	4282	2.75	9564	1.62	4719	4.35	5281	31
30	**9.78 4447**	2.75	**9.89 9467**	1.62	**9.88 4980**	4.37	**10.11 5020**	**30**
31	4612	2.73	9370	1.62	5242	4.37	4758	29
32	4776	2.75	9273	1.62	5504	4.35	4496	28
33	4941	2.73	9176	1.63	5765	4.35	4235	27
34	5105	2.73	9078	1.62	6026	4.37	3974	26
35	**9.78 5269**	2.73	**9.89 8981**	1.62	**9.88 6288**	4.35	**10.11 3712**	**25**
36	5433	2.73	8884	1.62	6549	4.37	3451	24
37	5597	2.73	8787	1.63	6811	4.35	3189	23
38	5761	2.73	8689	1.62	7072	4.35	2928	22
39	5925	2.73	8592	1.63	7333	4.35	2667	21
40	**9.78 6089**	2.72	**9.89 8494**	1.62	**9.88 7594**	4.35	**10.11 2406**	**20**
41	6252	2.73	8397	1.63	7855	4.35	2145	19
42	6416	2.72	8299	1.62	8116	4.37	1884	18
43	6579	2.72	8202	1.63	8378	4.35	1622	17
44	6742	2.73	8104	1.63	8639	4.35	1361	16
45	**9.78 6906**	2.72	**9.89 8006**	1.63	**9.88 8900**	4.35	**10.11 1100**	**15**
46	7069	2.72	7908	1.63	9161	4.33	0839	14
47	7232	2.72	7810	1.63	9421	4.35	0579	13
48	7395	2.70	7712	1.63	9682	4.35	0318	12
49	7557	2.72	7614	1.63	.88 9943	4.35	.11 0057	11
50	**9.78 7720**	2.72	**9.89 7516**	1.63	**9.89 0204**	4.35	**10.10 9796**	**10**
51	7883	2.70	7418	1.63	0465	4.33	9535	9
52	8045	2.72	7320	1.63	0725	4.35	9275	8
53	8208	2.70	7222	1.65	0986	4.35	9014	7
54	8370	2.70	7123	1.63	1247	4.33	8753	6
55	**9.78 8532**	2.70	**9.89 7025**	1.65	**9.89 1507**	4.35	**10.10 8493**	**5**
56	8694	2.70	6926	1.63	1768	4.33	8232	4
57	8856	2.70	6828	1.65	2028	4.35	7972	3
58	9018	2.70	6729	1.63	2289	4.33	7711	2
59	9180	2.70	6631	1.65	2549	4.35	7451	1
60	**9.78 9342**		**9.89 6532**		**9.89 2810**		**10.10 7190**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

127° 52°

38° TABLE XII.—*Continued.* 141°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.78 9342**	2.70	**9.89 6532**	1.65	**9.89 2810**	4.33	**10.10 7190**	**60**
1	9504	2.68	6433	1.63	3070	4.35	6930	59
2	9665	2.70	6335	1.65	3331	4.33	6669	58
3	9827	2.68	6236	1.65	3591	4.33	6409	57
4	.78 9988	2.68	6137	1.65	3851	4.33	6149	56
5	**9.79 0149**	2.68	**9.89 6038**	1.65	**9.89 4111**	4.35	**10.10 5889**	**55**
6	0310	2.68	5939	1.65	4372	4.33	5628	54
7	0471	2.68	5840	1.65	4632	4.33	5368	53
8	0632	2.68	5741	1.67	4892	4.33	5108	52
9	0793	2.68	5641	1.65	5152	4.33	4848	51
10	**9.79 0954**	2.68	**9.89 5542**	1.65	**9.89 5412**	4.33	**10.10 4588**	**50**
11	1115	2.67	5443	1.67	5672	4.33	4328	49
12	1275	2.68	5343	1.65	5932	4.33	4068	48
13	1436	2.67	5244	1.65	6192	4.33	3808	47
14	1596	2.68	5145	1.67	6452	4.33	3548	46
15	**9.79 1757**	2.67	**9.89 5045**	1.67	**9.89 6712**	4.32	**10.10 3288**	**45**
16	1917	2.67	4945	1.65	6971	4.33	3029	44
17	2077	2.67	4846	1.67	7231	4.33	2769	43
18	2237	2.67	4746	1.67	7491	4.33	2509	42
19	2397	2.67	4646	1.67	7751	4.32	2249	41
20	**9.79 2557**	2.65	**9.89 4546**	1.67	**9.89 8010**	4.33	**10.10 1990**	**40**
21	2716	2.67	4446	1.67	8270	4.33	1730	39
22	2876	2.65	4346	1.67	8530	4.32	1470	38
23	3035	2.67	4246	1.67	8789	4.33	1211	37
24	3195	2.65	4146	1.67	9049	4.32	0951	36
25	**9.79 3354**	2.67	**9.89 4046**	1.67	**9.89 9308**	4.33	**10.10 0692**	**35**
26	3514	2.65	3946	1.67	9568	4.32	0432	34
27	3673	2.65	3846	1.68	.89 9827	4.33	.10 0173	33
28	3832	2.65	3745	1.67	.90 0087	4.32	.09 9913	32
29	3991	2.65	3645	1.68	0346	4.32	9654	31
30	**9.79 4150**	2.63	**9.89 3544**	1.67	**9.90 0605**	4.32	**10.09 9395**	**30**
31	4308	2.65	3444	1.68	0864	4.33	9136	29
32	4467	2.65	3343	1.67	1124	4.32	8876	28
33	4626	2.63	3243	1.68	1383	4.32	8617	27
34	4784	2.63	3142	1.68	1642	4.32	8358	26
35	**9.79 4942**	2.65	**9.89 3041**	1.68	**9.90 1901**	4.32	**10.09 8099**	**25**
36	5101	2.63	2940	1.68	2160	4.33	7840	24
37	5259	2.63	2839	1.67	2420	4.32	7580	23
38	5417	2.63	2739	1.68	2679	4.32	7321	22
39	5575	2.63	2638	1.70	2938	4.32	7062	21
40	**9.79 5733**	2.63	**9.89 2536**	1.68	**9.90 3197**	4.32	**10.09 6803**	**20**
41	5891	2.63	2435	1.68	3456	4.30	6544	19
42	6049	2.62	2334	1.68	3714	4.32	6286	18
43	6206	2.63	2233	1.68	3973	4.32	6027	17
44	6364	2.62	2132	1.70	4232	4.32	5768	16
45	**9.79 6521**	2.63	**9.89 2030**	1.68	**9.90 4491**	4.32	**10.09 5509**	**15**
46	6679	2.62	1929	1.70	4750	4.30	5250	14
47	6836	2.62	1827	1.68	5008	4.32	4992	13
48	6993	2.62	1726	1.70	5267	4.32	4733	12
49	7150	2.62	1624	1.68	5526	4.32	4474	11
50	**9.79 7307**	2.62	**9.89 1523**	1.70	**9.90 5785**	4.30	**10.09 4215**	**10**
51	7464	2.62	1421	1.70	6043	4.32	3957	9
52	7621	2.60	1319	1.70	6302	4.30	3698	8
53	7777	2.62	1217	1.70	6560	4.32	3440	7
54	7934	2.62	1115	1.70	6819	4.30	3181	6
55	**9.79 8091**	2.60	**9.89 1013**	1.70	**9.90 7077**	4.32	**10.09 2923**	**5**
56	8247	2.60	0911	1.70	7336	4.30	2664	4
57	8403	2.62	0809	1.70	7594	4.32	2406	3
58	8560	2.60	0707	1.70	7853	4.30	2147	2
59	8716	2.60	0605	1.70	8111	4.30	1889	1
60	**9.79 8872**		**9.89 0503**		**9.90 8369**		**10.09 1631**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

128° 51°

39° TABLE XII.—*Continued.* 140°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.79 8872	2.60	9.89 0503	1.72	9.90 8369	4.32	10.09 1631	60
1	9028	2.60	0400	1.70	8628	4.30	1372	59
2	9184	2.58	0298	1.72	8886	4.30	1114	58
3	9339	2.60	0195	1.70	9144	4.30	0856	57
4	9495	2.60	.89 0093	1.72	9402	4.30	0598	56
5	9.79 9651	2.58	9.88 9990	1.70	9.90 9660	4.30	10.09 0340	55
6	9806	2.60	9888	1.72	.90 9918	4.32	.09 0082	54
7	.79 9962	2.58	9785	1.72	.91 0177	4.30	.08 9823	53
8	.80 0117	2.58	9682	1.72	0435	4.30	9565	52
9	0272	2.58	9579	1.70	0693	4.30	9307	51
10	9.80 0427	2.58	9.88 9477	1.72	9.91 0951	4.30	10.08 9049	50
11	0582	2.58	9374	1.72	1209	4.30	8791	49
12	0737	2.58	9271	1.72	1467	4.30	8533	48
13	0892	2.58	9168	1.73	1725	4.28	8275	47
14	1047	2.57	9064	1.72	1982	4.30	8018	46
15	9.80 1201	2.58	9.88 8961	1.72	9.91 2240	4.30	10.08 7760	45
16	1356	2.58	8858	1.72	2498	4.30	7502	44
17	1511	2.57	8755	1.73	2756	4.30	7244	43
18	1665	2.57	8651	1.72	3014	4.28	6986	42
19	1819	2.57	8548	1.73	3271	4.30	6729	41
20	9.80 1973	2.58	9.88 8444	1.72	9.91 3529	4.30	10.08 6471	40
21	2128	2.57	8341	1.73	3787	4.28	6213	39
22	2282	2.57	8237	1.72	4044	4.30	5956	38
23	2436	2.55	8134	1.73	4302	4.30	5698	37
24	2589	2.57	8030	1.73	4560	4.28	5440	36
25	9.80 2743	2.57	9.88 7926	1.73	9.91 4817	4.30	10.08 5183	35
26	2897	2.55	7822	1.73	5075	4.28	4925	34
27	3050	2.57	7718	1.73	5332	4.30	4668	33
28	3204	2.55	7614	1.73	5590	4.28	4410	32
29	3357	2.57	7510	1.73	5847	4.28	4153	31
30	9.80 3511	2.55	9.88 7406	1.73	9.91 6104	4.30	10.08 3896	30
31	3664	2.55	7302	1.73	6362	4.28	3638	29
32	3817	2.55	7198	1.75	6619	4.30	3381	28
33	3970	2.55	7093	1.73	6877	4.28	3123	27
34	4123	2.55	6989	1.73	7134	4.28	2866	26
35	9.80 4276	2.55	9.88 6885	1.75	9.91 7391	4.28	10.08 2609	25
36	4428	2.55	6780	1.73	7648	4.30	2352	24
37	4581	2.55	6676	1.75	7906	4.28	2094	23
38	4734	2.53	6571	1.75	8163	4.28	1837	22
39	4886	2.55	6466	1.73	8420	4.28	1580	21
40	9.80 5039	2.53	9.88 6362	1.75	9.91 8677	4.28	10.08 1323	20
41	5191	2.53	6257	1.75	8934	4.28	1066	19
42	5343	2.53	6152	1.75	9191	4.28	0809	18
43	5495	2.53	6047	1.75	9448	4.28	0552	17
44	5647	2.53	5942	1.75	9705	4.28	0295	16
45	9.80 5799	2.53	9.88 5837	1.75	9.91 9962	4.28	10.08 0038	15
46	5951	2.53	5732	1.75	.92 0219	4.28	.07 9781	14
47	6103	2.52	5627	1.75	0476	4.28	9524	13
48	6254	2.53	5522	1.77	0733	4.28	9267	12
49	6406	2.52	5416	1.75	0990	4.28	9010	11
50	9.80 6557	2.53	9.88 5311	1.77	9.92 1247	4.27	10.07 8753	10
51	6709	2.52	5205	1.75	1503	4.28	8497	9
52	6860	2.52	5100	1.77	1760	4.28	8240	8
53	7011	2.53	4994	1.75	2017	4.28	7983	7
54	7163	2.52	4889	1.77	2274	4.27	7726	6
55	9.80 7314	2.52	9.88 4783	1.77	9.92 2530	4.28	10.07 7470	5
56	7465	2.50	4677	1.75	2787	4.28	7213	4
57	7615	2.52	4572	1.77	3044	4.27	6956	3
58	7766	2.52	4466	1.77	3300	4.28	6700	2
59	7917	2.50	4360	1.77	3557	4.28	6443	1
60	9.80 8067		9.88 4254		9.92 3814		10.07 6186	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

129° 50°

40° Table XII.—*Continued.* 139°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.80 8067**	2.52	**9.88 4254**	1.77	**9.92 3814**	4.27	**10.07 6136**	60
1	8218	2.50	4148	1.77	4070	4.28	5930	59
2	8368	2.52	4042	1.77	4327	4.27	5673	58
3	8519	2.50	3936	1.78	4583	4.28	5417	57
4	8669	2.50	3829	1.77	4840	4.27	5160	56
5	**9.80 8819**	2.50	**9.88 3723**	1.77	**9.92 5096**	4.27	**10.07 4904**	**55**
6	8969	2.50	3617	1.78	5352	4.28	4648	54
7	9119	2.50	3510	1.77	5609	4.27	4391	53
8	9269	2.50	3404	1.78	5865	4.28	4135	52
9	9419	2.50	3297	1.77	6122	4.27	3878	51
10	**9.80 9569**	2.48	**9.88 3191**	1.78	**9.92 6378**	4.27	**10.07 3622**	**50**
11	9718	2.50	3084	1.78	6634	4.27	3366	49
12	.80 9868	2.48	2977	1.77	6890	4.28	3110	48
13	.81 0017	2.50	2871	1.78	7147	4.27	2853	47
14	0167	2.48	2764	1.78	7403	4.27	2597	46
15	**9.81 0316**	2.48	**9.88 2657**	1.78	**9.92 7659**	4.27	**10.07 2341**	**45**
16	0465	2.48	2550	1.78	7915	4.27	2085	44
17	0614	2.48	2443	1.78	8171	4.27	1829	43
18	0763	2.48	2336	1.78	8427	4.28	1573	42
19	0912	2.48	2229	1.80	8684	4.27	1316	41
20	**9.81 1061**	2.48	**9.88 2121**	1.78	**9.92 8940**	4.27	**10.07 1060**	**40**
21	1210	2.47	2014	1.78	9196	4.27	0804	39
22	1358	2.48	1907	1.80	9452	4.27	0548	38
23	1507	2.47	1799	1.78	9708	4.27	0292	37
24	1655	2.48	1692	1.80	.92 9964	4.27	.07 0036	36
25	**9.81 1804**	2.47	**9.88 1584**	1.78	**9.93 0220**	4.25	**10.06 9780**	**35**
26	1952	2.47	1477	1.80	0475	4.27	9525	34
27	2100	2.47	1369	1.80	0731	4.27	9269	33
28	2248	2.47	1261	1.80	0987	4.27	9013	32
29	2396	2.47	1153	1.78	1243	4.27	8757	31
30	**9.81 2544**	2.47	**9.88 1046**	1.80	**9.93 1499**	4.27	**10.06 8501**	**30**
31	2692	2.47	0938	1.80	1755	4.25	8245	29
32	2840	2.47	0830	1.80	2010	4.27	7990	28
33	2988	2.45	0722	1.82	2266	4.27	7734	27
34	3135	2.47	0613	1.80	2522	4.27	7478	26
35	**9.81 3283**	2.45	**9.88 0505**	1.80	**9.93 2778**	4.25	**10.06 7222**	**25**
36	3430	2.47	0397	1.80	3033	4.27	6967	24
37	3578	2.45	0289	1.82	3289	4.27	6711	23
38	3725	2.45	0180	1.80	3545	4.25	6455	22
39	3872	2.45	.88 0072	1.82	3800	4.27	6200	21
40	**9.81 4019**	2.45	**9.87 9963**	1.80	**9.93 4056**	4.25	**10.06 5944**	**20**
41	4166	2.45	9855	1.82	4311	4.27	5689	19
42	4313	2.45	9746	1.82	4567	4.25	5433	18
43	4460	2.45	9637	1.80	4822	4.27	5178	17
44	4607	2.43	9529	1.82	5078	4.25	4922	16
45	**9.81 4753**	2.45	**9.87 9420**	1.82	**9.93 5333**	4.27	**10.06 4667**	**15**
46	4900	2.43	9311	1.82	5589	4.25	4411	14
47	5046	2.45	9202	1.82	5844	4.27	4156	13
48	5193	2.43	9093	1.82	6100	4.25	3900	12
49	5339	2.43	8984	1.82	6355	4.27	3645	11
50	**9.81 5485**	2.45	**9.87 8875**	1.82	**9.93 6611**	4.25	**10.06 3389**	**10**
51	5632	2.43	8766	1.83	6866	4.25	3134	9
52	5778	2.43	8656	1.82	7121	4.27	2879	8
53	5924	2.42	8547	1.82	7377	4.25	2623	7
54	6069	2.43	8438	1.83	7632	4.25	2368	6
55	**9.81 6215**	2.43	**9.87 8328**	1.82	**9.93 7887**	4.25	**10.06 2113**	**5**
56	6361	2.43	8219	1.83	8142	4.27	1858	4
57	6507	2.42	8109	1.83	8398	4.25	1602	3
58	6652	2.43	7999	1.82	8653	4.25	1347	2
59	6798	2.42	7890	1.83	8908	4.25	1092	1
60	**9.81 6943**		**9.87 7780**		**9.93 9163**		**10.06 0837**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

130° 49°

41° **Table XII.**—*Continued.* 138°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.81 6943**	2.42	**9.87 7780**	1.83	**9.93 9163**	4.25	**10.06 0837**	**60**
1	7088	2.42	7670	1.83	9418	4.25	0582	59
2	7233	2.43	7560	1.83	9673	4.25	0327	58
3	7379	2.42	7450	1.83	.93 9928	4.25	.06 0072	57
4	7524	2.40	7340	1.83	.94 0183	4.27	.05 9817	56
5	**9.81 7668**	2.42	**9.87 7230**	1.83	**9.94 0439**	4.25	**10.05 9561**	**55**
6	7813	2.42	7120	1.83	0694	4.25	9306	54
7	7958	2.42	7010	1.85	0949	4.25	9051	53
8	8103	2.40	6899	1.83	1204	4.25	8796	52
9	8247	2.42	6789	1.85	1459	4.23	8541	51
10	**9.81 8392**	2.40	**9.87 6678**	1.83	**9.94 1713**	4.25	**10.05 8287**	**50**
11	8536	2.42	6568	1.85	1968	4.25	8032	49
12	8681	2.40	6457	1.83	2223	4.25	7777	48
13	8825	2.40	6347	1.85	2478	4.25	7522	47
14	8969	2.40	6236	1.85	2733	4.25	7267	46
15	**9.81 9113**	2.40	**9.87 6125**	1.85	**9.94 2988**	4.25	**10.05 7012**	**45**
16	9257	2.40	6014	1.83	3243	4.25	6757	44
17	9401	2.40	5904	1.85	3498	4.23	6502	43
18	9545	2.40	5793	1.85	3752	4.25	6248	42
19	9689	2.38	5682	1.85	4007	4.25	5993	41
20	**9.81 9832**	2.40	**9.87 5571**	1.87	**9.94 4262**	4.25	**10.05 5738**	**40**
21	.81 9976	2.40	5459	1.85	4517	4.23	5483	39
22	.82 0120	2.38	5348	1.85	4771	4.25	5229	38
23	0263	2.38	5237	1.85	5026	4.25	4974	37
24	0406	2.40	5126	1.87	5281	4.23	4719	36
25	**9.82 0550**	2.38	**9.87 5014**	1.85	**9.94 5535**	4.25	**10.05 4465**	**35**
26	0693	2.38	4903	1.87	5790	4.25	4210	34
27	0836	2.38	4791	1.85	6045	4.23	3955	33
28	0979	2.38	4680	1.87	6299	4.25	3701	32
29	1122	2.38	4568	1.87	6554	4.23	3446	31
30	**9.82 1265**	2.37	**9.87 4456**	1.87	**9.94 6808**	4.25	**10.05 3192**	**30**
31	1407	2.38	4344	1.87	7063	4.25	2937	29
32	1550	2.38	4232	1.85	7318	4.23	2682	28
33	1693	2.37	4121	1.87	7572	4.25	2428	27
34	1835	2.37	4009	1.88	7827	4.23	2173	26
35	**9.82 1977**	2.38	**9.87 3896**	1.87	**9.94 8081**	4.23	**10.05 1919**	**25**
36	2120	2.37	3784	1.87	8335	4.25	1665	24
37	2262	2.37	3672	1.87	8590	4.23	1410	23
38	2404	2.37	3560	1.87	8844	4.25	1156	22
39	2546	2.37	3448	1.88	9099	4.23	0901	21
40	**9.82 2688**	2.37	**9.87 3335**	1.87	**9.94 9353**	4.25	**10.05 0647**	**20**
41	2830	2.37	3223	1.88	9608	4.23	0392	19
42	2972	2.37	3110	1.87	.94 9862	4.23	.05 0138	18
43	3114	2.35	2998	1.88	.95 0116	4.25	.04 9884	17
44	3255	2.37	2885	1.88	0371	4.23	9629	16
45	**9.82 3397**	2.37	**9.87 2772**	1.88	**9.95 0625**	4.23	**10.04 9375**	**15**
46	3539	2.35	2659	1.87	0879	4.23	9121	14
47	3680	2.35	2547	1.88	1133	4.25	8867	13
48	3821	2.37	2434	1.88	1388	4.23	8612	12
49	3963	2.35	2321	1.88	1642	4.23	8358	11
50	**9.82 4104**	2.35	**9.87 2208**	1.88	**9.95 1896**	4.23	**10.04 8104**	**10**
51	4245	2.35	2095	1.90	2150	4.25	7850	9
52	4386	2.35	1981	1.88	2405	4.23	7595	8
53	4527	2.35	1868	1.88	2659	4.23	7341	7
54	4668	2.33	1755	1.90	2913	4.23	7087	6
55	**9.82 4808**	2.35	**9.87 1641**	1.88	**9.95 3167**	4.23	**10.04 6833**	**5**
56	4949	2.35	1528	1.90	3421	4.23	6579	4
57	5090	2.33	1414	1.88	3675	4.23	6325	3
58	5230	2.35	1301	1.90	3929	4.23	6071	2
59	5371	2.33	1187	1.90	4183	4.23	5817	1
60	**9.82 5511**		**9.87 1073**		**9.95 4437**		**10.04 5563**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

131° 48°

42° TABLE XII.—*Continued.* 137°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.82 5511**	2.33	**9.87 1073**	1.88	**9.95 4437**	4.23	**10.04 5563**	**60**
1	5651	2.33	0960	1.90	4691	4.25	5309	59
2	5791	2.33	0846	1.90	4946	4.23	5054	58
3	5931	2.33	0732	1.90	5200	4.23	4800	57
4	6071	2.33	0618	1.90	5454	4.23	4546	56
5	**9.82 6211**	2.33	**9.87 0504**	1.90	**9.95 5708**	4.22	**10.04 4292**	**55**
6	6351	2.33	0390	1.90	5961	4.23	4039	54
7	6491	2.33	0276	1.92	6215	4.23	3785	53
8	6631	2.32	0161	1.90	6469	4.23	3531	52
9	6770	2.33	.87 0047	1.90	6723	4.23	3277	51
10	**9.82 6910**	2.32	**9.86 9933**	1.92	**9.95 6977**	4.23	**10.04 3023**	**50**
11	7049	2.33	9818	1.90	7231	4.23	2769	49
12	7189	2.32	9704	1.92	7485	4.23	2515	48
13	7328	2.32	9589	1.92	7739	4.23	2261	47
14	7467	2.32	9474	1.90	7993	4.23	2007	46
15	**9.82 7606**	2.32	**9.86 9360**	1.92	**9.95 8247**	4.22	**10.04 1753**	**45**
16	7745	2.32	9245	1.92	8500	4.23	1500	44
17	7884	2.32	9130	1.92	8754	4.23	1246	43
18	8023	2.32	9015	1.92	9008	4.23	0992	42
19	8162	2.32	8900	1.92	9262	4.23	0738	41
20	**9.82 8301**	2.30	**9.86 8785**	1.92	**9.95 9516**	4.22	**10.04 0484**	**40**
21	8439	2.32	8670	1.92	.95 9769	4.23	.04 0231	39
22	8578	2.30	8555	1.92	.96 0023	4.23	.03 9977	38
23	8716	2.32	8440	1.93	0277	4.22	9723	37
24	8855	2.30	8324	1.92	0530	4.23	9470	36
25	**9.82 8993**	2.30	**9.86 8209**	1.93	**9.96 0784**	4.23	**10.03 9216**	**35**
26	9131	2.30	8093	1.92	1038	4.23	8962	34
27	9269	2.30	7978	1.93	1292	4.22	8708	33
28	9407	2.30	7862	1.92	1545	4.23	8455	32
29	9545	2.30	7747	1.93	1799	4.22	8201	31
30	**9.82 9683**	2.30	**9.86 7631**	1.93	**9.96 2052**	4.23	**10.03 7948**	**30**
31	9821	2.30	7515	1.93	2306	4.23	7694	29
32	.82 9959	2.30	7399	1.93	2560	4.22	7440	28
33	.83 0097	2.28	7283	1.93	2813	4.23	7187	27
34	0234	2.30	7167	1.93	3067	4.22	6933	26
35	**9.83 0372**	2.28	**9.86 7051**	1.93	**9.96 3320**	4.23	**10.03 6680**	**25**
36	0509	2.28	6935	1.93	3574	4.23	6426	24
37	0646	2.30	6819	1.93	3828	4.22	6172	23
38	0784	2.28	6703	1.95	4081	4.23	5919	22
39	0921	2.28	6586	1.93	4335	4.22	5665	21
40	**9.83 1058**	2.28	**9.86 6470**	1.95	**9.96 4588**	4.23	**10.03 5412**	**20**
41	1195	2.28	6353	1.93	4842	4.22	5158	19
42	1332	2.28	6237	1.95	5095	4.23	4905	18
43	1469	2.28	6120	1.93	5349	4.22	4651	17
44	1606	2.27	6004	1.95	5602	4.22	4398	16
45	**9.83 1742**	2.28	**9.86 5887**	1.95	**9.96 5855**	4.23	**10.03 4145**	**15**
46	1879	2.27	5770	1.95	6109	4.22	3891	14
47	2015	2.28	5653	1.95	6362	4.23	3638	13
48	2152	2.27	5536	1.95	6616	4.22	3384	12
49	2288	2.28	5419	1.95	6869	4.23	3131	11
50	**9.83 2425**	2.27	**9.86 5302**	1.95	**9.96 7123**	4.22	**10.03 2877**	**10**
51	2561	2.27	5185	1.95	7376	4.22	2624	9
52	2697	2.27	5068	1.97	7629	4.23	2371	8
53	2833	2.27	4950	1.95	7883	4.22	2117	7
54	2969	2.27	4833	1.95	8136	4.22	1864	6
55	**9.83 3105**	2.27	**9.86 4716**	1.97	**9.96 8389**	4.23	**10.03 1611**	**5**
56	3241	2.27	4598	1.95	8643	4.22	1357	4
57	3377	2.25	4481	1.97	8896	4.22	1104	3
58	3512	2.27	4363	1.97	9149	4.23	0851	2
59	3648	2.25	4245	1.97	9403	4.22	0597	1
60	**9.83 3783**		**9.86 4127**		**9.96 9656**		**10.03 0344**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

132° 47°

43° TABLE XII.—*Continued.* 136°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.83 3783	2.27	9.86 4127	1.95	9.96 9656	4.22	10.03 0344	60
1	3919	2.25	4010	1.97	.96 9909	4.22	.03 0091	59
2	4054	2.25	3892	1.97	.97 0162	4.23	.02 9838	58
3	4189	2.27	3774	1.97	0416	4.22	9584	57
4	4325	2.25	3656	1.97	0669	4.22	9331	56
5	9.83 4460	2.25	9.86 3538	1.98	9.97 0922	4.22	10.02 9078	55
6	4595	2.25	3419	1.97	1175	4.23	8825	54
7	4730	2.25	3301	1.97	1429	4.22	8571	53
8	4865	2.23	3183	1.98	1682	4.22	8318	52
9	4999	2.25	3064	1.97	1935	4.22	8065	51
10	9.83 5134	2.25	9.86 2946	1.98	9.97 2188	4.22	10.02 7812	50
11	5269	2.23	2827	1.97	2441	4.23	7559	49
12	5403	2.25	2709	1.98	2695	4.22	7305	48
13	5538	2.23	2590	1.98	2948	4.22	7052	47
14	5672	2.25	2471	1.97	3201	4.22	6799	46
15	9.83 5807	2.23	9.86 2353	1.98	9.97 3454	4.22	10.02 6546	45
16	5941	2.23	2234	1.98	3707	4.22	6293	44
17	6075	2.23	2115	1.98	3960	4.22	6040	43
18	6209	2.23	1996	1.98	4213	4.22	5787	42
19	6343	2.23	1877	1.98	4466	4.23	5534	41
20	9.83 6477	2.23	9.86 1758	2.00	9.97 4720	4.22	10.02 5280	40
21	6611	2.23	1638	1.98	4973	4.22	5027	39
22	6745	2.22	1519	1.98	5226	4.22	4774	38
23	6878	2.23	1400	2.00	5479	4.22	4521	37
24	7012	2.23	1280	1.98	5732	4.22	4268	36
25	9.83 7146	2.22	9.86 1161	2.00	9.97 5985	4.22	10.02 4015	35
26	7279	2.22	1041	1.98	6238	4.22	3762	34
27	7412	2.23	0922	2.00	6491	4.22	3509	33
28	7546	2.22	0802	2.00	6744	4.22	3256	32
29	7679	2.22	0682	2.00	6997	4.22	3003	31
30	9.83 7812	2.22	9.86 0562	2.00	9.97 7250	4.22	10.02 2750	30
31	7945	2.22	0442	2.00	7503	4.22	2497	29
32	8078	2.22	0322	2.00	7756	4.22	2244	28
33	8211	2.22	0202	2.00	8009	4.22	1991	27
34	8344	2.22	.86 0082	2.00	8262	4.22	1738	26
35	9.83 8477	2.22	9.85 9962	2.00	9.97 8515	4.22	10.02 1485	25
36	8610	2.20	9842	2.02	8768	4.22	1232	24
37	8742	2.22	9721	2.00	9021	4.22	0979	23
38	8875	2.20	9601	2.02	9274	4.22	0726	22
39	9007	2.22	9480	2.00	9527	4.22	0473	21
40	9.83 9140	2.20	9.85 9360	2.02	9.97 9780	4.22	10.02 0220	20
41	9272	2.20	9239	2.00	.98 0033	4.22	.01 9967	19
42	9404	2.20	9119	2.02	0286	4.20	9714	18
43	9536	2.20	8998	2.02	0538	4.22	9462	17
44	9668	2.20	8877	2.02	0791	4.22	9209	16
45	9.83 9800	2.20	9.85 8756	2.02	9.98 1044	4.22	10.01 8956	15
46	.83 9932	2.20	8635	2.02	1297	4.22	8703	14
47	.84 0064	2.20	8514	2.02	1550	4.22	8450	13
48	0196	2.20	8393	2.02	1803	4.22	8197	12
49	0328	2.18	8272	2.02	2056	4.22	7944	11
50	9.84 0459	2.20	9.85 8151	2.03	9.98 2309	4.22	10.01 7691	10
51	0591	2.18	8029	2.02	2562	4.20	7438	9
52	0722	2.20	7908	2.03	2814	4.22	7186	8
53	0854	2.18	7786	2.02	3067	4.22	6933	7
54	0985	2.18	7665	2.03	3320	4.22	6680	6
55	9.84 1116	2.18	9.85 7543	2.02	9.98 3573	4.22	10.01 6427	5
56	1247	2.18	7422	2.03	3826	4.22	6174	4
57	1378	2.18	7300	2.03	4079	4.22	5921	3
58	1509	2.18	7178	2.03	4332	4.20	5668	2
59	1640	2.18	7056	2.03	4584	4.22	5416	1
60	9.84 1771		9.85 6934		9.98 4837		10.01 5163	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

133° 46°

44° **Table XII.**—*Concluded.* 135°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.84 1771**	2.18	**9.85 6934**	2.03	**9.98 4337**	4.22	**10.01 5163**	**60**
1	1902	2.18	6812	2.03	5090	4.22	4910	59
2	2033	2.17	6690	2.03	5343	4.22	4657	58
3	2163	2.18	6568	2.03	5596	4.20	4404	57
4	2294	2.17	6446	2.05	5848	4.22	4152	56
5	**9.84 2424**	2.18	**9.85 6323**	2.03	**9.98 6101**	4.22	**10.01 3899**	**55**
6	2555	2.17	6201	2.05	6354	4.22	3646	54
7	2685	2.17	6078	2.03	6607	4.22	3393	53
8	2815	2.18	5956	2.05	6860	4.20	3140	52
9	2946	2.17	5833	2.03	7112	4.22	2888	51
10	**9.84 3076**	2.17	**9.85 5711**	2.05	**9.98 7365**	4.22	**10.01 2635**	**50**
11	3206	2.17	5588	2.05	7618	4.22	2382	49
12	3336	2.17	5465	2.05	7871	4.20	2129	48
13	3466	2.15	5342	2.05	8123	4.22	1877	47
14	3595	2.17	5219	2.05	8376	4.22	1624	46
15	**9.84 3725**	2.17	**9.85 5096**	2.05	**9.98 8629**	4.22	**10.01 1371**	**45**
16	3855	2.15	4973	2.05	8882	4.20	1118	44
17	3984	2.17	4850	2.05	9134	4.22	0866	43
18	4114	2.15	4727	2.07	9387	4.22	0613	42
19	4243	2.15	4603	2.05	9640	4.22	0360	41
20	**9.84 4372**	2.17	**9.85 4480**	2.07	**9.98 9893**	4.20	**10.01 0107**	**40**
21	4502	2.15	4356	2.05	.99 0145	4.22	.00 9855	39
22	4631	2.15	4233	2.07	0398	4.22	9602	38
23	4760	2.15	4109	2.05	0651	4.20	9349	37
24	4889	2.15	3986	2.07	0903	4.22	9097	36
25	**9.84 5018**	2.15	**9.85 3862**	2.07	**9.99 1156**	4.22	**10.00 8844**	**35**
26	5127	2.15	3738	2.07	1409	4.22	8591	34
27	5276	2.15	3614	2.07	1662	4.20	8338	33
28	5405	2.13	3490	2.07	1914	4.22	8086	32
29	5533	2.15	3366	2.07	2167	4.22	7833	31
30	**9.84 5662**	2.13	**9.85 3242**	2.07	**9.99 2420**	4.20	**10.00 7580**	**30**
31	5790	2.15	3118	2.07	2672	4.22	7328	29
32	5919	2.13	2994	2.08	2925	4.22	7075	28
33	6047	2.13	2869	2.07	3178	4.22	6822	27
34	6175	2.15	2745	2.08	3431	4.20	6569	26
35	**9.84 6304**	2.13	**9.85 2620**	2.07	**9.99 3683**	4.22	**10.00 6317**	**25**
36	6432	2.13	2496	2.08	3936	4.22	6064	24
37	6560	2.13	2371	2.07	4189	4.20	5811	23
38	6688	2.13	2247	2.08	4441	4.22	5559	22
39	6816	2.13	2122	2.08	4694	4.22	5306	21
40	**9.84 6944**	2.12	**9.85 1997**	2.08	**9.99 4947**	4.20	**10.00 5053**	**20**
41	7071	2.13	1872	2.08	5199	4.22	4801	19
42	7199	2.13	1747	2.08	5452	4.22	4548	18
43	7327	2.12	1622	2.08	5705	4.20	4295	17
44	7454	2.13	1497	2.08	5957	4.22	4043	16
45	**9.84 7582**	2.12	**9.85 1372**	2.10	**9.99 6210**	4.22	**10.00 3790**	**15**
46	7709	2.12	1246	2.08	6463	4.20	3537	14
47	7836	2.13	1121	2.08	6715	4.22	3285	13
48	7964	2.12	0996	2.10	6968	4.22	3032	12
49	8091	2.12	0870	2.08	7221	4.20	2779	11
50	**9.84 8218**	2.12	**9.85 0745**	2.10	**9.99 7473**	4.22	**10.00 2527**	**10**
51	8345	2.12	0619	2.10	7726	4.22	2274	9
52	8472	2.12	0493	2.08	7979	4.20	2021	8
53	8599	2.12	0368	2.10	8231	4.22	1769	7
54	8726	2.10	0242	2.10	8484	4.22	1516	6
55	**9.84 8852**	2.12	**9.85 0116**	2.10	**9.99 8737**	4.20	**10.00 1263**	**5**
56	8979	2.12	84 9990	2.10	8989	4.22	1011	4
57	9106	2.10	9864	2.10	9242	4.22	0758	3
58	9232	2.12	9738	2.12	9495	4.20	0505	2
59	9359	2.10	9611	2.10	9.99 9747	4.22	0253	1
60	**9.84 9485**		**9.84 9485**		10.00 0000		10.00 0000	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

134° 45°

TABLE XIII

	0°				0°				0°		
′	SINE	COSINE	′	′	SINE	COSINE	′	′	SINE	COSINE	′
0	.00000	1	60	21	.00611	.99998	39	41	.01193	.99993	19
1	.00029	1	59	22	.00640	.99998	38	42	.01222	.99993	18
2	.00058	1	58	23	.00669	.99998	37	43	.01251	.99992	17
3	.00087	1	57	24	.00698	.99998	36	44	.01280	.99992	16
4	.00116	1	56	25	.00727	.99997	35	45	.01309	.99991	15
5	.00145	1	55	26	.00756	.99997	34	46	.01338	.99991	14
6	.00175	1	54	27	.00785	.99997	33	47	.01367	.99991	13
7	.00204	1	53	28	.00814	.99997	32	48	.01396	.99990	12
8	.00233	1	52	29	.00844	.99996	31	49	.01425	.99990	11
9	.00262	1	51	30	.00873	.99996	30	50	.01454	.99989	10
10	.00291	1	50	31	.00902	.99996	29	51	.01483	.99989	9
11	.00320	.99999	49	32	.00931	.99996	28	52	.01513	.99989	8
12	.00349	.99999	48	33	.00960	.99995	27	53	.01542	.99988	7
13	.00378	.99999	47	34	.00989	.99995	26	54	.01571	.99988	6
14	.00407	.99999	46	35	.01018	.99995	25	55	.01600	.99987	5
15	.00436	.99999	45	36	.01047	.99995	24	56	.01629	.99987	4
16	.00465	.99999	44	37	.01076	.99994	23	57	.01658	.99986	3
17	.00495	.99999	43	38	.01105	.99994	22	58	.01687	.99986	2
18	.00524	.99999	42	39	.01134	.99994	21	59	.01716	.99985	1
19	.00553	.99998	41	40	.01164	.99993	20	60	.01745	.99985	0
20	.00582	.99998	40								
′	COSINE	SINE	′	′	COSINE	SINE	′	′	COSINE	SINE	′
	89°				89°				89°		

TABLE XIII.—*Continued*

′	1°		2°		3°		4°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	88°		87°		86°		85°		

TABLE XIII.—*Continued*

′	5°		6°		7°		8°		′
	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	0
′	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	′
	84°		83°		82°		81°		

TABLE XIII.—*Continued*

′	9°		10°		11°		12°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.15643	.98769	.17365	.98481	.19081	.98163	.20791	.97815	60
1	.15672	.98764	.17393	.98476	.19109	.98157	.20820	.97809	59
2	.15701	.98760	.17422	.98471	.19138	.98152	.20848	.97803	58
3	.15730	.98755	.17451	.98466	.19167	.98146	.20877	.97797	57
4	.15758	.98751	.17479	.98461	.19195	.98140	.20905	.97791	56
5	.15787	.98746	.17508	.98455	.19224	.98135	.20933	.97784	55
6	.15816	.98741	.17537	.98450	.19252	.98129	.20962	.97778	54
7	.15845	.98737	.17565	.98445	.19281	.98124	.20990	.97772	53
8	.15873	.98732	.17594	.98440	.19309	.98118	.21019	.97766	52
9	.15902	.98728	.17623	.98435	.19338	.98112	.21047	.97760	51
10	.15931	.98723	.17651	.98430	.19366	.98107	.21076	.97754	50
11	.15959	.98718	.17680	.98425	.19395	.98101	.21104	.97748	49
12	.15988	.98714	.17708	.98420	.19423	.98096	.21132	.97742	48
13	.16017	.98709	.17737	.98414	.19452	.98090	.21161	.97735	47
14	.16046	.98704	.17766	.98409	.19481	.98084	.21189	.97729	46
15	.16074	.98700	.17794	.98404	.19509	.98079	.21218	.97723	45
16	.16103	.98695	.17823	.98399	.19538	.98073	.21246	.97717	44
17	.16132	.98690	.17852	.98394	.19566	.98067	.21275	.97711	43
18	.16160	.98689	.17880	.98389	.19595	.98061	.21303	.97705	42
19	.16189	.98681	.17909	.98383	.19623	.98056	.21331	.97698	41
20	.16218	.98676	.17937	.98378	.19652	.98050	.21360	.97692	40
21	.16246	.98671	.17966	.98373	.19680	.98044	.21388	.97686	39
22	.16275	.98667	.17995	.98368	.19709	.98039	.21417	.97680	38
23	.16304	.98662	.18023	.98362	.19737	.98033	.21445	.97673	37
24	.16333	.98657	.18052	.98357	.19766	.98027	.21474	.97667	36
25	.16361	.98652	.18081	.98352	.19794	.98021	.21502	.97661	35
26	.16390	.98648	.18109	.98347	.19823	.98016	.21530	.97655	34
27	.16419	.98643	.18138	.98341	.19851	.98010	.21559	.97648	33
28	.16447	.98638	.18166	.98336	.19880	.98004	.21587	.97642	32
29	.16476	.98633	.18195	.98331	.19908	.97987	.21616	.97636	31
30	.16505	.98629	.18224	.98325	.19937	.97992	.21644	.97630	30
31	.16533	.98624	.18252	.98320	.19965	.97987	.21672	.97623	29
32	.16562	.98619	.18281	.98315	.19994	.97981	.21701	.97617	28
33	.16591	.98614	.18309	.98310	.20022	.97975	.21729	.97611	27
34	.16620	.98609	.18338	.98304	.20051	.97969	.21758	.97604	26
35	.16648	.98604	.18367	.98299	.20079	.97963	.21786	.97598	25
36	.16677	.98600	.18395	.98294	.20108	.97958	.21814	.97592	24
37	.16706	.98595	.18424	.98288	.20136	.97952	.21843	.97585	23
38	.16734	.98590	.18452	.98283	.20165	.97946	.21871	.97579	22
39	.16763	.98585	.18481	.98277	.20193	.97940	.21899	.97573	21
40	.16792	.98580	.18509	.98272	.20222	.97934	.21928	.97566	20
41	.16820	.98575	.18538	.98267	.20250	.97928	.21956	.97560	19
42	.16849	.98570	.18567	.98261	.20279	.97922	.21985	.97553	18
43	.16878	.98565	.18595	.98256	.20307	.97916	.22013	.97547	17
44	.16906	.98561	.18624	.98250	.20336	.97910	.22041	.97541	16
45	.16935	.98556	.18652	.98245	.20364	.97905	.22070	.97534	15
46	.16964	.98551	.18681	.98240	.20393	.97899	.22098	.97528	14
47	.16992	.98546	.18710	.98234	.20421	.97893	.22126	.97521	13
48	.17021	.98541	.18738	.98229	.20450	.97887	.22155	.97515	12
49	.17050	.98536	.18767	.98223	.20478	.97881	.22183	.97508	11
50	.17078	.98531	.18795	.98218	.20507	.97875	.22212	.97502	10
51	.17107	.98526	.18824	.98212	.20535	.97869	.22240	.97496	9
52	.17136	.98521	.18852	.98207	.20563	.97863	.22268	.97489	8
53	.17164	.98516	.18881	.98201	.20592	.97857	.22297	.97483	7
54	.17193	.98511	.18910	.98196	.20620	.97851	.22325	.97476	6
55	.17222	.98506	.18938	.98190	.20649	.97845	.22353	.97470	5
56	.17250	.98501	.18967	.98185	.20677	.97839	.22382	.97463	4
57	.17279	.98496	.18995	.98179	.20706	.97833	.22410	.97457	3
58	.17308	.98491	.19024	.98174	.20734	.97827	.22438	.97450	2
59	.17336	.98486	.19052	.98168	.20763	.97821	.22467	.97444	1
60	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	80°		79°		78°		77°		

TABLE XIII.—*Continued*

′	13°		14°		15°		16°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.22495	.97437	.24192	.97030	.25882	.96593	.27564	.96126	60
1	.22523	.97430	.24220	.97023	.25910	.96585	.27592	.96118	59
2	.22552	.97424	.24249	.97015	.25938	.96578	.27620	.96110	58
3	.22580	.97417	.24277	.97008	.25966	.96570	.27648	.96102	57
4	.22608	.97411	.24305	.97001	.25994	.96562	.27676	.96094	56
5	.22637	.97404	.24333	.96994	.26022	.96555	.27704	.96086	55
6	.22665	.97398	.24362	.96987	.26050	.96547	.27731	.96078	54
7	.22693	.97391	.24390	.96980	.26079	.96540	.27759	.96070	53
8	.22722	.97384	.24418	.96973	.26107	.96532	.27787	.96062	52
9	.22750	.97378	.24446	.96966	.26135	.96524	.27815	.96054	51
10	.22778	.97371	.24474	.96959	.26163	.96517	.27843	.96046	50
11	.22807	.97365	.24503	.96952	.26191	.96509	.27871	.96037	49
12	.22835	.97358	.24531	.96945	.26219	.96502	.27899	.96029	48
13	.22863	.97351	.24559	.96937	.26247	.96494	.27927	.96021	47
14	.22892	.97345	.24587	.96930	.26275	.96486	.27955	.96013	46
15	.22920	.97338	.24615	.96923	.26303	.96479	.27983	.96005	45
16	.22948	.97331	.24644	.96916	.26331	.96471	.28011	.95997	44
17	.22977	.97325	.24672	.96909	.26359	.96463	.28039	.95989	43
18	.23005	.97318	.24700	.96902	.26387	.96456	.28067	.95981	42
19	.23033	.97311	.24728	.96894	.26415	.96448	.28095	.95972	41
20	.23062	.97304	.24756	.96887	.26443	.96440	.28123	.95964	40
21	.23090	.97298	.24784	.96880	.26471	.96433	.28150	.95956	39
22	.23118	.97291	.24813	.96873	.26500	.96425	.28178	.95948	38
23	.23146	.97284	.24841	.96866	.26528	.96417	.28206	.95940	37
24	.23175	.97278	.24869	.96858	.26556	.96410	.28234	.95931	36
25	.23203	.97271	.24897	.96851	.26584	.96402	.28262	.95923	35
26	.23231	.97264	.24925	.96844	.26612	.96394	.28290	.95915	34
27	.23260	.97257	.24954	.96837	.26640	.96386	.28318	.95907	33
28	.23288	.97251	.24982	.96829	.26668	.96379	.28346	.95898	32
29	.23316	.97244	.25010	.96822	.26696	.96371	.28374	.95890	31
30	.23345	.97237	.25038	.96815	.26724	.96363	.28402	.95882	30
31	.23373	.97230	.25066	.96807	.26752	.96355	.28429	.95874	29
32	.23401	.97223	.25094	.96800	.26780	.96347	.28457	.95865	28
33	.23429	.97217	.25122	.96793	.26808	.96340	.28485	.95857	27
34	.23458	.97210	.25151	.96786	.26836	.96332	.28513	.95849	26
35	.23486	.97203	.25179	.96778	.26864	.96324	.28541	.95841	25
36	.23514	.97196	.25207	.96771	.26892	.96316	.28569	.95832	24
37	.23542	.97189	.25235	.96764	.26920	.96308	.28597	.95824	23
38	.23571	.97182	.25263	.96756	.26948	.96301	.28625	.95816	22
39	.23599	.97176	.25291	.96749	.26976	.96293	.28652	.95807	21
40	.23627	.97169	.25320	.96742	.27004	.96285	.28680	.95799	20
41	.23656	.97162	.25348	.96734	.27032	.96277	.28708	.95791	19
42	.23684	.97155	.25376	.96727	.27060	.96269	.28736	.95782	18
43	.23712	.97148	.25404	.96719	.27088	.96261	.28764	.95774	17
44	.23740	.97141	.25432	.96712	.27116	.96253	.28792	.95766	16
45	.23769	.97134	.25460	.96705	.27144	.96246	.28820	.95757	15
46	.23797	.97127	.25488	.96697	.27172	.96238	.28847	.95749	14
47	.23825	.97120	.25516	.96690	.27200	.96230	.28875	.95740	13
48	.23853	.97113	.25545	.96682	.27228	.96222	.28903	.95732	12
49	.23882	.97106	.25573	.96675	.27256	.96214	.28931	.95724	11
50	.23910	.97100	.25601	.96667	.27284	.96206	.28959	.95715	10
51	.23938	.97093	.25629	.96660	.27312	.96198	.28987	.95707	9
52	.23966	.97086	.25657	.96653	.27340	.96190	.29015	.95698	8
53	.23995	.97079	.25685	.96645	.27368	.96182	.29042	.95690	7
54	.24023	.97072	.25713	.96638	.27396	.96174	.29070	.95681	6
55	.24051	.97065	.25741	.96630	.27424	.96166	.29098	.95673	5
56	.24079	.97058	.25769	.96623	.27452	.96158	.29126	.95664	4
57	.24108	.97051	.25798	.96615	.27480	.96150	.29154	.95656	3
58	.24136	.97044	.25826	.96608	.27508	.96142	.29182	.95647	2
59	.24164	.97037	.25854	.96600	.27536	.96134	.29209	.95639	1
60	.24192	.97030	.25882	.96593	.27564	.96126	.29237	.95630	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	76°		75°		74°		73°		

TABLE XIII.—*Continued*

′	17°		18°		19°		20°		′
	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	
0	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	60
1	.29265	.95622	.30929	.95097	.32584	.94542	.34229	.93959	59
2	.29293	.95613	.30957	.95088	.32612	.94533	.34257	.93949	58
3	.29321	.95605	.30985	.95079	.32639	.94523	.34284	.93939	57
4	.29348	.95596	.31012	.95070	.32667	.94514	.34311	.93929	56
5	.29376	.95588	.31040	.95061	.32694	.94504	.34339	.93919	55
6	.29404	.95579	.31068	.95052	.32722	.94495	.34366	.93909	54
7	.29432	.95571	.31095	.95043	.32749	.94485	.34393	.93899	53
8	.29460	.95562	.31123	.95033	.32777	.94476	.34421	.93889	52
9	.29487	.95554	.31151	.95024	.32804	.94466	.34448	.93879	51
10	.29515	.95545	.31178	.95015	.32832	.94457	.34475	.93869	50
11	.29543	.95536	.31206	.95006	.32859	.94447	.34503	.93859	49
12	.29571	.95528	.31233	.94997	.32887	.94438	.34530	.93849	48
13	.29599	.95519	.31261	.94988	.32914	.94428	.34557	.93839	47
14	.29626	.95511	.31289	.94979	.32942	.94418	.34584	.93829	46
15	.29654	.95502	.31316	.94970	.32969	.94409	.34612	.93819	45
16	.29682	.95493	.31344	.94961	.32997	.94399	.34639	.93809	44
17	.29710	.95485	.31372	.94952	.33024	.94390	.34666	.93799	43
18	.29737	.95476	.31399	.94943	.33051	.94380	.34694	.93789	42
19	.29765	.95467	.31427	.94933	.33079	.94370	.34721	.93779	41
20	.29793	.95459	.31454	.94924	.33106	.94361	.34748	.93769	40
21	.29821	.95450	.31482	.94915	.33134	.94351	.34775	.93759	39
22	.29849	.95441	.31510	.94906	.33161	.94342	.34803	.93748	38
23	.29876	.95433	.31537	.94897	.33189	.94332	.34830	.93738	37
24	.29904	.95424	.31565	.94888	.33216	.94322	.34857	.93728	36
25	.29932	.95415	.31593	.94878	.33244	.94313	.34884	.93718	35
26	.29960	.95407	.31620	.94869	.33271	.94303	.34912	.93708	34
27	.29987	.95398	.31648	.94860	.33298	.94293	.34939	.93698	33
28	.30015	.95389	.31675	.94851	.33326	.94284	.34966	.93688	32
29	.30043	.95380	.31703	.94842	.33353	.94274	.34993	.93677	31
30	.30071	.95372	.31730	.94832	.33381	.94264	.35021	.93667	30
31	.30098	.95363	.31758	.94823	.33408	.94254	.35048	.93657	29
32	.30126	.95354	.31786	.94814	.33436	.94245	.35075	.93647	28
33	.30154	.95345	.31813	.94805	.33463	.94235	.35102	.93637	27
34	.30182	.95337	.31841	.94795	.33490	.94225	.35130	.93626	26
35	.30209	.95328	.31868	.94786	.33518	.94215	.35157	.93616	25
36	.30237	.95319	.31896	.94777	.33545	.94206	.35184	.93606	24
37	.30265	.95310	.31923	.94768	.33573	.94196	.35211	.93596	23
38	.30292	.95301	.31951	.94758	.33600	.94186	.35239	.93585	22
39	.30320	.95293	.31979	.94749	.33627	.94176	.35266	.93575	21
40	.30348	.95284	.32006	.94740	.33655	.94167	.35293	.93565	20
41	.30376	.95275	.32034	.94730	.33682	.94157	.35320	.93555	19
42	.30403	.95266	.32061	.94721	.33710	.94147	.35347	.93544	18
43	.30431	.95257	.32089	.94712	.33737	.94137	.35375	.93534	17
44	.30459	.95248	.32116	.94702	.33764	.94127	.35402	.93524	16
45	.30486	.95240	.32144	.94693	.33792	.94118	.35429	.93514	15
46	.30514	.95231	.32171	.94684	.33819	.94108	.35456	.93503	14
47	.30542	.95222	.32199	.94674	.33846	.94098	.35484	.93493	13
48	.30570	.95213	.32227	.94665	.33874	.94088	.35511	.93483	12
49	.30597	.95204	.32254	.94656	.33901	.94078	.35538	.93472	11
50	.30625	.95195	.32282	.94646	.33929	.94068	.35565	.93462	10
51	.30653	.95186	.32309	.94637	.33956	.94058	.35592	.93452	9
52	.30680	.95177	.32337	.94627	.33983	.94049	.35619	.93441	8
53	.30708	.95168	.32364	.94618	.34011	.94039	.35647	.93431	7
54	.30736	.95159	.32392	.94609	.34038	.94029	.35674	.93420	6
55	.30763	.95150	.32419	.94599	.34065	.94019	.35701	.93410	5
56	.30791	.95142	.32447	.94590	.34093	.94009	.35728	.93400	4
57	.30819	.95133	.32474	.94580	.34120	.93999	.35755	.93389	3
58	.30846	.95124	.32502	.94571	.34147	.93989	.35782	.93379	2
59	.30874	.95115	.32529	.94561	.34175	.93979	.35810	.93368	1
60	.30902	.95106	.32557	.94552	.34202	.93969	.35837	.93358	0
′	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	′
	72°		71°		70°		69°		

TABLE XIII.—*Continued*

′	21°		22°		23°		24°		′
	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	
0	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.36596	.93063	.38215	.92410	.39822	.91729	.41416	.91020	32
29	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.36894	.92945	.38510	.92287	.40115	.91601	.41707	.90887	21
40	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.37272	.92794	.38886	.92130	.40488	.91437	.42077	.90717	7
54	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	′
	68°		67°		66°		65°		

TABLE XIII.—*Continued*

′	25°		26°		27°		28°		′
	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	0
′	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	′
	64°		63°		62°		61°		

TABLE XIII.—*Continued*

′	29°		30°		31°		32°		′
	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	
0	.48481	.87462	.50000	.86603	.51504	.85717	.52992	.84805	60
1	.48506	.87448	.50025	.86588	.51529	.85702	.53017	.84789	59
2	.48532	.87434	.50050	.86573	.51554	.85687	.53041	.84774	58
3	.48557	.87420	.50076	.86559	.51579	.85672	.53066	.84759	57
4	.48583	.87406	.50101	.86544	.51604	.85657	.53091	.84743	56
5	.48608	.87391	.50126	.86530	.51628	.85642	.53115	.84728	55
6	.48634	.87377	.50151	.86515	.51653	.85627	.53140	.84712	54
7	.48659	.87363	.50176	.86501	.51678	.85612	.53164	.84697	53
8	.48684	.87349	.50201	.86486	.51703	.85597	.53189	.84681	52
9	.48710	.87335	.50227	.86471	.51728	.85582	.53214	.84666	51
10	.48735	.87321	.50252	.86457	.51753	.85567	.53238	.84650	50
11	.48761	.87306	.50277	.86442	.51778	.85551	.53263	.84635	49
12	.48786	.87292	.50302	.86427	.51803	.85536	.53288	.84619	48
13	.48811	.87278	.50327	.86413	.51828	.85521	.53312	.84604	47
14	.48837	.87264	.50352	.86398	.51852	.85506	.53337	.84588	46
15	.48862	.87250	.50377	.86384	.51877	.85491	.53361	.84573	45
16	.48888	.87235	.50403	.86369	.51902	.85476	.53386	.84557	44
17	.48913	.87221	.50428	.86354	.51927	.85461	.53411	.84542	43
18	.48938	.87207	.50453	.86340	.51952	.85446	.53435	.84526	42
19	.48964	.87193	.50478	.86325	.51977	.85431	.53460	.84511	41
20	.48989	.87178	.50503	.86310	.52002	.85416	.53484	.84495	40
21	.49014	.87164	.50528	.86295	.52026	.85401	.53509	.84480	39
22	.49040	.87150	.50553	.86281	.52051	.85385	.53534	.84464	38
23	.49065	.87136	.50578	.86266	.52076	.85370	.53558	.84448	37
24	.49090	.87121	.50603	.86251	.52101	.85355	.53583	.84433	36
25	.49116	.87107	.50628	.86237	.52126	.85340	.53607	.84417	35
26	.49141	.87093	.50654	.86222	.52151	.85325	.53632	.84402	34
27	.49166	.87079	.50679	.86207	.52175	.85310	.53656	.84386	33
28	.49192	.87064	.50704	.86192	.52200	.85294	.53681	.84370	32
29	.49217	.87050	.50729	.86178	.52225	.85279	.53705	.84355	31
30	.49242	.87036	.50754	.86163	.52250	.85264	.53730	.84339	30
31	.49268	.87021	.50779	.86148	.52275	.85249	.53754	.84324	29
32	.49293	.87007	.50804	.86133	.52299	.85234	.53779	.84308	28
33	.49318	.86993	.50829	.86119	.52324	.85218	.53804	.84292	27
34	.49344	.86978	.50854	.86104	.52349	.85203	.53828	.84277	26
35	.49369	.86964	.50879	.86089	.52374	.85188	.53853	.84261	25
36	.49394	.86949	.50904	.86074	.52399	.85173	.53877	.84245	24
37	.49419	.86935	.50929	.86059	.52423	.85157	.53902	.84230	23
38	.49445	.86921	.50954	.86045	.52448	.85142	.53926	.84214	22
39	.49470	.86906	.50979	.86030	.52473	.85127	.53951	.84198	21
40	.49495	.86892	.51004	.86015	.52498	.85112	.53975	.84182	20
41	.49521	.86878	.51029	.86000	.52522	.85096	.54000	.84167	19
42	.49546	.86863	.51054	.85985	.52547	.85081	.54024	.84151	18
43	.49571	.86849	.51079	.85970	.52572	.85066	.54049	.84135	17
44	.49596	.86834	.51104	.85956	.52597	.85051	.54073	.84120	16
45	.49622	.86820	.51129	.85941	.52621	.85035	.54097	.84104	15
46	.49647	.86805	.51154	.85926	.52646	.85020	.54122	.84088	14
47	.49672	.86791	.51179	.85911	.52671	.85005	.54146	.84072	13
48	.49697	.86777	.51204	.85896	.52696	.84989	.54171	.84057	12
49	.49723	.86762	.51229	.85881	.52720	.84974	.54195	.84041	11
50	.49748	.86748	.51254	.85866	.52745	.84959	.54220	.84025	10
51	.49773	.86733	.51279	.85851	.52770	.84943	.54244	.84009	9
52	.49798	.86719	.51304	.85836	.52794	.84928	.54269	.83994	8
53	.49824	.86704	.51329	.85821	.52819	.84913	.54293	.83978	7
54	.49849	.86690	.51354	.85806	.52844	.84897	.54317	.83962	6
55	.49874	.86675	.51379	.85792	.52869	.84882	.54342	.83946	5
56	.49899	.86661	.51404	.85777	.52893	.84866	.54366	.83930	4
57	.49924	.86646	.51429	.85762	.52918	.84851	.54391	.83915	3
58	.49950	.86632	.51454	.85747	.52943	.84836	.54415	.83899	2
59	.49975	.86617	.51479	.85732	.52967	.84820	.54440	.83883	1
60	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	0
′	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	′
	60°		59°		58°		57°		

TABLE XIII.—*Continued*

′	33° Sine	33° Cosine	34° Sine	34° Cosine	35° Sine	35° Cosine	36° Sine	36° Cosine	′
0	.54464	.83867	.55919	.82904	.57358	.81915	.58779	.80902	60
1	.54488	.83851	.55943	.82887	.57381	.81899	.58802	.80885	59
2	.54513	.83835	.55968	.82871	.57405	.81882	.58826	.80867	58
3	.54537	.83819	.55992	.82855	.57429	.81865	.58849	.80850	57
4	.54561	.83804	.56016	.82839	.57453	.81848	.58873	.80833	56
5	.54586	.83788	.56040	.82822	.57477	.81832	.58896	.80816	55
6	.54610	.83772	.56064	.82806	.57501	.81815	.58920	.80799	54
7	.54635	.83756	.56088	.82790	.57524	.81798	.58943	.80782	53
8	.54659	.83740	.56112	.82773	.57548	.81782	.58967	.80765	52
9	.54683	.83724	.56136	.82757	.57572	.81765	.58990	.80748	51
10	.54708	.83708	.56160	.82741	.57596	.81748	.59014	.80730	50
11	.54732	83692	.56184	.82724	.57619	.81731	.59037	.80713	49
12	.54756	.83676	.56208	.82708	.57643	.81714	.59061	.80696	48
13	.54781	.83660	.56232	.82692	.57667	.81698	.59084	.80679	47
14	.54805	.83645	.56256	.82675	.57691	.81681	.59108	.80662	46
15	.54829	.83629	.56280	.82659	.57715	.81664	.59131	.80644	45
16	.54854	.83613	.56305	.82643	.57738	.81647	.59154	.80627	44
17	.54878	.83597	.56329	.82626	.57762	.81631	.59178	.80610	43
18	.54902	.83581	.56353	.82610	.57786	.81614	.59201	.80593	42
19	.54927	.83565	.56377	.82593	.57810	.81597	.59225	.80576	41
20	.54951	.83549	.56401	.82577	.57833	.81580	.59248	.80558	40
21	.54975	.83533	.56425	.82561	.57857	.81563	.59272	.80541	39
22	.54999	.83517	.56449	.82544	.57881	.81546	.59295	.80524	38
23	.55024	.83501	.56473	.82528	.57904	.81530	.59318	.80507	37
24	.55048	.83485	.56497	.82511	.57928	.81513	.59342	.80489	36
25	.55072	.83469	.56521	.82495	.57952	.81496	.59365	.80472	35
26	.55097	.83453	.56545	.82478	.57976	.81479	.59389	.80455	34
27	.55121	.83437	.56569	.82462	.57999	.81462	.59412	.80438	33
28	.55145	.83421	.56593	.82446	.58023	.81445	.59436	.80420	32
29	.55169	.83405	.56617	.82429	.58047	.81428	.59459	.80403	31
30	.55194	.83389	.56641	.82413	.58070	.81412	.59482	.80386	30
31	.55218	.83373	.56665	.82396	.58094	.81395	.59506	.80368	29
32	.55242	.83356	.56689	.82340	.58118	.81378	.59529	.80351	28
33	.55266	.83340	.56713	.82363	.58141	.81361	.59552	.80334	27
34	.55291	.83324	.56736	.82347	.58165	.81344	.59576	.80316	26
35	.55315	.83308	.56760	.82330	.58189	.81327	.59599	.80299	25
36	.55339	.83292	.56784	.82314	.58212	.81310	.59622	.80282	24
37	.55363	.83276	.56808	.82297	.58236	.81293	.59646	.80264	23
38	.55388	.83260	.56832	.82281	.58260	.81276	.59669	.80247	22
39	.55412	.83244	.56856	.82264	.58283	.81259	.59693	.80230	21
40	.55436	.83228	.56880	.82248	.58307	.81242	.59716	.80212	20
41	.55460	.83212	.56904	.82231	.58330	.81225	.59739	.80195	19
42	.55484	.83195	.56928	.82214	.58354	.81208	.59763	.80178	18
43	.55509	.83179	.56952	.82198	.58378	.81191	.59786	.80160	17
44	.55533	.83163	.56976	.82181	.58401	.81174	.59809	.80143	16
45	.55557	.83147	.57000	.82165	.58425	.81157	.59832	.80125	15
46	.55581	.83131	.57024	.82148	.58449	.81140	.59856	.80108	14
47	.55605	.83115	.57047	.82132	.58472	.81123	.59879	.80091	13
48	.55630	.83098	.57071	.82115	.58496	.81106	.59902	.80073	12
49	.55654	.83082	.57095	.82098	.58519	.81089	.59926	.80056	11
50	.55678	.83066	.57119	.82082	.58543	.81072	.59949	.80038	10
51	.55702	.83050	.57143	.82065	.58567	.81055	.59972	.80021	9
52	.55726	.83034	.57167	.82048	.58590	.81038	.59995	.80003	8
53	.55750	.83017	.57191	.82032	.58614	.81021	.60019	.79986	7
54	.55775	.83001	.57215	.82015	.58637	.81004	.60042	.79968	6
55	.55799	.82985	.57238	.81999	.58661	.80987	.60065	.79951	5
56	.55823	.82969	.57262	.81982	.58684	.80970	.60089	.79934	4
57	.55847	.82953	.57286	.81965	.58708	.80953	.60112	.79916	3
58	.55871	.82936	.57310	.81949	.58731	.80936	.60135	.79899	2
59	.55895	.82920	.57334	.81932	.58755	.80919	.60158	.79881	1
60	.55919	.82904	.57358	.81915	.58779	.80902	.60182	.79864	0
′	Cosine 56°	Sine 56°	Cosine 55°	Sine 55°	Cosine 54°	Sine 54°	Cosine 53°	Sine 53°	′

TABLE XIII.—*Continued*

′	37°		38°		39°		40°		′
	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	
0	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	60
1	.60205	.79846	.61589	.78783	.62955	.77696	.64301	.76586	59
2	.60228	.79829	.61612	.78765	.62977	.77678	.64323	.76567	58
3	.60251	.79811	.61635	.78747	.63000	.77660	.64346	.76548	57
4	.60274	.79793	.61658	.78729	.63022	.77641	.64368	.76530	56
5	.60298	.79776	.61681	.78711	.63045	.77623	.64390	.76511	55
6	.60321	.79758	.61704	.78694	.63068	.77605	.64412	.76492	54
7	.60344	.79741	.61726	.78676	.63090	.77586	.64435	.76473	53
8	.60367	.79723	.61749	.78658	.63113	.77568	.64457	.76455	52
9	.60390	.79706	.61772	.78640	.63135	.77550	.64479	.76436	51
10	.60414	.79688	.61795	.78622	.63158	.77531	.64501	.76417	50
11	.60437	.79671	.61818	.78604	.63180	.77513	.64524	.76398	49
12	.60460	.79653	.61841	.78586	.63203	.77494	.64546	.76380	48
13	.60483	.79635	.61864	.78568	.63225	.77476	.64568	.76361	47
14	.60506	.79618	.61887	.78550	.63248	.77458	.64590	.76342	46
15	.60529	.79600	.61909	.78532	.63271	.77439	.64612	.76323	45
16	.60553	.79583	.61932	.78514	.63293	.77421	.64635	.76304	44
17	.60576	.79565	.61955	.78496	.63316	.77402	.64657	.76286	43
18	.60599	.79547	.61978	.78478	.63338	.77384	.64679	.76267	42
19	.60622	.79530	.62001	.78460	.63361	.77366	.64701	.76248	41
20	.60645	.79512	.62024	.78442	.63383	.77347	.64723	.76229	40
21	.60668	.79494	.62046	.78424	.63406	.77329	.64746	.76210	39
22	.60691	.79477	.62069	.78405	.63428	.77310	.64768	.76192	38
23	.60714	.79459	.62092	.78387	.63451	.77292	.64790	.76173	37
24	.60738	.79441	.62115	.78369	.63473	.77273	.64812	.76154	36
25	.60761	.79424	.62138	.78351	.63496	.77255	.64834	.76135	35
26	.60784	.79406	.62160	.78333	.63518	.77236	.64856	.76116	34
27	.60807	.79388	.62183	.78315	.63540	.77218	.64878	.76097	33
28	.60830	.79371	.62206	.78297	.63563	.77199	.64901	.76078	32
29	.60853	.79353	.62229	.78279	.63585	.77181	.64923	.76059	31
30	.60876	.79335	.62251	.78261	.63608	.77162	.64945	.76041	30
31	.60899	.79318	.62274	.78243	.63630	.77144	.64967	.76022	29
32	.60922	.79300	.62297	.78225	.63653	.77125	.64989	.76003	28
33	.60945	.79282	.62320	.78206	.63675	.77107	.65011	.75984	27
34	.60968	.79264	.62342	.78188	.63698	.77088	.65033	.75965	26
35	.60991	.79247	.62365	.78170	.63720	.77070	.65055	.75946	25
36	.61015	.79229	.62388	.78152	.63742	.77051	.65077	.75927	24
37	.61038	.79211	.62411	.78134	.63765	.77033	.65100	.75908	23
38	.61061	.79193	.62433	.78116	.63787	.77014	.65122	.75889	22
39	.61084	.79176	.62456	.78098	.63810	.76996	.65144	.75870	21
40	.61107	.79158	.62479	.78079	.63832	.76977	.65166	.75851	20
41	.61130	.79140	.62502	.78061	.63854	.76959	.65188	.75832	19
42	.61153	.79122	.62524	.78043	.63877	.76940	.65210	.75813	18
43	.61176	.79105	.62547	.78025	.63899	.76921	.65232	.75794	17
44	.61199	.79087	.62570	.78007	.63922	.76903	.65254	.75775	16
45	.61222	.79069	.62592	.77988	.63944	.76884	.65276	.75756	15
46	.61245	.79051	.62615	.77970	.63966	.76866	.65298	.75738	14
47	.61268	.79033	.62638	.77952	.63989	.76847	.65320	.75719	13
48	.61291	.79016	.62660	.77934	.64011	.76828	.65342	.75700	12
49	.61314	.78998	.62683	.77916	.64033	.76810	.65364	.75680	11
50	.61337	.78980	.62706	.77897	.64056	.76791	.65386	.75661	10
51	.61360	.78962	.62728	.77879	.64078	.76772	.65408	.75642	9
52	.61383	.78944	.62751	.77861	.64100	.76754	.65430	.75623	8
53	.61406	.78926	.62774	.77843	.64123	.76735	.65452	.75604	7
54	.61429	.78908	.62796	.77824	.64145	.76717	.65474	.75585	6
55	.61451	.78891	.62819	.77806	.64167	.76698	.65496	.75566	5
56	.61474	.78873	.62842	.77788	.64190	.76679	.65518	.75547	4
57	.61497	.78855	.62864	.77769	.64212	.76661	.65540	.75528	3
58	.61520	.78837	.62887	.77751	.64234	.76642	.65562	.75509	2
59	.61543	.78819	.62909	.77733	.64256	.76623	.65584	.75490	1
60	.61566	.78801	.62932	.77715	.64279	.76604	.65606	.75471	0
′	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	′
	52°		51°		50°		49°		

TABLE XIII.—*Concluded*

′	41° Sine	41° Cosine	42° Sine	42° Cosine	43° Sine	43° Cosine	44° Sine	44° Cosine	′
0	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.66306	.74857	.67602	.73688	.68878	.72497	.70132	.71284	28
33	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.66436	.74741	.67730	.73570	.69004	.72377	.70257	.71162	22
39	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.66870	.74353	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	Cosine 48°	Sine 48°	Cosine 47°	Sine 47°	Cosine 46°	Sine 46°	Cosine 45°	Sine 45°	′

TABLE XIV

′	0°		1°		2°		3°		′
	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	
0	.00000	Infinite.	.01746	57.2900	.03492	28.6363	.05241	19.0811	60
1	.00029	3437.750	.01775	56.3506	.03521	28.3994	.05270	18.9755	59
2	.00058	1718.870	.01804	55.4415	.03550	28.1604	.05299	18.8711	58
3	.00087	1145.920	.01833	54.5613	.03579	27.9372	.05328	18.7678	57
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	33
28	.00814	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	15
46	.01338	74.7292	.03084	32.4213	.04832	20.6932	.06584	15.1893	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	9
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	0
′	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN	TAN.	CO-TAN	TAN.	′
	89°		88°		87°		86°		

TABLE XIV.—*Continued*

′	4°		5°		6°		7°		′
	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	
0	.06993	14.3007	.08749	11.4301	.10510	9.51436	.12278	8.14435	60
1	.07022	14.2411	.08778	11.3919	.10540	9.48781	.12308	8.12481	59
2	.07051	14.1821	.08807	11.3540	.10569	9.46141	.12338	8.10536	58
3	.07080	14.1235	.08837	11.3163	.10599	9.43515	.12367	8.08600	57
4	.07110	14.0655	.08866	11.2789	.10628	9.40904	.12397	8.06674	56
5	.07139	14.0079	.08895	11.2417	.10657	9.38307	.12426	8.04756	55
6	.07168	13.9507	.08925	11.2048	.10687	9.35724	.12456	8.02848	54
7	.07197	13.8940	.08954	11.1681	.10716	9.33154	.12485	8.00948	53
8	.07227	13.8378	.08983	11.1316	.10746	9.30599	.12515	7.99058	52
9	.07256	13.7821	.09013	11.0954	.10775	9.28058	.12544	7.97176	51
10	.07285	13.7267	.09042	11.0594	.10805	9.25530	.12574	7.95302	50
11	.07314	13.6719	.09071	11.0237	.10834	9.23016	.12603	7.93438	49
12	.07344	13.6174	.09101	10.9882	.10863	9.20516	.12633	7.91582	48
13	.07373	13.5634	.09130	10.9529	.10893	9.18028	.12662	7.89734	47
14	.07402	13.5098	.09159	10.9178	.10922	9.15554	.12692	7.87895	46
15	.07431	13.4566	.09189	10.8829	.10952	9.13093	.12722	7.86064	45
16	.07461	13.4039	.09218	10.8483	.10981	9.10646	.12751	7.84242	44
17	.07490	13.3515	.09247	10.8139	.11011	9.08211	.12781	7.82428	43
18	.07519	13.2996	.09277	10.7797	.11040	9.05789	.12810	7.80622	42
19	.07548	13.2480	.09306	10.7457	.11070	9.03379	.12840	7.78825	41
20	.07578	13.1969	.09335	10.7119	.11099	9.00983	.12869	7.77035	40
21	.07607	13.1461	.09365	10.6783	.11128	8.98598	.12899	7.75254	39
22	.07636	13.0958	.09394	10.6450	.11158	8.96227	.12929	7.73480	38
23	.07665	13.0458	.09423	10.6118	.11187	8.93867	.12958	7.71715	37
24	.07695	12.9962	.09453	10.5789	.11217	8.91520	.12988	7.69957	36
25	.07724	12.9469	.09482	10.5462	.11246	8.89185	.13017	7.68208	35
26	.07753	12.8981	.09511	10.5136	.11276	8.86862	.13047	7.66466	34
27	.07782	12.8496	.09541	10.4813	.11305	8.84551	.13076	7.64732	33
28	.07812	12.8014	.09570	10.4491	.11335	8.82252	.13106	7.63005	32
29	.07841	12.7536	.09600	10.4172	.11364	8.79964	.13136	7.61287	31
30	.07870	12.7062	.09629	10.3854	.11394	8.77689	.13165	7.59575	30
31	.07899	12.6591	.09658	10.3538	.11423	8.75425	.13195	7.57872	29
32	.07929	12.6124	.09688	10.3224	.11452	8.73172	.13224	7.56176	28
33	.07958	12.5660	.09717	10.2913	.11482	8.70931	.13254	7.54487	27
34	.07987	12.5199	.09746	10.2602	.11511	8.68701	.13284	7.52806	26
35	.08017	12.4742	.09776	10.2294	.11541	8.66482	.13313	7.51132	25
36	.08046	12.4288	.09805	10.1988	.11570	8.64275	.13343	7.49465	24
37	.08075	12.3838	.09834	10.1683	.11600	8.62078	.13372	7.47806	23
38	.08104	12.3390	.09864	10.1381	.11629	8.59893	.13402	7.46154	22
39	.08134	12.2946	.09893	10.1080	.11659	8.57718	.13432	7.44509	21
40	.08163	12.2505	.09923	10.0780	.11688	8.55555	.13461	7.42871	20
41	.08192	12.2067	.09952	10.0483	.11718	8.53402	.13491	7.41240	19
42	.08221	12.1632	.09981	10.0187	.11747	8.51259	.13521	7.39616	18
43	.08251	12.1201	.10011	9.98931	.11777	8.49128	.13550	7.37999	17
44	.08280	12.0772	.10040	9.96007	.11806	8.47007	.13580	7.36389	16
45	.08309	12.0346	.10069	9.93101	.11836	8.44896	.13609	7.34786	15
46	.08339	11.9923	.10099	9.90211	.11865	8.42795	.13639	7.33190	14
47	.08368	11.9504	.10128	9.87338	.11895	8.40705	.13669	7.31600	13
48	.08397	11.9087	.10158	9.84482	.11924	8.38625	.13698	7.30018	12
49	.08427	11.8673	.10187	9.81641	.11954	8.36555	.13728	7.28442	11
50	.08456	11.8262	.10216	9.78817	.11983	8.34496	.13758	7.26873	10
51	.08485	11.7853	.10246	9.76009	.12013	8.32446	.13787	7.25310	9
52	.08514	11.7448	.10275	9.73217	.12042	8.30406	.13817	7.23754	8
53	.08544	11.7045	.10305	9.70441	.12072	8.28376	.13846	7.22204	7
54	.08573	11.6645	.10334	9.67680	.12101	8.26355	.13876	7.20661	6
55	.08602	11.6248	.10363	9.64935	.12131	8.24345	.13906	7.19125	5
56	.08632	11.5853	.10393	9.62205	.12160	8.22344	.13935	7.17594	4
57	.08661	11.5461	.10422	9.59490	.12190	8.20352	.13965	7.16071	3
58	.08690	11.5072	.10452	9.56791	.12219	8.18370	.13995	7.14553	2
59	.08720	11.4685	.10481	9.54106	.12249	8.16398	.14024	7.13042	1
60	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	0
′	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	′
	85°		84°		83°		82°		

TABLE XIV.—*Continued*

′	8°		9°		10°		11°		′
	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	
0	.14054	7.11537	.15838	6.31375	.17633	5.67128	.19438	5.14455	60
1	.14084	7.10038	.15868	6.30189	.17663	5.66165	.19468	5.13658	59
2	.14113	7.08546	.15898	6.29007	.17693	5.65205	.19498	5.12862	58
3	.14143	7.07059	.15928	6.27829	.17723	5.64248	.19529	5.12069	57
4	.14173	7.05579	.15958	6.26655	.17753	5.63295	.19559	5.11279	56
5	.14202	7.04105	.15988	6.25486	.17783	5.62344	.19589	5.10490	55
6	.14232	7.02637	.16017	6.24321	.17813	5.61397	.19619	5.09704	54
7	.14262	7.01174	.16047	6.23160	.17843	5.60452	.19649	5.08921	53
8	.14291	6.99718	.16077	6.22003	.17873	5.59511	.19680	5.08139	52
9	.14321	6.98268	.16107	6.20851	.17903	5.58573	.19710	5.07360	51
10	.14351	6.96823	.16137	6.19703	.17933	5.57638	.19740	5.06584	50
11	.14381	6.95385	.16167	6.18559	.17963	5.56706	.19770	5.05809	49
12	.14410	6.93952	.16196	6.17419	.17993	5.55777	.19801	5.05037	48
13	.14440	6.92525	.16226	6.16283	.18023	5.54851	.19831	5.04267	47
14	.14470	6.91104	.16256	6.15151	.18053	5.53927	.19861	5.03499	46
15	.14499	6.89688	.16286	6.14023	.18083	5.53007	.19891	5.02734	45
16	.14529	6.88278	.16316	6.12899	.18113	5.52090	.19921	5.01971	44
17	.14559	6.86874	.16346	6.11779	.18143	5.51176	.19952	5.01210	43
18	.14588	6.85475	.16376	6.10664	.18173	5.50264	.19982	5.00451	42
19	.14618	6.84082	.16405	6.09552	.18203	5.49356	.20012	4.99695	41
20	.14648	6.82694	.16435	6.08444	.18233	5.48451	.20042	4.98940	40
21	.14678	6.81312	.16465	6.07340	.18263	5.47548	.20073	4.98188	39
22	.14707	6.79936	.16495	6.06240	.18293	5.46648	.20103	4.97438	38
23	.14737	6.78564	.16525	6.05143	.18323	5.45751	.20133	4.96690	37
24	.14767	6.77199	.16555	6.04051	.18353	5.44857	.20164	4.95945	36
25	.14796	6.75838	.16585	6.02962	.18383	5.43966	.20194	4.95201	35
26	.14826	6.74483	.16615	6.01878	.18414	5.43077	.20224	4.94460	34
27	.14856	6.73133	.16645	6.00797	.18444	5.42192	.20254	4.93721	33
28	.14886	6.71789	.16674	5.99720	.18474	5.41309	.20285	4.92984	32
29	.14915	6.70450	.16704	5.98646	.18504	5.40429	.20315	4.92249	31
30	.14945	6.69116	.16734	5.97576	.18534	5.39552	.20345	4.91516	30
31	.14975	6.67787	.16764	5.96510	.18564	5.38677	.20376	4.90785	29
32	.15005	6.66463	.16794	5.95448	.18594	5.37805	.20406	4.90056	28
33	.15034	6.65144	.16824	5.94390	.18624	5.36936	.20436	4.89330	27
34	.15064	6.63831	.16854	5.93335	.18654	5.36070	.20466	4.83605	26
35	.15094	6.62523	.16884	5.92283	.18684	5.35206	.20497	4.87882	25
36	.15124	6.61219	.16914	5.91235	.18714	5.34345	.20527	4.87162	24
37	.15153	6.59921	.16944	5.90191	.18745	5.33487	.20557	4.86444	23
38	.15183	6.58627	.16974	5.89151	.18775	5.32631	.20588	4.85727	22
39	.15213	6.57339	.17004	5.88114	.18805	5.31778	.20618	4.85013	21
40	.15243	6.56055	.17033	5.87080	.18835	5.30928	.20648	4.84300	20
41	.15272	6.54777	.17063	5.86051	.18865	5.30080	.20679	4.83590	19
42	.15302	6.53503	.17093	5.85024	.18895	5.29235	.20709	4.82882	18
43	.15332	6.52234	.17123	5.84001	.18925	5.28393	.20739	4.82175	17
44	.15362	6.50970	.17153	5.82982	.18955	5.27553	.20770	4.81471	16
45	.15391	6.49710	.17183	5.81966	.18986	5.26715	.20800	4.80769	15
46	.15421	6.48456	.17213	5.80953	.19016	5.25880	.20830	4.80068	14
47	.15451	6.47206	.17243	5.79944	.19046	5.25048	.20861	4.79370	13
48	.15481	6.45961	.17273	5.78938	.19076	5.24218	.20891	4.78673	12
49	.15511	6.44720	.17303	5.77936	.19106	5.23391	.20921	4.77978	11
50	.15540	6.43484	.17333	5.76937	.19136	5.22566	.20952	4.77286	10
51	.15570	6.42253	.17363	5.75941	.19166	5.21744	.20982	4.76595	9
52	.15600	6.41026	.17393	5.74949	.19197	5.20925	.21013	4.75906	8
53	.15630	6.39804	.17423	5.73960	.19227	5.20107	.21043	4.75219	7
54	.15660	6.38587	.17453	5.72974	.19257	5.19293	.21073	4.74534	6
55	.15689	6.37374	.17483	5.71992	.19287	5.18480	.21104	4.73851	5
56	.15719	6.36165	.17513	5.71013	.19317	5.17671	.21134	4.73170	4
57	.15749	6.34961	.17543	5.70037	.19347	5.16863	.21164	4.72490	3
58	.15779	6.33761	.17573	5.69064	.19378	5.16058	.21195	4.71813	2
59	.15809	6.32566	.17603	5.68094	.19408	5.15256	.21225	4.71137	1
60	.15838	6.31375	.17633	5.67128	.19438	5.14455	.21256	4.70463	0
′	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	′
	81°		80°		79°		78°		

TABLE XIV.—*Continued*

′	12°		13°		14°		15°		′
	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	
0	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	60
1	.21286	4.69791	.23117	4.32573	.24964	4.00582	.26826	3.72771	59
2	.21316	4.69121	.23148	4.32001	.24995	4.00086	.26857	3.72338	58
3	.21347	4.68452	.23179	4.31430	.25026	3.99592	.26888	3.71907	57
4	.21377	4.67786	.23209	4.30860	.25056	3.99099	.26920	3.71476	56
5	.21408	4.67121	.23240	4.30291	.25087	3.98607	.26951	3.71046	55
6	.21438	4.66458	.23271	4.29724	.25118	3.98117	.26982	3.70616	54
7	.21469	4.65797	.23301	4.29159	.25149	3.97627	.27013	3.70188	53
8	.21499	4.65138	.23332	4.28595	.25180	3.97139	.27044	3.69761	52
9	.21529	4.64480	.23363	4.28032	.25211	3.96651	.27076	3.69335	51
10	.21560	4.63825	.23393	4.27471	.25242	3.96165	.27107	3.68909	50
11	.21590	4.63171	.23424	4.26911	.25273	3.95680	.27138	3.68485	49
12	.21621	4.62518	.23455	4.26352	.25304	3.95196	.27169	3.68061	48
13	.21651	4.61868	.23485	4.25795	.25335	3.94713	.27201	3.67638	47
14	.21682	4.61219	.23516	4.25239	.25366	3.94232	.27232	3.67217	46
15	.21712	4.60572	.23547	4.24685	.25397	3.93751	.27263	3.66796	45
16	.21743	4.59927	.23578	4.24132	.25428	3.93271	.27294	3.66376	44
17	.21773	4.59283	.23608	4.23580	.25459	3.92793	.27326	3.65957	43
18	.21804	4.58641	.23639	4.23030	.25490	3.92316	.27357	3.65538	42
19	.21834	4.58001	.23670	4.22481	.25521	3.91839	.27388	3.65121	41
20	.21864	4.57363	.23700	4.21933	.25552	3.91364	.27419	3.64705	40
21	.21895	4.56726	.23731	4.21387	.25583	3.90890	.27451	3.64289	39
22	.21925	4.56091	.23762	4.20842	.25614	3.90417	.27482	3.63874	38
23	.21956	4.55458	.23793	4.20298	.25645	3.89945	.27513	3.63461	37
24	.21986	4.54826	.23823	4.19756	.25676	3.89474	.27545	3.63048	36
25	.22017	4.54196	.23854	4.19215	.25707	3.89004	.27576	3.62636	35
26	.22047	4.53568	.23885	4.18675	.25738	3.88536	.27607	3.62224	34
27	.22078	4.52941	.23916	4.18137	.25769	3.88068	.27638	3.61814	33
28	.22108	4.52316	.23946	4.17600	.25800	3.87601	.27670	3.61405	32
29	.22139	4.51693	.23977	4.17064	.25831	3.87136	.27701	3.60996	31
30	.22169	4.51071	.24008	4.16530	.25862	3.86671	.27732	3.60588	30
31	.22200	4.50451	.24039	4.15997	.25893	3.86208	.27764	3.60181	29
32	.22231	4.49832	.24069	4.15465	.25924	3.85745	.27795	3.59775	28
33	.22261	4.49215	.24100	4.14934	.25955	3.85284	.27826	3.59370	27
34	.22292	4.48600	.24131	4.14405	.25986	3.84824	.27858	3.58966	26
35	.22322	4.47986	.24162	4.13877	.26017	3.84364	.27889	3.58562	25
36	.22353	4.47374	.24193	4.13350	.26048	3.83906	.27920	3.58160	24
37	.22383	4.46764	.24223	4.12825	.26079	3.83449	.27952	3.57758	23
38	.22414	4.46155	.24254	4.12301	.26110	3.82992	.27983	3.57357	22
39	.22444	4.45548	.24285	4.11778	.26141	3.82537	.28015	3.56957	21
40	.22475	4.44942	.24316	4.11256	.26172	3.82083	.28046	3.56557	20
41	.22505	4.44338	.24347	4.10736	.26203	3.81630	.28077	3.56159	19
42	.22536	4.43735	.24377	4.10216	.26235	3.81177	.28109	3.55761	18
43	.22567	4.43134	.24408	4.09699	.26266	3.80726	.28140	3.55364	17
44	.22597	4.42534	.24439	4.09182	.26297	3.80276	.28172	3.54968	16
45	.22628	4.41936	.24470	4.08666	.26328	3.79827	.28203	3.54573	15
46	.22658	4.41340	.24501	4.08152	.26359	3.79378	.28234	3.54179	14
47	.22689	4.40745	.24532	4.07639	.26390	3.78931	.28266	3.53785	13
48	.22719	4.40152	.24562	4.07127	.26421	3.78485	.28297	3.53393	12
49	.22750	4.39560	.24593	4.06616	.26452	3.78040	.28329	3.53001	11
50	.22781	4.38969	.24624	4.06107	.26483	3.77595	.28360	3.52609	10
51	.22811	4.38381	.24655	4.05599	.26515	3.77152	.28391	3.52219	9
52	.22842	4.37793	.24686	4.05092	.26546	3.76709	.28423	3.51829	8
53	.22872	4.37207	.24717	4.04586	.26577	3.76268	.28454	3.51441	7
54	.22903	4.36623	.24747	4.04081	.26608	3.75828	.28486	3.51053	6
55	.22934	4.36040	.24778	4.03578	.26639	3.75388	.28517	3.50666	5
56	.22964	4.35459	.24809	4.03075	.26670	3.74950	.28549	3.50279	4
57	.22995	4.34879	.24840	4.02574	.26701	3.74512	.28580	3.49894	3
58	.23026	4.34300	.24871	4.02074	.26733	3.74075	.28612	3.49509	2
59	.23056	4.33723	.24902	4.01576	.26764	3.73640	.28643	3.49125	1
60	.23087	4.33148	.24933	4.01078	.26795	3.73205	.28675	3.48741	0
′	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	′
	77°		76°		75°		74°		

TABLE XIV.—*Continued*

′	16°		17°		18°		19°		′
	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	
0	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35019	2.85555	42
19	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.29337	3.40869	.31242	3.20079	.33160	3.01489	.35117	2.84758	39
22	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.30097	3.32264	.32010	3.12400	.33945	2.94590	.35904	2.78523	15
46	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	′
	73°		72°		71°		70°		

Table XIV.—*Continued*

′	20°		21°		22°		23°		′
	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	39
22	.37124	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43239	2.31271	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	33
28	.37322	2.67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	27
34	.37521	2.66516	.39526	2.53001	.41558	2.40629	.43620	2.29254	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	18
43	.37820	2.64410	.39829	2.51076	.41865	2.38862	.43932	2.27626	17
44	.37853	2.64177	.39862	2.50864	.41899	2.38668	.43966	2.27447	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	8
53	.38153	2.62103	.40166	2.48967	.42207	2.36925	.44279	2.25840	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	5
56	.38253	2.61418	.40267	2.48340	.42310	2.36349	.44384	2.25309	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2.24780	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	0
′	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	′
	69°		68°		67°		66°		

TABLE XIV.—*Continued*

′	24°		25°		26°		27°		
	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	
0	.44523	2.24604	.46631	2.14451	.48773	2.05030	.50953	1.96261	60
1	.44558	2.24428	.46666	2.14288	.48809	2.04879	.50989	1.96120	59
2	.44593	2.24252	.46702	2.14125	.48845	2.04728	.51026	1.95979	58
3	.44627	2.24077	.46737	2.13963	.48881	2.04577	.51063	1.95838	57
4	.44662	2.23902	.46772	2.13801	.48917	2.04426	.51099	1.95698	56
5	.44697	2.23727	.46808	2.13639	.48953	2.04276	.51136	1.95557	55
6	.44732	2.23553	.46843	2.13477	.48989	2.04125	.51173	1.95417	54
7	.44767	2.23378	.46879	2.13316	.49026	2.03975	.51209	1.95277	53
8	.44802	2.23204	.46914	2.13154	.49062	2.03825	.51246	1.95137	52
9	.44837	2.23030	.46950	2.12993	.49098	2.03675	.51283	1.94997	51
10	.44872	2.22857	.46985	2.12832	.49134	2.03526	.51319	1.94858	50
11	.44907	2.22683	.47021	2.12671	.49170	2.03376	.51356	1.94718	49
12	.44942	2.22510	.47056	2.12511	.49206	2.03227	.51393	1.94579	48
13	.44977	2.22337	.47092	2.12350	.49242	2.03078	.51430	1.94440	47
14	.45012	2.22164	.47128	2.12190	.49278	2.02929	.51467	1.94301	46
15	.45047	2.21992	.47163	2.12030	.49315	2.02780	.51503	1.94162	45
16	.45082	2.21819	.47199	2.11871	.49351	2.02631	.51540	1.94023	44
17	.45117	2.21647	.47234	2.11711	.49387	2.02483	.51577	1.93885	43
18	.45152	2.21475	.47270	2.11552	.49423	2.02335	.51614	1.93746	42
19	.45187	2.21304	.47305	2.11392	.49459	2.02187	.51651	1.93608	41
20	.45222	2.21132	.47341	2.11233	.49495	2.02039	.51688	1.93470	40
21	.45257	2.20961	.47377	2.11075	.49532	2.01891	.51724	1.93332	39
22	.45292	2.20790	.47412	2.10916	.49568	2.01743	.51761	1.93195	38
23	.45327	2.20619	.47448	2.10758	.49604	2.01596	.51798	1.93057	37
24	.45362	2.20449	.47483	2.10600	.49640	2.01449	.51835	1.92920	36
25	.45397	2.20278	.47519	2.10442	.49677	2.01302	.51872	1.92782	35
26	.45432	2.20108	.47555	2.10284	.49713	2.01155	.51909	1.92645	34
27	.45467	2.19938	.47590	2.10126	.49749	2.01008	.51946	1.92508	33
28	.45502	2.19769	.47626	2.09969	.49786	2.00862	.51983	1.92371	32
29	.45537	2.19599	.47662	2.09811	.49822	2.00715	.52020	1.92235	31
30	.45573	2.19430	.47698	2.09654	.49858	2.00569	.52057	1.92098	30
31	.45608	2.19261	.47733	2.09498	.49894	2.00423	.52094	1.91962	29
32	.45643	2.19092	.47769	2.09341	.49931	2.00277	.52131	1.91826	28
33	.45678	2.18923	.47805	2.09184	.49967	2.00131	.52168	1.91690	27
34	.45713	2.18755	.47840	2.09028	.50004	1.99986	.52205	1.91554	26
35	.45748	2.18587	.47876	2.08872	.50040	1.99841	.52242	1.91418	25
36	.45784	2.18419	.47912	2.08716	.50076	1.99695	.52279	1.91282	24
37	.45819	2.18251	.47948	2.08560	.50113	1.99550	.52316	1.91147	23
38	.45854	2.18084	.47984	2.08405	.50149	1.99406	.52353	1.91012	22
39	.45889	2.17916	.48019	2.08250	.50185	1.99261	.52390	1.90876	21
40	.45924	2.17749	.48055	2.08094	.50222	1.99116	.52427	1.90741	20
41	.45960	2.17582	.48091	2.07939	.50258	1.98972	.52464	1.90607	19
42	.45995	2.17416	.48127	2.07785	.50295	1.98828	.52501	1.90472	18
43	.46030	2.17249	.48163	2.07630	.50331	1.98684	.52538	1.90337	17
44	.46065	2.17083	.48198	2.07476	.50368	1.98540	.52575	1.90203	16
45	.46101	2.16917	.48234	2.07321	.50404	1.98396	.52613	1.90069	15
46	.46136	2.16751	.48270	2.07167	.50441	1.98253	.52650	1.89935	14
47	.46171	2.16585	.48306	2.07014	.50477	1.98110	.52687	1.89801	13
48	.46206	2.16420	.48342	2.06860	.50514	1.97966	.52724	1.89667	12
49	.46242	2.16255	.48378	2.06706	.50550	1.97823	.52761	1.89533	11
50	.46277	2.16090	.48414	2.06553	.50587	1.97680	.52798	1.89400	10
51	.46312	2.15925	.48450	2.06400	.50623	1.97538	.52836	1.89266	9
52	.46348	2.15760	.48486	2.06247	.50660	1.97395	.52873	1.89133	8
53	.46383	2.15596	.48521	2.06094	.50696	1.97253	.52910	1.89000	7
54	.46418	2.15432	.48557	2.05942	.50733	1.97111	.52947	1.88867	6
55	.46454	2.15268	.48593	2.05790	.50769	1.96969	.52984	1.88734	5
56	.46489	2.15104	.48629	2.05637	.50806	1.96827	.53022	1.88602	4
57	.46525	2.14940	.48665	2.05485	.50843	1.96685	.53059	1.88469	3
58	.46560	2.14777	.48701	2.05333	.50879	1.96544	.53096	1.88337	2
59	.46595	2.14614	.48737	2.05182	.50916	1.96402	.53134	1.88205	1
60	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	0
′	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	′
	65°		64°		63°		62°		

TABLE XIV.—*Continued*

	28°		29°		30°		31°		
	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	
0	.53171	1.88073	.55431	1.80405	.57735	1.73205	.60086	1.66428	60
1	.53208	1.87941	.55469	1.80281	.57774	1.73089	.60126	1.66318	59
2	.53246	1.87809	.55507	1.80158	.57813	1.72973	.60165	1. 6209	58
3	.53283	1.87677	.55545	1.80034	.57851	1.72857	.60205	1.66099	57
4	.53320	1.87546	.55583	1.79911	.57890	1.72741	.60245	1.65990	56
5	.53358	1.87415	.55621	1.79788	.57929	1.72625	.60284	1.65881	55
6	.53395	1.87283	.55659	1.79665	.57968	1.72509	.60324	1.65772	54
7	.53432	1.87152	.55697	1.79542	.58007	1.72393	.60364	1.65663	53
8	.53470	1.87021	.55736	1.79419	.58046	1.72278	.60403	1.65534	52
9	.53507	1.86891	.55774	1.79296	.58085	1.72163	.60443	1.65445	51
10	.53545	1.86760	.55812	1.79174	.58124	1.72047	.60483	1.65337	50
11	.53582	1.86630	.55850	1.79051	.58162	1.71932	.60522	1.65228	49
12	.53620	1.86499	.55888	1.78929	.58201	1.71817	.60562	1.65120	48
13	.53657	1.86369	.55926	1.78807	.58240	1.71702	.60602	1.65011	47
14	.53694	1.86239	.55964	1.78685	.58279	1.71588	.60642	1.64903	46
15	.53732	1.86109	.56003	1.78563	.58318	1.71473	.60681	1.64795	45
16	.53769	1.85979	.56041	1.78441	.58357	1.71358	.60721	1.64687	44
17	.53807	1.85850	.56079	1.78319	.58396	1.71244	.60761	1.64579	43
18	.53844	1.85720	.56117	1.78198	.58435	1.71129	.60801	1.64471	42
19	.53882	1.85591	.56156	1.78077	.58474	1.71015	.60841	1.64363	41
20	.53920	1.85462	.56194	1.77955	.58513	1.70901	.60881	1.64256	40
21	.53957	1.85333	.56232	1.77834	.58552	1.70787	.60921	1.64148	39
22	.53995	1.85204	.56270	1.77713	.58591	1.70673	.60960	1.64041	38
23	.54032	1.85075	.56309	1.77592	.58631	1.70560	.61000	1.63934	37
24	.54070	1.84946	.56347	1.77471	.58670	1.70446	.61040	1.63826	36
25	.54107	1.84818	.56385	1.77351	.58709	1.70332	.61080	1.63719	35
26	.54145	1.84689	.56424	1.77230	.58748	1.70219	.61120	1.63612	34
27	.54183	1.84561	.56462	1.77110	.58787	1.70106	.61160	1.63505	33
28	.54220	1.84433	.56500	1.76990	.58826	1.69992	.61200	1.63398	32
29	.54258	1.84305	.56539	1.76869	.58865	1.69879	.61240	1.63292	31
30	.54296	1.84177	.56577	1.76749	.58904	1.69766	.61280	1.63185	30
31	.54333	1.84049	.56616	1.76630	.58944	1.69653	.61320	1.63079	29
32	.54371	1.83922	.56654	1.76510	.58983	1.69541	.61360	1.62972	28
33	.54409	1.83794	.56693	1.76390	.59022	1.69428	.61400	1.62866	27
34	.54446	1.83667	.56731	1.76271	.59061	1.69316	.61440	1.62760	26
35	.54484	1.83540	.56769	1.76151	.59101	1.69203	.61480	1.62654	25
36	.54522	1.83413	.56808	1.76032	.59140	1.69091	.61520	1.62548	24
37	.54560	1.83286	.56846	1.75913	.59179	1.68979	.61561	1.62442	23
38	.54597	1.83159	.56885	1.75794	.59218	1.68866	.61601	1.62336	22
39	.54635	1.83033	.56923	1.75675	.59258	1.68754	.61641	1.62230	21
40	.54673	1.82906	.56962	1.75556	.59297	1.68643	.61681	1.62125	20
41	.54711	1.82780	.57000	1.75437	.59336	1.68531	.61721	1.62019	19
42	.54748	1.82654	.57039	1.75319	.59376	1.68419	.61761	1.61914	18
43	.54786	1.82528	.57078	1.75200	.59415	1.68308	.61801	1.61808	17
44	.54824	1.82402	.57116	1.75082	.59454	1.68196	.61842	1.61703	16
45	.54862	1.82276	.57155	1.74964	.59494	1.68085	.61882	1.61598	15
46	.54900	1.82150	.57193	1.74846	.59533	1.67974	.61922	1.61493	14
47	.54938	1.82025	.57232	1.74728	.59573	1.67863	.61962	1.61388	13
48	.54975	1.81899	.57271	1.74610	.59612	1.67752	.62003	1.61283	12
49	.55013	1.81774	.57309	1.74492	.59651	1.67641	.62043	1.61179	11
50	.55051	1.81649	.57348	1.74375	.59691	1.67530	.62083	1.61074	10
51	.55089	1.81524	.57386	1.74257	.59730	1.67419	.62124	1.60970	9
52	.55127	1.81399	.57425	1.74140	.59770	1.67309	.62164	1.60865	8
53	.55165	1.81274	.57464	1.74022	.59809	1.67198	.62204	1.60761	7
54	.55203	1.81150	.57503	1.73905	.59849	1.67088	.62245	1.60657	6
55	.55241	1.81025	.57541	1.73788	.59888	1.66978	.62285	1.60553	5
56	.55279	1.80901	.57580	1.73671	.59928	1.66867	.62325	1.60449	4
57	.55317	1.80777	.57619	1.73555	.59967	1.66757	.62366	1.60345	3
58	.55355	1.80653	.57657	1.73438	.60007	1.66647	.62406	1.60241	2
59	.55393	1.80529	.57696	1.73321	.60046	1.66538	.62446	1.60137	1
60	.55431	1.80405	.57735	1.73205	.60086	1.66428	.62487	1.60033	0
	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	
	61°		60°		59°		58°		

TABLE XIV.—*Continued*

′	32°		33°		34°		35°		′
	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	
0	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	60
1	.62527	1.59930	.64982	1.53888	.67493	1.48163	.70064	1.42726	59
2	.62568	1.59826	.65023	1.53791	.67536	1.48070	.70107	1.42638	58
3	.62608	1.59723	.65065	1.53693	.67578	1.47977	.70151	1.42550	57
4	.62649	1.59620	.65106	1.53595	.67620	1.47885	.70194	1.42462	56
5	.62689	1.59517	.65148	1.53497	.67663	1.47792	.70238	1.42374	55
6	.62730	1.59414	.65189	1.53400	.67705	1.47699	.70281	1.42286	54
7	.62770	1.59311	.65231	1.53302	.67748	1.47607	.70325	1.42198	53
8	.62811	1.59208	.65272	1.53205	.67790	1.47514	.70368	1.42110	52
9	.62852	1.59105	.65314	1.53107	.67832	1.47422	.70412	1.42022	51
10	.62892	1.59002	.65355	1.53010	.67875	1.47330	.70455	1.41934	50
11	.62933	1.58900	.65397	1.52913	.67917	1.47238	.70499	1.41847	49
12	.62973	1.58797	.65438	1.52816	.67960	1.47146	.70542	1.41759	48
13	.63014	1.58695	.65480	1.52719	.68002	1.47053	.70586	1.41672	47
14	.63055	1.58593	.65521	1.52622	.68045	1.46962	.70629	1.41584	46
15	.63095	1.58490	.65563	1.52525	.68088	1.46870	.70673	1.41497	45
16	.63136	1.58388	.65604	1.52429	.68130	1.46778	.70717	1.41409	44
17	.63177	1.58286	.65646	1.52332	.68173	1.46686	.70760	1.41322	43
18	.63217	1.58184	.65688	1.52235	.68215	1.46595	.70804	1.41235	42
19	.63258	1.58083	.65729	1.52139	.68258	1.46503	.70848	1.41148	41
20	.63299	1.57981	.65771	1.52043	.68301	1.46411	.70891	1.41061	40
21	.63340	1.57879	.65813	1.51946	.68343	1.46320	.70935	1.40974	39
22	.63380	1.57778	.65854	1.51850	.68386	1.46229	.70979	1.40887	38
23	.63421	1.57676	.65896	1.51754	.68429	1.46137	.71023	1.40800	37
24	.63462	1.57575	.65938	1.51658	.68471	1.46046	.71066	1.40714	36
25	.63503	1.57474	.65980	1.51562	.68514	1.45955	.71110	1.40627	35
26	.63544	1.57372	.66021	1.51466	.68557	1.45864	.71154	1.40540	34
27	.63584	1.57271	.66063	1.51370	.68600	1.45773	.71198	1.40454	33
28	.63625	1.57170	.66105	1.51275	.68642	1.45682	.71242	1.40367	32
29	.63666	1.57069	.66147	1.51179	.68685	1.45592	.71285	1.40281	31
30	.63707	1.56969	.66189	1.51084	.68728	1.45501	.71329	1.40195	30
31	.63748	1.56868	.66230	1.50988	.68771	1.45410	.71373	1.40109	29
32	.63789	1.56767	.66272	1.50893	.68814	1.45320	.71417	1.40022	28
33	.63830	1.56667	.66314	1.50797	.68857	1.45229	.71461	1.39936	27
34	.63871	1.56566	.66356	1.50702	.68900	1.45139	.71505	1.39850	26
35	.63912	1.56466	.66398	1.50607	.68942	1.45049	.71549	1.39764	25
36	.63953	1.56366	.66440	1.50512	.68985	1.44958	.71593	1.39679	24
37	.63994	1.56265	.66482	1.50417	.69028	1.44868	.71637	1.39593	23
38	.64035	1.56165	.66524	1.50322	.69071	1.44778	.71681	1.39507	22
39	.64076	1.56065	.66566	1.50228	.69114	1.44688	.71725	1.39421	21
40	.64117	1.55966	.66608	1.50133	.69157	1.44598	.71769	1.39336	20
41	.64158	1.55866	.66650	1.50038	.69200	1.44508	.71813	1.39250	19
42	.64199	1.55766	.66692	1.49944	.69243	1.44418	.71857	1.39165	18
43	.64240	1.55666	.66734	1.49849	.69286	1.44329	.71901	1.39079	17
44	.64281	1.55567	.66776	1.49755	.69329	1.44239	.71946	1.38994	16
45	.64322	1.55467	.66818	1.49661	.69372	1.44149	.71990	1.38909	15
46	.64363	1.55368	.66860	1.49566	.69416	1.44060	.72034	1.38824	14
47	.64404	1.55269	.66902	1.49472	.69459	1.43970	.72078	1.38738	13
48	.64446	1.55170	.66944	1.49378	.69502	1.43881	.72122	1.38653	12
49	.64487	1.55071	.66986	1.49284	.69545	1.43792	.72166	1.38568	11
50	.64528	1.54972	.67028	1.49190	.69588	1.43703	.72211	1.38484	10
51	.64569	1.54873	.67071	1.49097	.69631	1.43614	.72255	1.38399	9
52	.64610	1.54774	.67113	1.49003	.69675	1.43525	.72299	1.38314	8
53	.64652	1.54675	.67155	1.48909	.69718	1.43436	.72344	1.38229	7
54	.64693	1.54576	.67197	1.48816	.69761	1.43347	.72388	1.38145	6
55	.64734	1.54478	.67239	1.48722	.69804	1.43258	.72432	1.38060	5
56	.64775	1.54379	.67282	1.48629	.69847	1.43169	.72477	1.37976	4
57	.64817	1.54281	.67324	1.48536	.69891	1.43080	.72521	1.37891	3
58	.64858	1.54183	.67366	1.48442	.69934	1.42992	.72565	1.37807	2
59	.64899	1.54085	.67409	1.48349	.69977	1.42903	.72610	1.37722	1
60	.64941	1.53986	.67451	1.48256	.70021	1.42815	.72054	1.37638	0
′	Co-tan.	Tan.	Co-tan	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	′
	57°		56°		55°		54°		

TABLE XIV.—*Continued*

′	36°		37°		38°		39°		′
	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	
0	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	60
1	.72699	1.37554	.75401	1.32624	.78175	1.27917	.81027	1.23416	59
2	.72743	1.37470	.75447	1.32544	.78222	1.27841	.81075	1.23343	58
3	.72788	1.37386	.75492	1.32464	.78269	1.27764	.81123	1.23270	57
4	.72832	1.37302	.75538	1.32384	.78316	1.27688	.81171	1.23196	56
5	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	55
6	.72921	1.37134	.75629	1.32224	.78410	1.27535	.81268	1.23050	54
7	.72966	1.37050	.75675	1.32144	.78457	1.27458	.81316	1.22977	53
8	.73010	1.36967	.75721	1.32064	.78504	1.27382	.81364	1.22904	52
9	.73055	1.36883	.75767	1.31984	.78551	1.27306	.81413	1.22831	51
10	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	50
11	.73144	1.36716	.75858	1.31825	.78645	1.27153	.81510	1.22685	49
12	.73189	1.36633	.75904	1.31745	.78692	1.27077	.81558	1.22612	48
13	.73234	1.36549	.75950	1.31666	.78739	1.27001	.81606	1.22539	47
14	.73278	1.36466	.75996	1.31586	.78786	1.26925	.81655	1.22467	46
15	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	45
16	.73368	1.36300	.76088	1.31427	.78881	1.26774	.81752	1.22321	44
17	.73413	1.36217	.76134	1.31348	.78928	1.26698	.81800	1.22249	43
18	.73457	1.36133	.76180	1.31269	.78975	1.26622	.81849	1.22176	42
19	.73502	1.36051	.76226	1.31190	.79022	1.26546	.81898	1.22104	41
20	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	40
21	.73592	1.35885	.76318	1.31031	.79117	1.26395	.81995	1.21959	39
22	.73637	1.35802	.76364	1.30952	.79164	1.26319	.82044	1.21886	38
23	.73681	1.35719	.76410	1.30873	.79212	1.26244	.82092	1.21814	37
24	.73726	1.35637	.76456	1.30795	.79259	1.26169	.82141	1.21742	36
25	.73771	1.35554	.76502	1.30716	.79306	1.26093	.82190	1.21670	35
26	.73816	1.35472	.76548	1.30637	.79354	1.26018	.82238	1.21598	34
27	.73861	1.35389	.76594	1.30558	.79401	1.25943	.82287	1.21526	33
28	.73906	1.35307	.76640	1.30480	.79449	1.25867	.82336	1.21454	32
29	.73951	1.35224	.76686	1.30401	.79496	1.25792	.82385	1.21382	31
30	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	30
31	.74041	1.35060	.76779	1.30244	.79591	1.25642	.82483	1.21238	29
32	.74086	1.34978	.76825	1.30166	.79639	1.25567	.82531	1.21166	28
33	.74131	1.34896	.76871	1.30087	.79686	1.25492	.82580	1.21094	27
34	.74176	1.34814	.76918	1.30009	.79734	1.25417	.82629	1.21023	26
35	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	25
36	.74267	1.34650	.77010	1.29853	.79829	1.25268	.82727	1.20879	24
37	.74312	1.34568	.77057	1.29775	.79877	1.25193	.82776	1.20808	23
38	.74357	1.34487	.77103	1.29696	.79924	1.25118	.82825	1.20736	22
39	.74402	1.34405	.77149	1.29618	.79972	1.25044	.82874	1.20665	21
40	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	20
41	.74492	1.34242	.77242	1.29463	.80067	1.24895	.82972	1.20522	19
42	.74538	1.34160	.77289	1.29385	.80115	1.24820	.83022	1.20451	18
43	.74583	1.34079	.77335	1.29307	.80163	1.24746	.83071	1.20379	17
44	.74628	1.33998	.77382	1.29229	.80211	1.24672	.83120	1.20308	16
45	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	15
46	.74719	1.33835	.77475	1.29074	.80306	1.24523	.83218	1.20166	14
47	.74764	1.33754	.77521	1.28997	.80354	1.24449	.83268	1.20095	13
48	.74810	1.33673	.77568	1.28919	.80402	1.24375	.83317	1.20024	12
49	.74855	1.33592	.77615	1.28842	.80450	1.24301	.83366	1.19953	11
50	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	10
51	.74946	1.33430	.77708	1.28687	.80546	1.24153	.83465	1.19811	9
52	.74991	1.33349	.77754	1.28610	.80594	1.24079	.83514	1.19740	8
53	.75037	1.33268	.77801	1.28533	.80642	1.24005	.83564	1.19669	7
54	.75082	1.33187	.77848	1.28456	.80690	1.23931	.83613	1.19599	6
55	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	5
56	.75173	1.33026	.77941	1.28302	.80786	1.23784	.83712	1.19457	4
57	.75219	1.32946	.77988	1.28225	.80834	1.23710	.83761	1.19387	3
58	.75264	1.32865	.78035	1.28148	.80882	1.23637	.83811	1.19316	2
59	.75310	1.32785	.78082	1.28071	.80930	1.23563	.83860	1.19246	1
60	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19175	0
′	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan	′
	53°		52°		51°		50°		

TABLE XIV.—*Continued*

′	40° Tan.	40° Co-tan.	41° Tan.	41° Co-tan.	42° Tan.	42° Co-tan.	43° Tan.	43° Co-tan.	′
0	.83910	1.19175	.86929	1.15037	.90040	1.11061	.93252	1.07237	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	58
3	.84059	1.18964	.87082	1.14834	.90199	1.10867	.93415	1.07049	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	33
28	.85307	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	28
33	.85559	1.16378	.88628	1.12831	.91794	1.08940	.95062	1.05194	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	25
36	.85710	1.16672	.88784	1.12633	.91955	1.08749	.95229	1.05010	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	13
48	.86318	1.15851	.89410	1.11844	.92691	1.07990	.95897	1.04279	12
49	.86368	1.15783	.89463	1.11778	.92655	1.07927	.95952	1.04218	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	0
′	Co-tan. 49°	Tan. 49°	Co-tan. 48°	Tan. 48°	Co-tan. 47°	Tan. 47°	Co-tan. 46°	Tan. 46°	′

TABLE XIV.—*Concluded*

	44°				44°				44°		
′	TAN.	CO-TAN.	′	′	TAN.	CO-TAN.	′	′	TAN.	CO-TAN.	′
0	.96569	1.03553	60	21	.97756	1.02295	39	41	.98901	1.01112	19
1	.96625	1.03493	59	22	.97813	1.02236	38	42	.98958	1.01053	18
2	.96681	1.03433	58	23	.97870	1.02176	37	43	.99016	1.00994	17
3	.96738	1.03372	57	24	.97927	1.02117	36	44	.99073	1.00935	16
4	.96794	1.03312	56	25	.97984	1.02057	35	45	.99131	1.00876	15
5	.96850	1.03252	55	26	.98041	1.01998	34	46	.99189	1.00818	14
6	.96907	1.03192	54	27	.98098	1.01939	33	47	.99247	1.00759	13
7	.96963	1.03132	53	28	.98155	1.01879	32	48	.99304	1.00701	12
8	.97020	1.03072	52	29	.98213	1.01820	31	49	.99362	1.00642	11
9	.97076	1.03012	51	30	.98270	1.01761	30	50	.99420	1.00583	10
10	.97133	1.02952	50	31	.98327	1.01702	29	51	.99478	1.00525	9
11	.97189	1.02892	49	32	.98384	1.01642	28	52	.99536	1.00467	8
12	.97246	1.02832	48	33	.98441	1.01583	27	53	.99594	1.00408	7
13	.97302	1.02772	47	34	.98499	1.01524	26	54	.99652	1.00350	6
14	.97359	1.02713	46	35	.98556	1.01465	25	55	.99710	1.00291	5
15	.97416	1.02653	45	36	.98613	1.01406	24	56	.99768	1.00233	4
16	.97472	1.02593	44	37	.98671	1.01347	23	57	.99826	1.00175	3
17	.97529	1.02533	43	38	.98728	1.01288	22	58	.99884	1.00116	2
18	.97586	1.02474	42	39	.98786	1.01229	21	59	.99942	1.00058	1
19	.97643	1.02414	41	40	.98843	1.01170	20	60	1	1	0
20	.97700	1.02355	40								
′	CO-TAN.	TAN.	′	′	CO-TAN.	TAN.	′	′	CO-TAN.	TAN.	′
	45°				45°				45°		

TABLE XV.—CONVERGENCY OF MERIDIANS, SIX MILES LONG AND SIX MILES APART, AND DIFFERENCES OF LATITUDE AND LONGITUDE

Lat.	Convergency		Difference of longitude per range		Difference of latitude for—	
	On the parallel	Angle	In arc	In time	1 mi.	1 Tp.
°	*Lks.*	′ ″	′ ″	*Seconds*	′	′
25	33.9	2 25	5 44.34	22.96		
26	35.4	2 32	5 47.20	23.15		
27	37.0	2 39	5 50.22	23.35	0.871	5.229
28	38.6	2 46	5 53.40	23.56		
29	40.2	2 53	5 56.74	23.78		
30	41.9	3 0	6 0.26	24.02		
31	43.6	3 7	6 3.97	24.26		
32	45.4	3 15	6 7.87	24.52	0.871	5.225
33	47.2	3 23	6 11.96	24.80		
34	49.1	3 30	6 16.26	25.08		
35	50.9	3 38	6 20.78	25.39		
36	52.7	3 46	6 25.53	25.70		
37	54.7	3 55	6 30.52	26.03	0.870	5.221
38	56.8	4 4	6 35.76	26.38		
39	58.8	4 13	6 41.27	26.75		
40	60.9	4 22	6 47.06	27.14		
41	63.1	4 31	6 53.15	27.54		
42	65.4	4 41	6 59.56	27.97	0.869	5.216
43	67.7	4 51	7 6.29	28.42		
44	70.1	5 1	7 13.39	28.89		
45	72.6	5 12	7 20.86	29.39		
46	75.2	5 23	7 28.74	29.92		
47	77.8	5 34	7 37.04	30.47	0.869	5.211
48	80.6	5 46	7 45.80	31.05		
49	83.5	5 59	7 55.05	31.67		
50	86.4	6 12	8 4.83	32.32		
51	89.6	6 25	8 15.17	33.03		
52	92.8	6 39	8 26.13	33.74	0.868	5.207
53	96.2	6 54	8 37.75	34.52		
54	99.8	7 9	8 50.07	35.34		
55	103.5	7 25	9 3.18	36.22		
56	107.5	7 42	9 17.12	37.14		
57	111.6	8 0	9 31.97	38.13	0.867	5.202
58	116.0	8 19	9 47.83	39.19		
59	120.6	8 38	10 4.78	40.32		
60	125.5	8 59	10 22.94	41.52		
61	130.8	9 22	10 42.42	42.83		
62	136.3	9 46	11 3.38	44.22	0.866	5.198
63	142.2	10 11	11 25.97	45.73		
64	148.6	10 38	11 50.37	47.36		
65	155.0	11 8	12 16.82	49.12		
66	162.8	11 39	12 45.55	51.04		
67	170.7	12 13	13 16.88	53.12	0.866	5.195
68	179.3	12 51	13 51.15	55.41		
69	188.7	13 31	14 28.77	57.92		
70	199.1	14 15	15 10.26	60.68	0.866	5.193

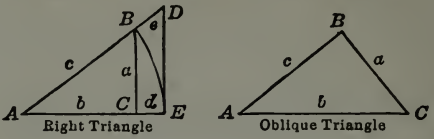
TABLE XVI.—AZIMUTHS OF THE SECANT

Lat.	0 mi.	1 mi.	2 mi.	3 mi.	Deflection angle 6 mi.
°	° ′	° ′	° ′		′ ″
25	89 58.8	89 59.2	89 59.6	90°	2 25
26	58.7	59.2	59.6	E or W.	2 32
27	58.7	59.1	59.6	" " "	2 39
28	58.6	59.1	59.5	" " "	2 46
29	58.6	59.0	59.5	" " "	2 53
30	58.5	59.0	59.5	" " "	3 0
31	58.4	59.0	59.5	" " "	3 7
32	58.4	58.9	59.5	" " "	3 15
33	58.3	58.9	59.4	" " "	3 23
34	58.2	58.8	59.4	" " "	3 30
35	58.2	58.8	59.4	" " "	3 38
36	58.1	58.7	59.4	" " "	3 46
37	58.0	58.7	59.3	" " "	3 55
38	58.0	58.6	59.3	" " "	4 4
39	57.9	58.6	59.3	" " "	4 13
40	57.8	58.5	59.3	" " "	4 22
41	57.7	58.5	59.2	" " "	4 31
42	57.7	58.4	59.2	" " "	4 41
43	57.6	58.4	59.2	" " "	4 51
44	57.5	58.3	59.2	" " "	5 1
45	57.4	58.3	59.1	" " "	5 12
46	57.3	58.2	59.1	" " "	5 23
47	57.2	58.1	59.1	" " "	5 34
48	57.1	58.1	59.0	" " "	5 46
49	57.0	58.0	59.0	" " "	5 59
50	56.9	57.9	59.0	" " "	6 12
51	56.8	57.9	58.9	" " "	6 25
52	56.7	57.8	58.9	" " "	6 39
53	56.6	57.7	58.8	" " "	6 54
54	56.4	57.6	58.8	" " "	7 9
55	56.3	57.5	58.8	" " "	7 25
56	56.2	57.4	58.7	" " "	7 42
57	56.0	57.3	58.7	" " "	8 0
58	55.8	57.2	58.6	" " "	8 19
59	55.7	57.1	58.6	" " "	8 38
60	55.5	57.0	58.5	" " "	8 59
61	55.3	56.9	58.4	" " "	9 22
62	55.1	56.7	58.4	" " "	9 46
63	54.9	56.6	58.3	" " "	10 11
64	54.7	56.5	58.2	" " "	10 38
65	54.4	56.3	58.1	" " "	11 8
66	54.2	56.1	58.1	" " "	11 39
67	53.9	55.9	58.0	" " "	12 13
68	53.6	55.7	57.9	" " "	12 51
69	53.2	55.5	57.8	" " "	13 31
70	89° 52′.9	89° 55′.3	89° 57′.6	" " "	14′ 15″
	6 mi.	5 mi.	4 mi.	3 mi.	

TABLE XVII.—OFFSETS, IN LINKS, FROM THE SECANT TO THE PARALLEL

Lat.	0 mi.	½ mi.	1 mi.	1½ mi.	2 mi.	2½ mi.	3 mi.
°							
25	2 N.	1 N.	0	1 S.	1 S.	2 S.	2 S.
26	2	1	0	1	1	2	2
27	3	1	0	1	2	2	2
28	3	1	0	1	2	2	2
29	3	1	0	1	2	2	2
30	3	1	0	1	2	2	2
31	3	1	0	1	2	2	2
32	3	1	0	1	2	2	3
33	3	1	0	1	2	2	3
34	3	2	0	1	2	3	3
35	4	2	0	1	2	3	3
36	4	2	0	1	2	3	3
37	4	2	0	1	2	3	3
38	4	2	0	1	2	3	3
39	4	2	0	1	2	3	3
40	4	2	0	1	3	3	3
41	4	2	0	2	3	3	4
42	5	2	0	2	3	3	4
43	5	2	0	2	3	4	4
44	5	2	0	2	3	4	4
45	5	2	0	2	3	4	4
46	5	2	0	2	3	4	4
47	5	2	0	2	3	4	4
48	6	3	0	2	3	4	4
49	6	3	0	2	3	4	5
50	6	3	0	2	4	4	5
51	6	3	0	2	4	5	5
52	6	3	0	2	4	5	5
53	7	3	0	2	4	5	5
54	7	3	0	2	4	5	6
55	7	3	0	3	4	5	6
56	7	3	0	3	4	6	6
57	8	3	0	3	5	6	6
58	8	4	0	3	5	6	6
59	8	4	0	3	5	6	7
60	9	4	0	3	5	7	7
61	9	4	0	3	5	7	7
62	9	4	0	3	6	7	8
63	10	4	0	3	6	7	8
64	10	5	0	4	6	8	8
65	11	5	0	4	6	8	9
66	11	5	0	4	7	8	9
67	12	5	0	4	7	9	9
68	12	6	0	4	7	9	10
69	13	6	0	5	8	10	10
70	14 N.	6 N.	0	5 S.	8 S.	10 S.	11 S.
	6 mi.	5½ mi.	5 mi.	4½ mi.	4 mi.	3½ mi.	3 mi.

TABLE XVIII.—TRIGONOMETRIC FORMULAS



Right Triangle Oblique Triangle

RIGHT TRIANGLES

$$\sin A = \frac{a}{c} = \cos B \qquad \sec A = \frac{c}{b} = \operatorname{cosec} B$$

$$\cos A = \frac{b}{c} = \sin B \qquad \operatorname{cosec} A = \frac{c}{a} = \sec B$$

$$\tan A = \frac{a}{b} = \cot B \qquad \operatorname{vers} A = \frac{c-b}{c} = \frac{d}{c}$$

$$\cot A = \frac{b}{a} = \tan B \qquad \operatorname{exsec} A = \frac{e}{c}$$

$$a = c \sin A = c \cos B = b \tan A = b \cot B = \sqrt{c^2 - b^2}$$

$$b = c \cos A = c \sin B = a \cot A = a \tan B = \sqrt{c^2 - a^2}$$

$$c = \frac{a}{\sin A} = \frac{a}{\cos B} = \frac{b}{\sin B} = \frac{b}{\cos A} = \frac{d}{\operatorname{vers} A} = \frac{e}{\operatorname{exsec} A} = \sqrt{a^2 + b^2}$$

$$d = c \operatorname{vers} A \qquad e = c \operatorname{exsec} A$$

OBLIQUE TRIANGLES

Given	Sought	Formulas
A, B, a	b, c	$b = \frac{a}{\sin A} \cdot \sin B \qquad c = \frac{a}{\sin A} \cdot \sin (A + B)$
A, a, b	B, c	$\sin B = \frac{\sin A}{a} \cdot b \qquad c = \frac{a}{\sin A} \cdot \sin C$
C, a, b	$\frac{1}{2}(A + B)$	$\frac{1}{2}(A + B) = 90^\circ - \frac{1}{2}C$
	$\frac{1}{2}(A - B)$	$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \cdot \tan \frac{1}{2}(A + B)$
a, b, c	A	If $s = \frac{1}{2}(a + b + c)$, $\sin \frac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}}$ $\cos \frac{1}{2}A = \sqrt{\frac{s(s - a)}{bc}}$, $\tan \frac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}$ $\sin A = 2 \frac{\sqrt{s(s - a)(s - b)(s - c)}}{bc}$ $\operatorname{vers} A = \frac{2(s - b)(s - c)}{bc}$
	area	$area = \sqrt{s(s - a)(s - b)(s - c)}$
C, a, b	*area*	$area = \frac{1}{2}ab \sin C$
A, B, C, a	*area*	$area = \frac{a^2 \sin B \cdot \sin C}{2 \sin A}$

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